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Variations on a theorem of Cowling and Price with applications to nilpotent Lie groups

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VARIATIONS ON A THEOREM OF COWLING AND PRICE WITH APPLICATIONS TO NILPOTENT LIE GROUPS

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ABSTRACT. In this paper we prove a new version of Cowling–Price theorem for Fourier transform on \mathbb{R}^n using which we formulate and prove an uncertainty principle for operators. This leads to an analogue of Cowling–Price theorem for nilpotent Lie groups. We also prove an exact analogue of Cowling–Price theorem for the Heisenberg group.

1. INTRODUCTION

Consider functions f on \mathbb{R}^n which satisfy estimates of the form

$$|f(x)| \leq c e^{-a|x|^2}, |\hat{f}(\xi)| \leq c e^{-b|\xi|^2}$$

for some $a, b > 0$ where \hat{f} is the Fourier transform of f defined by

$$(1.1) \quad \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx.$$

A classical theorem of Hardy [8] proved way back in 1933 states that for nontrivial f the product ab is at most $\frac{1}{4}$ and the maximum value is attained precisely when $f(x) = c e^{-a|x|^2}$. Since $ab > \frac{1}{4}$ implies $f = 0$ the result of Hardy is an example of uncertainty principle for the Fourier transform. The case $ab = \frac{1}{4}$ is considered as a characterisation of the Gaussian.

In 1983 Cowling and Price [5] obtained a generalization of Hardy’s theorem. Let us define $\phi_a(x) = e^{-a|x|^2}$. Then we have

Theorem 1.1. (Cowling–Price) For $1 \leq p, q \leq \infty$ let $f\phi_a^{-1} \in L^p(\mathbb{R}^n)$ and $\hat{f}\phi_b^{-1} \in L^q(\mathbb{R}^n)$. Then $f = 0$ whenever $ab > \frac{1}{4}$.

Note that $p = q = \infty$ is Hardy’s theorem with $ab > \frac{1}{4}$. The above result is true even if $ab = \frac{1}{4}$ under the added assumption that $\min(p, q) < \infty$. A further generalization of Cowling–Price theorem for the case $ab = \frac{1}{4}$ has been recently obtained by Bonami et al [4].

Analogues of Hardy and Cowling–Price theorems have received considerable attention during the last decade, see the monograph [16] and the references there. In this paper we are mainly concerned with Cowling–Price theorem for the group Fourier transform on nilpotent Lie groups. Such a theorem can be proved for the simplest case of the Heisenberg group.

Let H^n be the $(2n + 1)$ dimensional Heisenberg group and let $\hat{f}(\lambda)$, $\lambda \in \mathbb{R} \setminus \{0\}$ be the group Fourier transform of a function f on H^n . The role of the Gaussian ϕ_a will be played by the heat kernel $q_a(z, t)$ associated to the sublaplacian on H^n . Let $H(\lambda) = -\Delta + \lambda^2|x|^2$ be the scaled Hermite operator. Then we have the following result.

Theorem 1.2. Let f be a function on H^n which satisfies $f q_a^{-1} \in L^p(H^n)$, $1 \leq p \leq \infty$ and $\hat{f}(\lambda) e^{bH(\lambda)}$ is a bounded operator in $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$. Then $f = 0$ whenever $a < b$.

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The case $p = \infty$ is Hardy's theorem for the Heisenberg group which can be found in [14]. We are interested in finding an analogue of the above result for stratified nilpotent groups. An examination of the proof of the above theorem, given in section 3, reveals that we need explicit formula for the heat kernel and good estimates. Due to the lack of such information in the general case, we look for alternative versions of the Cowling–Price theorem.

Returning to the Euclidean case consider the Gaussian $\phi_a(x)$ for which $\hat{\phi}_a = C\phi_b$ with $b = \frac{1}{4a}$. In view of the Plancherel theorem we have

$$\|\partial^\alpha \hat{\phi}_a\|_2^2 = \int_{\mathbb{R}^n} |x^\alpha \phi_a(x)|^2 dx = C \prod_{j=1}^n \int_0^\infty t^{2\alpha_j} e^{-2at^2} dt$$

which gives the estimate

$$(1.2) \quad \|\partial^\alpha \hat{\phi}_a\|_2^2 = C \prod_{j=1}^n \Gamma\left(\alpha_j + \frac{1}{2}\right) (2a)^{-|\alpha|} \leq C \alpha!(2b)^{|\alpha|}.$$

If a function f satisfies $|f(x)| \leq C \phi_a(x)$ then the derivatives of \hat{f} satisfy the estimates

$$(1.3) \quad \|\partial^\alpha \hat{f}\|_2^2 \leq C \alpha!(2a)^{-|\alpha|}.$$

Replacing the pointwise estimate $|f(x)| \leq C \phi_a(x)$ by the slightly weaker estimates (1.3) we get the following uncertainty principle.

Theorem 1.3. *Let f be a function on \mathbb{R}^n such that $|\hat{f}(\xi)| \leq C e^{-b|\xi|^2}$, and for every $\alpha \in \mathbb{N}^n$, $\|\partial^\alpha \hat{f}\|_2^2 \leq C \alpha!(2a)^{-|\alpha|}$. Then $f = 0$ whenever $ab > \frac{1}{4}$, when $ab = \frac{1}{4}$, $\hat{f}(\xi) = \phi(\xi)e^{-b|\xi|^2}$ where ϕ is a bounded function.*

We remark that Theorem 1.3 (case $ab > \frac{1}{4}$) is equivalent to Theorem 1.1 (case $p = 2, q = \infty$). Thus Theorem 1.3 is another version of Cowling–Price theorem. Stated in this form it has natural extension to the case of nilpotent Lie groups.

The group Fourier transform on a nilpotent Lie group G is operator valued. Given an irreducible unitary representation π of G and a function f on G the operator $\hat{f}(\pi) = \pi(f)$ is realised on $L^2(\mathbb{R}^n)$ for a suitable n . In order to formulate an analogue of Theorem 1.3 we need such a result for operators.

Given a bounded linear operator T on $L^2(\mathbb{R}^n)$ we define certain noncommutative derivations of T by

$$(1.4) \quad \delta_j T = [A_j, T], \quad \bar{\delta}_j T = [T, A_j^*]$$

where $[T, S] = TS - ST$ is the commutator and $A_j = \frac{\partial}{\partial \xi_j} + \xi_j$, $A_j^* = -\frac{\partial}{\partial \xi_j} + \xi_j$ are the annihilation and creation operators. The above derivations were introduced by Mauceri [10] and the second author has used them on several occasions, see [15, 16]. For multiindices α, β we define $\delta^\alpha T$ and $\bar{\delta}^\beta T$ iteratively. Let $H = -\Delta + |x|^2$ be the Hermite operator on \mathbb{R}^n which generates the semigroup e^{-tH} , $t > 0$. We denote by \mathcal{S}_q ($q > 0$) the set of all linear operators T on $L^2(\mathbb{R}^n)$ such that $\text{tr}(|T|^q) < \infty$. For $q \geq 1$, \mathcal{S}_q endowed with the norm $\|T\|_q := (\text{tr}(|T|^q))^{\frac{1}{q}}$ is a complete subalgebra of the set of all bounded operators on $L^2(\mathbb{R}^n)$. In particular for $q = 2$, \mathcal{S}_2 is the Hilbert space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$ equipped with the inner product $(T, S) = \text{tr}(TS^*)$. Let $\|T\|_{\text{HS}}$ be the norm of T in this Hilbert space. With these notations we are ready to state our operator analogue of Theorem 1.3.

