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Weyl multipliers for invariant Sobolev spaces

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Abstract. A concrete characterization for the L^p-multipliers $(1 < p < \infty)$ for the Weyl transform is obtained. This is used to study the Weyl multipliers for Laguerre Sobolev spaces $W^{m,p}(\mathbb{C}^n)$. A dual space characterization is obtained for the Weyl multiplier class $M_W(W_L^{m,1}(\mathbb{C}^n))$.

Keywords. Heisenberg group; Hilbert–Schmidt operator; multiplier; Sobolev space; special Hermite functions; twisted convolution; Weyl transform.

1. Introduction

The Laguerre Sobolev spaces $W_L^{s,2}(\mathbb{C}^n)$ were introduced by Peetre and Sparr in [6]. They were also studied by Thangavelu [9] in connection with the spherical means of the Heisenberg group. This space has an invariant property which is not shared by the usual Sobolev space $W^{s,2}(\mathbb{R}^n)$ namely, it is invariant under the symplectic Fourier transform. The details and the relation between this space and the usual Sobolev space can be found in [10].

Fourier multipliers for ordinary Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, ($m \geqslant 0$, an integer), $1 \leqslant p < \infty$ have been characterized by Poornima in [7]. The purpose of this paper is to consider a similar problem for Weyl multipliers for the Laguerre Sobolev spaces $W_L^{m,p}(\mathbb{C}^n)$.

This paper is organized in the following way: In §2, we give the required notations and collect the necessary background. In §3, we obtain a concrete characterization for the L^p-multipliers $(1 < p < \infty)$ for the Weyl transform. In §4, we characterize the Weyl multipliers for $W_L^{m,p}(\mathbb{C}^n)$, based on the result which we obtain in §3. In §5, a dual space characterization is obtained for the space $M_W(W^{m,1}(\mathbb{C}^n))$.

2. Notations and preliminaries

Characterization of Fourier multipliers of L^p-spaces is one of the important problems in multiplier theory. For definition, examples and sufficient conditions for L^p-multipliers on $\mathbb{R}^n$, we refer to Stein [8]. A necessary condition, namely if m is a multiplier for $L^p(\mathbb{R}^n)$, then there exists a pseudo measure σ such that $T_m f = \sigma * f$(* denotes convolution) is also known. In fact, this result is proved for any locally compact abelian group G in place of $\mathbb{R}^n$. This is based on the development of the works of Hormander [3] and Gaudry [2]. The details can be found in [4].

The Weyl transform $W(f)$ of a function $f \in L^1(\mathbb{C}^n)$ is defined by

$$W(f)\varphi(\xi) = \int_{C^n} f(z)\exp(ix(y/2+\xi))\varphi(\xi+y)\,\mathrm{d}z, \quad \varphi \in L^2(\mathbb{R}^n)$$

where $z=x+iy$. The map W from $L^1(\mathbb{C}^n)$ to the space of bounded operators on $L^2(\mathbb{R}^n)$, defined as above, extends uniquely to a bijection from $S'(\mathbb{C}^n)$ to the space of continuous linear maps from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$. Moreover, W maps $L^2(\mathbb{C}^n)$ unitarily onto the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. In other words, we have the Plancherel formula for the Weyl transform, given by

$$\|f\|_2^2=(2\pi)^{-n}\|W(f)\|_{HS}^2.$$

The inversion formula is given by

$$f(z)=(2\pi)^{-n}\operatorname{tr}(W(z)^* W(f)),$$

where $W(z)$ is the operator valued function

$$W(z)\varphi(\xi)=\exp(ix(y/2+\xi))\varphi(\xi+y).$$

For details, we refer to Folland [1].

The twisted convolution of two functions $f, g \in L^1(\mathbb{C}^n)$ is defined by

$$f\times g(z)=\int_{C^n} f(z-w)g(w)\exp(i\operatorname{Im} z\bar{w}/2)dw.$$

Under this, $L^1(\mathbb{C}^n)$ becomes an algebra. Like the ordinary convolution, twisted convolution also extends from $L^1(\mathbb{C}^n)$ to other $L^p(\mathbb{C}^n)$ and satisfies the Young's inequality

$$\|f\times g\|_r\leqslant\|f\|_p\|g\|_q,\quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1.$$

Though the twisted convolution is not commutative, it has better behaviour with respect to L^p estimates. For example, we have the following.

Theorem 2.1. *For f and g in $L^2(\mathbb{C}^n)$, $f\times g$ is also in $L^2(\mathbb{C}^n)$ and*

$$\|f\times g\|_2\leqslant(2\pi)^{n/2}\|f\|_2\|g\|_2.$$

Further, we have $W(f\times g)=W(f)W(g)$.

A bounded operator $M\in\mathscr{B}(L^2(\mathbb{R}^n))$ is called a (left) Weyl multiplier of $L^p(\mathbb{C}^n)$ if the operator T_M defined on $f\in L^1\cap L^p(\mathbb{C}^n)$ by $W(T_M f)=MW(f)$ extends to a bounded operator on $L^p(\mathbb{C}^n)$. We denote the Weyl multiplier class by M_W. The space $M_W(L^1(\mathbb{C}^n))$ is identified with $\mathscr{M}(\mathbb{C}^n)$, the Banach algebra of finite Borel measures on $\mathbb{C}^n$, and $M_W((L^2(\mathbb{C}^n))$ is the algebra $\mathscr{B}(L^2(\mathbb{R}^n)$ of all bounded operators on $L^2(\mathbb{R}^n)$. For any p, $1<p<\infty$, a sufficient condition for L^p-Weyl multipliers has been proved by Mauceri in [5]. However, for the necessary part, only the following is known (see Mauceri [5]).

Let $M\in M_W(L^p(\mathbb{C}^n))$, $1\leqslant p<\infty$. Then there exists a tempered distribution $\rho\in S'(\mathbb{C}^n)$ such that for $f\in\mathscr{S}(\mathbb{C}^n)$, $T_M f=\rho\times f$.

In §3, we try to obtain such a characterization for $M\in M_W(L^p(\mathbb{C}^n))$, through elements in the dual space of a concrete function space, which we call pseudo measures.

Given a function f in $L^p(\mathbb{C}^n)$, $1\leqslant p<\infty$, we have the special Hermite expansion, given by

$$f=(2\pi)^{-n}\sum_{k=0}^{\infty} f\times\varphi_k. \tag{2.1}$$

Here φ_k stands for the Laguerre function

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)\exp(-|z|^2/4),$$

where L_k^{n-1} is the kth Laguerre polynomial of type $(n-1)$. For various results concerning the special Hermite expansions, we refer to [11].

Let L be the special Hermite operator defined by

$$L = -\Delta + \tfrac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right).$$

Then the special Hermite functions are eigenfunctions of the operator L and the series (2.1) is the eigenfunction expansion associated to L. In view of this and spectral theorem one can define L^s(s real), by

$$L^s f = (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)^s f\times\varphi_k.$$

We make use of these operators in the study of Weyl multipliers.

3. Weyl multipliers for $L^p(\mathbb{C}^n)$

Let $\mathscr{B} = \mathscr{B}(L^2(\mathbb{R}^n))$. We denote by $\mathscr{B}_2 = \mathscr{B}_2(L^2(\mathbb{R}^n))$, the Hilbert space of Hilbert–Schmidt operators on $(L^2(\mathbb{R}^n)$, with the norm $\|.\|_2$ and $\mathscr{B}_1 = \mathscr{B}_1(L^2(\mathbb{R}