Theorem 1.4. *Let $T \in \mathcal{S}_2$ satisfy the estimates*

$$T^* T \leq c e^{-2bH}, \quad \|\delta^\alpha \bar{\delta}^\beta (T^* T)\|_{\text{HS}}^2 \leq c (\alpha + \beta)! a^{|\alpha| + |\beta|}$$

for some $a, b > 0$ for all $\alpha, \beta \in \mathbb{N}^n$. Then $T = 0$ whenever $a < 2 \tanh 2b$.

This theorem is an uncertainty principle for the Weyl transform for the following reason. As is well known every $T \in \mathcal{S}_2$ is of the form $W(f)$ for some $f \in L^2(\mathbb{C}^n)$; that is to say

$$(1.5) \quad T = W(f) = \int_{\mathbb{C}^n} f(z)\pi(z) dz$$

where $\pi(z) = \pi_1(z, 0)$, π_1 being the Schrödinger representation of H^n with parameter $\lambda = 1$. The following analogue of Cowling–Price theorem can be proved for the Weyl transform.

Theorem 1.5. *Suppose f on \mathbb{C}^n satisfies $fe^{a|z|^2} \in L^p(\mathbb{C}^n)$ and $W(f)e^{bH} \in \mathcal{S}_q$, where $1 \leq p, q \leq \infty$. Then $f = 0$ whenever $a \tanh b \geq \frac{1}{4}$ and $\min(p, q) < \infty$. If $p = q = \infty$ then $f = 0$ for $a \tanh b > \frac{1}{4}$ and $f(z) = Ce^{-a|z|^2}$ for $a \tanh b = \frac{1}{4}$.*

As an immediate corollary of Theorem 1.4 we obtain the following theorem for nilpotent Lie groups.

Theorem 1.6. *Let G be a connected, simply connected nilpotent Lie group and let Λ be a cross section for the generic coadjoint orbits parametrising the elements of \hat{G} which are relevant for the Plancherel theorem. For each $\lambda \in \Lambda$ let π_λ be the associated element of \hat{G} . Let $f \in L^1 \cap L^2(G)$ satisfy the following conditions:*

- (i) $\pi_\lambda(f)^* \pi_\lambda(f) \leq C e^{-2b(\lambda)H}$
- (ii) $\|\delta^\alpha \bar{\delta}^\beta (\pi_\lambda(f)^* \pi_\lambda(f))\|_{HS}^2 \leq C (\alpha + \beta)! a(\lambda)^{|\alpha|+|\beta|}$

where $a(\lambda), b(\lambda) > 0$. Then $f = 0$ whenever $a(\lambda) < 2 \tanh 2b(\lambda)$ for all $\lambda \in \Lambda$.

For the case of the Heisenberg group it can be easily checked using the explicit formula for the heat kernel that $|f^\lambda(z)| \leq C q_a^\lambda(z)$ leads to the estimates

$$(1.6) \quad \|\delta^\alpha \bar{\delta}^\beta (\hat{f}(\lambda)^* \hat{f}(\lambda))\|_{HS}^2 \leq C (\alpha + \beta)! (a|\lambda|)^{|\alpha|+|\beta|}.$$

Thus condition (ii) in the above theorem is a suitable alternative which compensates for the absence of a good formula for the heat kernel. In the case of the Heisenberg group we can replace the condition (i) by $\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C e^{-2bH(\lambda)}$. Note that $e^{-bH(\lambda)} = \hat{q}_b(\lambda)$ and so it is a natural candidate for measuring the decay of $\hat{f}(\lambda)$. As $H(\lambda)$ is unitarily equivalent to $|\lambda|H$ the condition (i) is natural. The same comment applies to the case of all step two groups as the scaled Hermite operator is related to the sublaplacian even in that case. In the case of general nilpotent groups, there is no canonical way of measuring the decay of $\pi_\lambda(f)$. Therefore, we have used $e^{-b(\lambda)H}$ to measure the decay of the Fourier transform since we don't have any other choice.

We conclude this section with the following remarks. Different versions of Hardy's theorem and Cowling–Price theorem for nilpotent Lie groups have been proved in [1, 2, 3, 9] and [13]. In each paper the conditions are in terms of the Hilbert–Schmidt norm of $\pi_\lambda(f)$ and as such they are in some sense results for the central variable. This remark is easily justified if one considers functions of the form $f(z, t) = g(z)h(t)$ on the Heisenberg group. The right analogue of Hardy's theorem for H^n was proved in [14] and in this paper we have tried to formulate such an analogue of Cowling–Price theorem.

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2. COWLING–PRICE ON \mathbb{R}^n REVISITED

As we have mentioned in the introduction we will show that Theorem 1.3 (case $ab > \frac{1}{4}$) and Theorem 1.1 (case $p = 2, q = \infty$) are equivalent. However, we first give an independent proof of Theorem 1.3. We start with the following lemma which allows us to get pointwise estimates on $\partial^\alpha \hat{f}$ when we have estimates on $\|\partial^\alpha \hat{f}\|_2$.

Lemma 2.1. *Suppose we have*

$$\|\partial^\alpha \hat{f}\|_2^2 \leq C \alpha! (2a)^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$. Then we also have

$$|\partial^\alpha \hat{f}(\xi)|^2 \leq C \prod_{j=1}^n (\alpha_j + n)! (2a)^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$.

Proof. In view of Sobolev embedding theorem

$$|\partial^\alpha \hat{f}(\xi)| \leq C \sum_{|\beta| \leq n} \|\partial^{\alpha+\beta} \hat{f}\|_2$$

which gives the estimate

$$\begin{aligned} |\partial^\alpha \hat{f}(\xi)|^2 &\leq C \sum_{|\beta| \leq n} \|\partial^{\alpha+\beta} \hat{f}\|_2^2 \\ &\leq C \sum_{|\beta| \leq n} (2a)^{-(|\alpha|+|\beta|)} (\alpha + \beta)! \\ &\leq C \prod_{j=1}^n (\alpha_j + n)! (2a)^{-|\alpha|}. \end{aligned}$$

□

In view of this lemma we only need to prove the following version of Theorem 1.3.

Theorem 2.2. *The conclusion of Theorem 1.3 are valid if we replace the estimates on $\|\partial^\alpha \hat{f}\|_2$ by*

$$|\partial^\alpha \hat{f}(\xi)|^2 \leq C \prod_{j=1}^n (\alpha_j + n)! (2a)^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$.

As an application of the above theorem we have the following corollary:

Corollary 2.3. *Let \hat{f} be a smooth function such that*

$$\|\partial^\alpha \hat{f}\|_p^2 \leq C \alpha! (2a)^{-|\alpha|}, |\hat{f}(\xi)| \leq C e^{-b|\xi|^2}$$

where $1 \leq p < \infty$ then $f = 0$ for $ab > \frac{1}{4}$.

Proof. Choose $t > 0$ such that $a > a - t > \frac{1}{4b}$. Consider the function $F_t(x) = \hat{f} * p_t(x)$ where $p_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. Now

$$\begin{aligned} |\partial^\alpha F_t(x)|^2 &\leq \|p_t\|_{p'}^2 \|\partial^\alpha \hat{f}\|_p^2 \\ &\leq C \alpha! (2a)^{-|\alpha|} \end{aligned}$$

$$|F_t(x)| \leq C e^{-\frac{b}{4bt+1}|x|^2}$$

Therefore $F_t(x) = 0$ as $a \frac{b}{4bt+1} > \frac{1}{4}$. Hence $f = 0$ for $ab > \frac{1}{4}$ □

Let us complete the proof of Theorem 2.2. We first consider the case $ab > \frac{1}{4}$. We make use of the following lemma.

Lemma 2.4. *Let $F(\xi)$ be a smooth function on \mathbb{R}^n which satisfies*

$$|\partial^\alpha F(\xi)|^2 \leq C \prod_{j=1}^n (\alpha_j + n)! (2a)^{-|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$. Then F extends to \mathbb{C}^n as an entire function which satisfies $|F(\zeta)| \leq C e^{b|\Im\zeta|^2}$ for every $b > \frac{1}{4a}$.

Proof. For $b > \frac{1}{4a}$,

$$\begin{aligned} F(\xi + \eta) &= \sum_{\alpha} \frac{\partial^\alpha F(\xi)}{\alpha!} \eta^\alpha \\ &\leq \sum_{\alpha} \left| \frac{\partial^\alpha F(\xi)}{\alpha!} \right| |\eta|^{|\alpha|} \\ &= \sum_{\alpha} \frac{|\partial^\alpha F(\xi)|}{\alpha!^{\frac{1}{2}}} (2b)^{-\frac{|\alpha|}{2}} \cdot \frac{(2b)^{\frac{|\alpha|}{2}}}{\alpha!^{\frac{1}{2}}} |\eta|^{|\alpha|} \\ &\leq \left(\sum_{\alpha} \frac{|\partial^\alpha F(\xi)|^2}{\alpha!} (2b)^{-|\alpha|} \right)^{\frac{1}{2}} \left(\sum_{\alpha} \frac{(|\eta|^2 2b)^{|\alpha|}}{\alpha!} \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{\alpha} \frac{(\alpha + n)!}{\alpha!} \left(\frac{1}{4ab} \right)^{|\alpha|} \right)^{\frac{1}{2}} e^{b|\eta|^2} \\ &= C(n, b) e^{b|\eta|^2}. \end{aligned}$$

This shows that F can be extended as an entire function on \mathbb{C}^n and it satisfies

$$|F(\xi + i\eta)| \leq C(n, b) e^{b|\eta|^2}.$$

Coming to the proof of the case $ab > \frac{1}{4}$ choose b' such that $b > b' > \frac{1}{4a}$. By Lemma 2.4 we have

$$|\hat{f}(\zeta)| \leq C e^{b'|\Im\zeta|^2}.$$

As we have $|\hat{f}(\xi)| \leq C e^{-b|\xi|^2}$ and $b' < b$ we appeal to the following lemma to conclude that $\hat{f} = 0$. \square

Lemma 2.5. *Let $F(\zeta)$ be an entire function on \mathbb{C}^n which satisfies*

$$|F(\zeta)| \leq C e^{a|\Im\zeta|^2}, \quad |F(\xi)| \leq C e^{-b|\xi|^2}$$

for $\zeta \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$. Then $F = 0$ whenever $a < b$ and $F(\zeta) = C e^{-a\zeta^2}$ for $a = b$.

We now take up the equality case. Clearly it is enough to prove it when $n = 1$. Indeed if we have the result in the one dimensional case then by considering the function

$$F(\xi_n) = \hat{f}(\xi', \xi_n), \quad \xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$$

which satisfies the estimates

$$|F(\xi_n)| \leq C(\xi') e^{-b\xi_n^2} \quad |\partial^k F(\xi_n)|^2 \leq C(k+n)! (2a)^{-k}$$

we obtain

$$F(\xi_n) = C(\xi', \xi_n) e^{-b\xi_n^2}.$$

But now the function $C(\xi', \xi_n)$ satisfies the same estimates as \hat{f} on \mathbb{R}^{n-1} . By using induction we can obtain $\hat{f}(\xi) = \phi(\xi) e^{-b|\xi|^2}$ with ϕ bounded. For the one dimensional case using Lemma 2.4 \hat{f} can be

extended to \mathbb{C} as an entire function of order at most 2, \hat{f} can't decay in \mathbb{R} faster than its order. So its order is 2. Since we have the estimate

$$|\hat{f}(\zeta)| \leq C e^{b'|\Im\zeta|^2} \text{ for all } b' > \frac{1}{4a}$$

its type is $\frac{1}{4a}$. Now we apply the following result of Pfannschmidt [11] to the entire function $\hat{f}(\zeta)$.

Theorem 2.6. *Let F be an entire function of one variable ζ of order ρ (ρ integer) and type b . Let*

$$h(\theta) = \limsup_{r \rightarrow \infty} \frac{\log |F(re^{i\theta})|}{r^\rho}, \theta \in [0, \infty)$$

be its indicator and assume that

$$h\left(\frac{2\pi j}{\rho}\right) \leq -b, j = 0, 1, 2, \dots, \rho - 1.$$

Then $F(\zeta) = P(\zeta)e^{-b\zeta^\rho}$ where $P(\zeta)$ is an entire function at most of minimal type of order ρ .

The following remark is in order. In Theorem 1.3 with $ab = \frac{1}{4}$ we have concluded that $\hat{f}(\xi) = \phi(\xi)e^{-b|\xi|^2}$. It would be nice to say some thing about f itself. As $\phi(\zeta)$ is an entire function we have

$$\phi(\xi) = \sum_{|\alpha| \leq N} a_\alpha \xi^\alpha + \phi_N(\xi)$$

where $|\phi_N(\xi)| \leq c(1 + |\xi|)^N$. This shows that, in view of the inversion formula

$$(2.1) \quad f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \phi(\xi) e^{-b|\xi|^2} d\xi$$

f can be written as

$$f(x) = \left(\sum_{|\alpha| \leq N} c_\alpha x^\alpha \right) e^{-a|x|^2} + f_N(x)$$

where

$$\hat{f}_N(\xi) = \phi_N(\xi) e^{-b|\xi|^2}.$$

Finally it is an easy matter to prove the equivalence between Theorem 1.3 (case $ab > \frac{1}{4}$) and Theorem 1.1 (case $p = 2, q = \infty$). Assuming Theorem 1.1 consider Theorem 1.3 with $ab > \frac{1}{4}$. Choose $a' < a$ but satisfying $a'b > \frac{1}{4}$ and consider $f\phi_{a'}$. Expanding the Gaussian we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x)\phi_{a'}(x)|^2 dx \\ &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} |f(x)|^2 \frac{(2a')^k}{k!} |x|^{2k} dx \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!} \int_{\mathbb{R}^n} |f(x)|^2 x^{2\alpha} dx \right) (2a')^k. \end{aligned}$$

Under the hypothesis on $\|\partial^\alpha \hat{f}\|_2^2$ we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x)\phi_{a'}(x)|^2 dx \\ & \leq C \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!} \right) \left(\frac{a'}{a} \right)^k \\ & \leq C \sum_{k=0}^{\infty} k^{n-1} \left(\frac{a'}{a} \right)^k \\ & < \infty. \end{aligned}$$

Hence we can apply Theorem 1.1 to conclude that $f = 0$.

The above calculation shows that $f\phi_a^{-1} \in L^2$ implies the estimates

$$\|\partial^\alpha \hat{f}\|_2^2 \leq C \alpha! (2a)^{-|\alpha|}$$

for every $\alpha \in \mathbb{N}^n$ and therefore Theorem 1.3 implies Theorem 1.1 ($p = 2, q = \infty$).

We also have the following implication: Theorem 2.2 implies Theorem 1.1 ($p = 2, q = \infty$). In view of

$$(2.2) \quad \partial^\alpha \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} (-i)^{|\alpha|} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) x^\alpha dx$$

and by the Cauchy-Schwarz inequality, the assumption $f\phi_a^{-1} \in L^2$ gives,

$$\begin{aligned} |\partial^\alpha \hat{f}(\xi)|^2 & \leq C \|f\phi_a^{-1}\|_2^2 \|x^\alpha \phi_a\|_2^2 \\ & \leq C \alpha! (2a)^{-|\alpha|}. \end{aligned}$$

Hence the hypothesis of Theorem 2.2 are satisfied.

3. AN UNCERTAINTY PRINCIPLE FOR OPERATORS

In this section we prove Theorem 1.4 and Theorem 1.5. In order to prove those results we need to use several properties of the Weyl transform and special Hermite functions. We recall the relevant results referring to [17, 18] for details.

The Weyl transform is closely related to the Fourier transform on the Heisenberg group H^n . If π_λ is the Schrödinger representations on H^n with parameter λ . Explicitly $\pi_\lambda(z, t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is the unitary operator given by

$$(3.1) \quad \pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} \phi(\xi + y)$$

where $\phi \in L^2(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$ and $z = (x + iy)$. We define $\pi_\lambda(z) = \pi_\lambda(z, 0)$ so that $\pi_\lambda(z, t) = e^{i\lambda t} \pi_\lambda(z)$. For $f \in L^1(\mathbb{C}^n)$ its Weyl transform $W_\lambda(f)$ is the bounded operator on $L^2(\mathbb{R}^n)$ given by

$$(3.2) \quad W_\lambda(f)\phi = \int_{\mathbb{C}^n} f(z) \pi_\lambda(z) \phi dz.$$

It is clear that $\|W_\lambda(f)\| \leq \|f\|_1$ and for $f \in L^1 \cap L^2(\mathbb{C}^n)$, $W_\lambda(f)$ is Hilbert-Schmidt and we have the Plancherel theorem

$$(3.3) \quad \|W_\lambda(f)\|_{\text{HS}}^2 = (2\pi)^n |\lambda|^{-n} \int_{\mathbb{C}^n} |f(z)|^2 dz.$$

Thus W_λ is an isometric isomorphism between $L^2(\mathbb{C}^n)$ and \mathcal{S}_2 . For $f \in L^1(H^n)$ set

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t) dt$$

to be the inverse Fourier transform of f in the t -variable. Then from the definition of $\hat{f}(\lambda)$ it follows that $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. For $\lambda = 1$ we define $W(z) = \pi_1(z)$ and $W(f) = W_1(f)$.

Given $f \in L^1 \cap L^2(\mathbb{C}^n)$ we define the Fourier-Weyl transform of f as the operator valued function $\tilde{f}(u, v)$ on \mathbb{R}^{2n} given by

$$(3.4) \quad \tilde{f}(u, v) = W(u + iv)W(f)W(u + iv)^*.$$

As $W(z)$ is a projective representation of \mathbb{C}^n it is easily seen that

$$(3.5) \quad \tilde{f}(u, v) = \int_{\mathbb{C}^n} e^{i(x \cdot v - y \cdot u)} f(x + iy)W(x + iy) dx dy.$$

A simple calculation using the definition shows that $\delta_j W(f) = W(\bar{M}_j f)$ and $\bar{\delta}_j W(f) = W(M_j f)$ where $M_j f(z) = z_j f(z)$ and $\bar{M}_j f(z) = \bar{z}_j f(z)$. By iteration we get $\delta^\alpha W(f) = W(\bar{z}^\alpha f)$ and $\bar{\delta}^\beta W(f) = W(z^\beta f)$. Taking derivatives in u, v and using these relations we get

$$\partial_u^\alpha \partial_v^\beta \tilde{f}(u, v) = 2^{-(|\alpha| + |\beta|)} W(u + iv)(\delta + \bar{\delta})^\beta (\delta - \bar{\delta})^\alpha W(f)W(u + iv)^*.$$

This identity shows that

$$(3.6) \quad \|\partial_u^\alpha \partial_v^\beta \tilde{f}(u, v)\|_{\text{HS}}^2 \leq C (\alpha + \beta)! a^{|\alpha| + |\beta|}$$

whenever we have

$$(3.7) \quad \|\delta^\alpha \bar{\delta}^\beta W(f)\|_{\text{HS}}^2 \leq C (\alpha + \beta)! a^{|\alpha| + |\beta|}.$$

The special Hermite functions $\Phi_{\mu\nu}$ are defined as follows. Let $\Phi_\mu(x)$, $x \in \mathbb{R}^n$ be the Hermite functions on \mathbb{R}^n . Then we define

$$\Phi_{\mu\nu}(z) = (2\pi)^{\frac{n}{2}} (W(z)\Phi_\mu, \Phi_\nu).$$

These functions form an orthonormal basis for $L^2(\mathbb{C}^n)$ and they are expressible in terms of Laguerre functions. For our purposes we only require the formula

$$(3.8) \quad \Phi_{0,\mu}(z) = (2\pi)^{-\frac{n}{2}} \left(\frac{1}{\mu!}\right)^{\frac{1}{2}} \left(-i\frac{z}{\sqrt{2}}\right)^\mu e^{-\frac{1}{4}|z|^2}.$$

We refer to [18] for these and more on special Hermite functions.

With these preparations we embark on a proof of Theorem 1.4. For $T \in \mathcal{S}_2$ we define $\hat{T}(u, v) = W(u + iv)TW(u + iv)^*$ so that $\hat{T}(u, v) = \tilde{f}(u, v)$ if $T = W(f)$. Let F be the function on \mathbb{R}^{2n} defined by

$$(3.9) \quad F(u, v) = \left(\hat{T}(u, v)^* \hat{T}(u, v) \Phi_0, \Phi_0\right).$$

We claim that $F(u, v)$ satisfies the following two properties:

- (i) $F(u, v)$ extends to \mathbb{C}^{2n} as an entire function which satisfies the estimate $|F(\zeta)| \leq C e^{a_1 |\Im \zeta|^2}$ for some $a_1 < \frac{1}{2}(1 - e^{-4b})$,
- (ii) $F(u, v) \leq C e^{-\frac{1}{2}(1 - e^{-4b})(|u|^2 + |v|^2)}$.

Assuming this claim for a moment we appeal to the following lemma.

Lemma 3.1. *Let $F(\zeta)$ be an entire function on \mathbb{C}^n which satisfies $|F(\zeta)| \leq C e^{a |\Im \zeta|^2}$, $\zeta \in \mathbb{C}^n$ and $|F(\xi)| \leq C (1 + |\xi|^2)^m e^{-b |\xi|^2}$, $\xi \in \mathbb{R}^n$. Then $F = 0$ whenever $a < b$.*

The lemma shows that $F = 0$ whenever $a_1 < \frac{1}{2}(1 - e^{-4b})$. Since $a < 2 \tanh 2b$ we have $\coth 2b < \frac{2}{a}$ and so we can choose b_1 and b_2 such that $\coth 2b < 4b_1 < 4b_2 < \frac{2}{a}$. This gives $b_2 < \frac{1}{2a}$ and $1 + 4b_1 > 1 + \coth 2b = \frac{2}{1 - e^{-4b}}$ or $\frac{2}{1 + 4b_1} < (1 - e^{-4b})$. In our claim we can take $a_1 = \frac{1}{1 + 4b_1}$ so that $F = 0$.

Let $S = T^*T$. Then it follows that $\hat{S}(u, v) = \hat{T}(u, v)^* \hat{T}(u, v)$. If $S = W(f)$, $f \in L^2(\mathbb{C}^n)$ then we have

$$(3.10) \quad \begin{aligned} F(u, v) &= (W(u + iv)W(f)W(u + iv)^* \Phi_0, \Phi_0) \\ &= \int_{\mathbb{R}^{2n}} e^{i(x \cdot v - y \cdot u)} f(x, y) e^{-\frac{1}{4}(|x|^2 + |y|^2)} dx dy. \end{aligned}$$

Hence $F(u, v) = 0$ implies $f(x, y) = 0$ proving the theorem as $S = T^*T = W(f) = 0$.

It remains to prove the claim with $a_1 = \frac{1}{1+4b_1}$ where b_1 is chosen as above. As we have indicated earlier the estimates on $\delta^\alpha \bar{\delta}^\beta S$ give the estimates

$$(3.11) \quad \|\partial_u^\alpha \partial_v^\beta \tilde{f}(u, v)\|_{\text{HS}}^2 \leq C (\alpha + \beta)! a^{|\alpha|+|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}^n$ and hence

$$(3.12) \quad \|\partial_u^\alpha \partial_v^\beta F(u, v)\|_2^2 \leq C (\alpha + \beta)! a^{|\alpha|+|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}^n$.

Since $S = W(f)$ using the Plancherel theorem we have the estimates

$$\int_{\mathbb{R}^{2n}} |x^\alpha y^\beta f(x, y)|^2 dx dy \leq C (\alpha + \beta)! a^{(|\alpha|+|\beta|)}.$$

We claim that

$$(3.13) \quad \int_{\mathbb{R}^{2n}} |f(x, y)|^2 e^{2b_2(|x|^2+|y|^2)} dx dy < \infty$$

for any $b_2 < \frac{1}{2a}$. To see this consider the series

$$(3.14) \quad \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{2n}} |f(x, y)|^2 (2b_2)^k (|x|^2 + |y|^2)^k dx dy$$

which converges as long as $b_2 < \frac{1}{2a}$ as in section 2.

Now consider $F(u, v)$ which is given by

$$F(u, v) = \int_{\mathbb{R}^{2n}} f(x, y) e^{i(x \cdot v - y \cdot u)} e^{-\frac{1}{4}(|x|^2+|y|^2)} dx dy.$$

By Hölder's inequality

$$(3.15) \quad |\partial_u^\alpha \partial_v^\beta F(u, v)|^2 \leq C \int_{\mathbb{R}^{2n}} |x^\alpha y^\beta|^2 e^{-2(b_2 + \frac{1}{4})(|x|^2+|y|^2)} dx dy$$

which gives the estimate

$$(3.16) \quad |\partial_u^\alpha \partial_v^\beta F(u, v)|^2 \leq C (\alpha + \beta)! \left(2 \left(b_2 + \frac{1}{4} \right) \right)^{-(|\alpha|+|\beta|)}.$$

Appealing to Lemma 2.3 we see that $F(u, v)$ extends to an entire function of type a_2 where $a_2 = \frac{1}{1+4b_2}$.

Since $a_2 < a_1$ we get $|F(\zeta)| \leq C e^{a_1 |\Im \zeta|^2}$ which proves the claim (i).

The second claim is proved using the bound $T^*T \leq C e^{-2bH}$. We have

$$(3.17) \quad \begin{aligned} F(u, v) &= (W(u + iv) T^* T W(u + iv))^* \Phi_0, \Phi_0 \\ &\leq C (W(u + iv) e^{-2bH} W(u + iv))^* \Phi_0, \Phi_0. \end{aligned}$$

We now expand $W(u + iv)^* \Phi_0 = W(u - iv) \Phi_0$ in terms of Φ_μ :

$$(3.18) \quad W(u + iv)^* \Phi_0 = (2\pi)^{\frac{n}{2}} \sum \Phi_{0,\mu}(u - iv) \Phi_\mu.$$

Since

$$e^{-2bH} \Phi_\mu = e^{-2b(2|\mu|+n)} \Phi_\mu$$

we have, using Parseval's formula for Hermite expansions,

$$(3.19) \quad F(u, v) \leq C \sum e^{-2b(2|\mu|+n)} |\Phi_{0,\mu}(u - iv)|^2.$$

Now using the explicit formula for $\Phi_{0,\mu}$ we get

$$(3.20) \quad F(u, v) \leq C \sum e^{-2b(2|\mu|+n)} \frac{1}{\mu!} \left(\frac{1}{2}(|u|^2 + |v|^2) \right)^{|\mu|} e^{-\frac{1}{2}(|u|^2+|v|^2)}$$

which gives

$$(3.21) \quad F(u, v) \leq C e^{-\frac{1}{2}(1-e^{-4b})(|u|^2+|v|^2)}$$

as desired. This completes the proof of Theorem 1.4.

Let f be a function in $L^2(\mathbb{C}^n)$ which is invariant under the action on \mathbb{T}^n . Then f is called polyradial and it has expansion

$$f(z) = \sum_{\mu} (f, \Phi_{\mu\mu}) \Phi_{\mu\mu}(z).$$

Let d be a function on \mathbb{N} . For each $j \in \mathbb{N}$ we define the difference operators Δ_j^+ and Δ_j^- by

$$(\Delta_j^+ d)(\mu) = d(\mu + e_j) - d(\mu), (\Delta_j^- d)(\mu) = d(\mu) - d(\mu - e_j).$$

For multi-indices $\alpha, \beta \in \mathbb{N}$ we define

$$\Delta_+^{\alpha} = (\Delta_1^+)^{\alpha_1} (\Delta_2^+)^{\alpha_2} \dots (\Delta_n^+)^{\alpha_n}, \Delta_-^{\beta} = (\Delta_1^-)^{\beta_1} (\Delta_2^-)^{\beta_2} \dots (\Delta_n^-)^{\beta_n}.$$

With these notations we will prove the following corollary of Theorem 1.4 which can be considered as Cowling–Price theorem for eigenfunction expansion of polyradial functions.

Corollary 3.2. *Let f be a polyradial function which is in $L^2(\mathbb{C}^n)$, $C(\mu) = (f, \Phi_{\mu\mu})$ and $d(\mu) = |C(\mu)|^2$, $\mu \in \mathbb{N}^n$. If $C(\mu)$ satisfy the following conditions:*

- (i) $|C(\mu)| \leq C e^{-b(2|\mu|+n)}$
- (ii) $\sum_{\mu} \frac{(\mu+\alpha)!}{(\mu-\beta)!} \left| (\Delta_-^{\beta} \Delta_+^{\alpha} d)(\mu) \right|^2 \leq C(\alpha + \beta)! 2^{-(|\alpha|+|\beta|)} a^{(|\alpha|+|\beta|)}$

where $a, b > 0$, then $f = 0$ whenever $a < 2 \tanh 2b$.

Proof. Since f is polyradial

$$W(f)\phi = \sum_{\mu} (f, \Phi_{\mu\mu})(\phi, \Phi_{\mu}) \Phi_{\mu}.$$

Using the formulae

$$A_j \Phi_{\mu} = (2\mu_j + 2)^{\frac{1}{2}} \Phi_{\mu+e_j}, \quad A_j^* \Phi_{\mu} = (2\mu_j)^{\frac{1}{2}} \Phi_{\mu-e_j}$$

it is easy to see that

$$\left(\delta^{\alpha} \bar{\delta}^{\beta} W(f)^* W(f) \right) \Phi_{\mu} = (-1)^{|\alpha|+|\beta|} 2^{\frac{|\alpha|+|\beta|}{2}} \left(\frac{(\mu + \alpha)!}{(\mu - \beta)!} \right)^{\frac{1}{2}} \left(\Delta_-^{\beta} \Delta_+^{\alpha} d \right)(\mu) \Phi_{\mu+\alpha-\beta}.$$

Then using the above conditions we will get

$$W(f)^* W(f) \leq C e^{-2bH}, \quad \|\delta^{\alpha} \bar{\delta}^{\beta} (W(f)^* W(f))\|_{\text{HS}}^2 \leq C(\alpha + \beta)! a^{|\alpha|+|\beta|}.$$

So by Theorem 1.4, $W(f) = 0$ for $a < 2 \tanh 2b$ and hence $f = 0$. □

4. COWLING–PRICE THEOREM FOR HEISENBERG
GROUPS

In this section we prove Theorem 1.2 which is an analogue of Cowling–Price theorem for the Heisenberg group H^n . Before going to the proof of Theorem 1.2, let us prove the following theorem which can be considered as Cowling–Price theorem for Weyl transform and Theorem 1.5 will be proved as a special case $\lambda = 1$.

Theorem 4.1. *Let $f e^{a|z|^2} \in L^p(\mathbb{C}^n)$ and $W_\lambda(f) e^{bH(\lambda)} \in \mathcal{S}_q$ where $1 \leq p, q \leq \infty$. Then $f = 0$ a.e. whenever $a \frac{\tanh b\lambda}{\lambda} \geq \frac{1}{4}$ and $\min(p, q) < \infty$. If $p = q = \infty$ then $f = 0$ for $a \frac{\tanh b\lambda}{\lambda} > \frac{1}{4}$ and $f(z) = C e^{-a|z|^2}$ for $a \frac{\tanh b\lambda}{\lambda} = \frac{1}{4}$.*

Proof. Let \mathbb{T}^n denotes the subgroup of the unitary group $U(n)$ consisting of diagonal matrices. Then each element of \mathbb{T}^n can be written in the form $e^{i\theta} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ so that \mathbb{T}^n can be identified with n -torus. The action of the n -torus on \mathbb{C}^n is given by $e^{i\theta} \cdot z = (e^{i\theta_1} z_1, e^{i\theta_2} z_2 \dots e^{i\theta_n} z_n)$. Let m be an n -tuple of integers. We say g is m -homogeneous if $g(e^{i\theta} \cdot z) = e^{im \cdot \theta} g(z)$ for all θ . It is easy to see that when g is m -homogeneous and h is k -homogeneous then $(g, h) = 0$ unless $m = k$. Since $f e^{a|z|^2}$ is in $L^p(\mathbb{C}^n)$ it follows that $f_m e^{a|z|^2} \in L^p(\mathbb{C}^n)$ where

$$(4.1) \quad f_m(z) = \int_{\mathbb{T}^n} f(e^{i\theta} \cdot z) e^{-im \cdot \theta} d\theta.$$

Hence

$$\begin{aligned} W_\lambda(f_m) &= \int_{\mathbb{T}^n} f(e^{i\theta} \cdot z) \pi_\lambda(z) e^{-im \cdot \theta} d\theta \\ &= \int_{\mathbb{T}^n} f(e^{i\theta} \cdot z) \pi_\lambda(e^{-i\theta} z) e^{-im \cdot \theta} d\theta \\ &= \int_{\mathbb{T}^n} \mu_\lambda(\theta)^* W_\lambda(f) \mu_\lambda(\theta) e^{-im \cdot \theta} d\theta \end{aligned}$$

where $\mu_\lambda(\theta)$ is a unitary operator on $L^2(\mathbb{R}^n)$ such that

$$\pi_\lambda(e^{i\theta} \cdot z) = \mu_\lambda(\theta)^* \pi_\lambda(z) \mu_\lambda(\theta).$$

We refer to [17] for several properties of $\mu_\lambda(\theta)$. Using the above expression for $W_\lambda(f_m)$ we get

$$(4.2) \quad \begin{aligned} \|W_\lambda(f_m) \Phi_\alpha\|_2 &\leq \int_{\mathbb{T}^n} \|\mu_\lambda(\theta)^* W_\lambda(f) \mu_\lambda(\theta) \Phi_\alpha\|_2 d\theta \\ &= \int_{\mathbb{T}^n} \|W_\lambda(f) \mu_\lambda(\theta) \Phi_\alpha^\lambda\|_2 d\theta. \end{aligned}$$

Now $E_k^\lambda = \text{span of } \{\Phi_\alpha^\lambda : |\alpha| = k\}$ are invariant under the action of $\mu_\lambda(\theta)$. Therefore,

$$(4.3) \quad \mu_\lambda(\theta) \Phi_\alpha^\lambda = \sum_{|\beta|=k} C_{\alpha\beta}^\lambda(\theta) \Phi_\beta^\lambda$$

for all $\theta \in \mathbb{T}^n$. As $\mu_\lambda(\theta)$ is a unitary operator

$$(4.4) \quad \sum_{|\beta|=k} |C_{\alpha\beta}^\lambda(\theta)|^2 = 1$$

for all $\theta \in \mathbb{T}^n$. Since $W_\lambda(f) e^{bH(\lambda)} \in \mathcal{S}_q$ it is bounded and hence there exists $C > 0$ such that $\|W_\lambda(f) \Phi_\alpha^\lambda\|_2 \leq C e^{-b(2|\alpha|+n)|\lambda|}$ for all $\alpha \in \mathbb{N}^n$. Now using these estimates on $W_\lambda(f)$ and applying the Cauchy-Schwarz inequality we get

$$(4.5) \quad \|W_\lambda(f) \mu_\lambda(\theta) \Phi_\alpha^\lambda\|_2^2 \leq C \frac{(k+n-1)!}{k!(n-1)!} e^{-2b(2k+n)|\lambda|}$$

for all Φ_α^λ with $|\alpha| = k$. So for all α with $|\alpha| = k$

$$\begin{aligned}
|(f_m, \Phi_{\alpha, \alpha+m}^\lambda)| &= |(W_\lambda(f_m)\Phi_\alpha^\lambda, \Phi_{\alpha+m}^\lambda)| \\
&\leq \|W_\lambda(f_m)\Phi_\alpha^\lambda\|_2 \\
&\leq C_n \sqrt{\frac{\Gamma(k+n)}{\Gamma(k+1)}} e^{-b(2k+n)|\lambda|} \\
&\leq C_n k^n e^{-b(2k+n)|\lambda|} \\
(4.6) \qquad &\leq C_n \prod_{i=1}^n \left((2\alpha_i + 1)^n e^{-b(2\alpha_i+1)|\lambda|} \right).
\end{aligned}$$

Let $\mathcal{F}_\lambda f$ stand for the symplectic Fourier transform of a function f in \mathbb{C}^n defined by

$$(4.7) \qquad \mathcal{F}_\lambda f(z) = (2\pi)^{-n} \int_{\mathbb{C}^n} f(z-w) e^{i\frac{\lambda}{2}\Im(z \cdot \bar{w})} dw.$$

\mathcal{F}_λ is related to the ordinary Fourier transform by $\mathcal{F}_\lambda f(z) = \hat{f}(-i\frac{\lambda}{2}z)$. As f_m is m -homogeneous and $\Phi_{\alpha\beta}$ is $(\beta - \alpha)$ -homogeneous expanding f_m we get

$$f_m(z) = \sum_{\beta} (f_m, \Phi_{\beta, \beta+m}) \Phi_{\beta, \beta+m}.$$

As $\mathcal{F}_\lambda \Phi_{\alpha, \alpha+m}^\lambda = (-1)^{|\alpha|} \Phi_{\alpha, \alpha+m}^\lambda$,

$$(4.8) \qquad \mathcal{F}_\lambda f_m(z) = \sum_m (-1)^{|\alpha|} (f_m, \Phi_{\alpha, \alpha+m}^\lambda) \Phi_{\alpha, \alpha+m}^\lambda.$$

For a proof of this we refer to [17]. The functions $\Phi_{\alpha, \alpha+m}^\lambda$ are expressible as products of one dimensional Laguerre functions and so without loss of generality we can assume that $n = 1$ and m is an integer. Explicit formula for $\Phi_{k, k+m}^\lambda$ is given as follows

$$\begin{aligned}
\Phi_{k, k+m}^\lambda(z) &= \lambda^{\frac{1}{2}} \Phi_{k, k+m}(\sqrt{\lambda}z) \text{ for } \lambda > 0, \\
\Phi_{k, k+m}^\lambda(z) &= |\lambda|^{\frac{1}{2}} \bar{\Phi}_{k, k+m}(\sqrt{|\lambda|}z) \text{ for } \lambda < 0
\end{aligned}$$

where

$$\Phi_{k, k+m}(z) = (2\pi)^{-\frac{1}{2}} \left(\frac{-i}{\sqrt{2}} \right)^m \left(\frac{\Gamma(k+1)}{\Gamma(k+m+1)} \right)^{\frac{1}{2}} z^m \phi_k^m(z)$$

and $\phi_k^m(z)$ are the Laguerre functions on \mathbb{C}^n defined by

$$\phi_k^m(z) = L_k^m \left(\frac{1}{2}|z|^2 \right) e^{-\frac{1}{4}|z|^2}.$$

Therefore

$$\begin{aligned}
&|\mathcal{F}_\lambda f_m(z)| \\
&\leq C_n(\lambda) \sum_{k=0}^{\infty} (2k+1)|\lambda| e^{-b(2k+1)|\lambda|} |\Phi_{k, k+m}^\lambda(z)| \\
&\leq C_{n,m}(\lambda) |z|^m \sum_{k=0}^{\infty} (2k+m+1)|\lambda| e^{-b(2k+m+1)|\lambda|} \left(\frac{\Gamma(k+1)}{\Gamma(k+m+1)} \right)^{\frac{1}{2}} |\phi_{k,\lambda}^m(z)| \\
&\leq C_{n,m}(\lambda) |z|^m \left(\sum_{k=0}^{\infty} ((2k+m+1)|\lambda|)^4 e^{-2b(2k+m+1)|\lambda|} \left(\frac{\Gamma(k+1)}{\Gamma(k+m+1)} \right) (\phi_{k,\lambda}^m(z))^2 \right)^{\frac{1}{2}} \\
&\leq C_{n,m}(\lambda) (1 + |\lambda||z|^2)^l e^{-\left(\frac{1}{4}\lambda \tanh b\lambda\right)|z|^2}
\end{aligned}$$

for some positive integer $l > 0$. See page 93 of [16].

Using the relation

$$\mathcal{F}_\lambda f_m(z) = \hat{f}_m(-i\frac{\lambda}{2}z)$$

we have

$$|\hat{f}_m(z)| \leq C_{n,m}(\lambda)(1 + |z|^2)^l e^{-\frac{\tanh b\lambda}{\lambda}|z|^2}.$$

So by the Cowling–Price theorem for Fourier transform we conclude the following:

Case 1. if $p < \infty, q \leq \infty$ then for $a\frac{\tanh b\lambda}{\lambda} \geq \frac{1}{4}$, $f_m = 0$ for all m and hence $f = 0$ a.e.

Case 2. if $p = \infty, q < \infty$ then for $a\frac{\tanh b\lambda}{\lambda} > \frac{1}{4}$, $f_m = 0$ and for $a\frac{\tanh b\lambda}{\lambda} = \frac{1}{4}$ $f_m(z) = C_m e^{-a|z|^2}$ for all m . Since f_m is m -homogeneous $C_m = 0$ except $m = 0$ and hence $f(z) = f_0(z) = C e^{-a|z|^2}$ which yields $W_\lambda(f) = C e^{-bH(\lambda)}$. But this is not compatible with the condition $W_\lambda(f)e^{bH(\lambda)} \in \mathcal{S}_q$ and hence $f = 0$ a.e.

Case 3. if $p = q = \infty$ then $f_m = 0$ for all m whenever $a\frac{\tanh b\lambda}{\lambda} > \frac{1}{4}$ and for $a\frac{\tanh b\lambda}{\lambda} = \frac{1}{4}$ arguing as before $f(z) = f_0(z) = C e^{-a|z|^2}$. \square

The special Hermite operator L_λ is defined by the relation

$$\mathcal{L}(e^{i\lambda t} f(z)) = e^{i\lambda t} L_\lambda f(z)$$

where \mathcal{L} is the sublaplacian on the Heisenberg group. This operator plays an important role in harmonic analysis and we refer [16, 18] for details.

Corollary 4.2. *If $f(q_a^\lambda)^{-1} \in L^p(\mathbb{C}^n)$ and $W_\lambda(f)e^{bH(\lambda)} \in \mathcal{S}_q$, $1 \leq p, q \leq \infty$, where q_a^λ is the heat kernel corresponding to the special Hermite operator, then $f = 0$ a.e. whenever $a \leq b$ and $\min(p, q) < \infty$. If $p = q = \infty$ then $f = 0$ for $a < b$ and $f(z) = C(\lambda)q_a^\lambda(z)$ for $a = b$.*

Proof. The explicit formula for the heat kernel q_a^λ corresponding to the special Hermite operator is

$$q_a^\lambda(z) = (4\pi)^{-n} \left(\frac{\lambda}{\sinh \lambda a} \right)^n e^{-\frac{1}{4}\lambda(\coth a\lambda)|z|^2}.$$

Since $a \leq b$ and $\tanh(\cdot)$ is an increasing function, $\tanh a|\lambda| \leq \tanh b|\lambda|$. So we will get the desired result applying Theorem 4.1. \square

To prove the Theorem 1.2 let $g(z, t) = q_a^{-1}(z, t)f(z, t)$. Then $g \in L^p(H^n)$. Using the estimate

$$|q_a(z, t)| \leq C a^{-n-1} e^{-\frac{A}{a}(|z|^2 + |t|)}$$

for some $C, A > 0$ and the condition $g \in L^p(H^n)$ it is easy to see that

$$(4.9) \quad f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} q_a(z, t) g(z, t) dt$$

can be extended as a holomorphic function in the strip $|\Im \lambda| < \frac{A}{a}$ of the complex plane.

For $p = \infty$,

$$\begin{aligned} |f^\lambda(z)| &\leq \|g(z, \cdot)\|_\infty \int_{-\infty}^{\infty} |q(z, t)| dt \\ &\leq \|g(z, \cdot)\|_\infty e^{-\frac{1}{4a}|z|^2} \end{aligned}$$

For $1 \leq p \leq 2$,

$$\begin{aligned}
|f^\lambda(z)| &\leq \int_{-\infty}^{\infty} |q_a(z, t)| |g(z, t)| dt \\
&\leq \left(\int_{-\infty}^{\infty} |q_a(z, t)|^{p'} dt \right)^{\frac{1}{p'}} \left(\int_{-\infty}^{\infty} |g(z, t)|^p dt \right)^{\frac{1}{p}} \\
&\leq \left(\int_{-\infty}^{\infty} |q_a^\lambda(z)|^p d\lambda \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(z, t)|^p dt \right)^{\frac{1}{p}} \\
&\leq (4\pi)^{-n} \left(\int_{-\infty}^{\infty} \left(\frac{\lambda}{\sinh a\lambda} \right)^{pn} e^{-p\frac{\lambda}{4}(\coth a\lambda)|z|^2} d\lambda \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(z, t)|^p dt \right)^{\frac{1}{p}} \\
&\leq C e^{-\frac{1}{4a}|z|^2} \left(\int_{-\infty}^{\infty} |g(z, t)|^p dt \right)^{\frac{1}{p}}.
\end{aligned}$$

When $2 < p < \infty$, $1 < p' < 2$ write $\frac{1}{p'} = \frac{\nu}{1} + \frac{1-\nu}{2}$ for some $0 < \nu < 1$. Since $\|q_a(z, \cdot)\|_1 \leq e^{-\frac{1}{4a}|z|^2}$ and $\|q_a(z, \cdot)\|_2 \leq e^{-\frac{1}{4a}|z|^2}$, applying Hölder's inequality with the pair of conjugate exponents $\frac{1}{\nu p'}$ and $\frac{2}{(1-\nu)p'}$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} |q_a(z, t)|^{p'} dt &= \int_{-\infty}^{\infty} |q_a(z, t)|^{\nu p'} |q_a(z, t)|^{(1-\nu)p'} dt \\
&\leq \|q_a(z, \cdot)\|_1^{\nu p'} \|q_a(z, \cdot)\|_2^{(1-\nu)p'}
\end{aligned}$$

which gives $\|q_a(\cdot, t)\|_{p'} \leq e^{-\frac{1}{4a}|z|^2}$. Therefore $f^\lambda e^{\frac{1}{4a}|z|^2} \in L^p(\mathbb{C}^n)$ for $1 \leq p \leq \infty$. Also we have $W_\lambda(f^\lambda) e^{bH(\lambda)} = \hat{f}(\lambda) e^{bH(\lambda)}$ is a bounded operator on $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R} \setminus 0$.

Since $a < b$, we can choose $\delta > 0$ such that $a(e^{b\delta} + e^{-b\delta}) < 2b$. Now for $0 < \lambda < \delta$

$$\tanh b\lambda = \frac{e^{b\lambda} - e^{-b\lambda}}{e^{b\lambda} + e^{-b\lambda}} > \frac{2b\lambda}{e^{b\lambda} + e^{-b\lambda}} > \frac{2b\lambda}{e^{b\delta} + e^{-b\delta}} > a\lambda.$$

Using Theorem 4.1 we can conclude that $f^\lambda = 0$ for $0 < \lambda < \delta$ and hence $f^\lambda = 0$ for all λ as f^λ can be extended to a holomorphic function in the strip $|\Im \lambda| < \frac{\delta}{a}$. So $f = 0$ a.e.

Remark 4.3. We can assume $\hat{f}(\lambda) e^{bH(\lambda)} \in \mathcal{S}_q$, $1 \leq q \leq \infty$ in Theorem 1.2 as every member in \mathcal{S}_q is also a bounded operator.

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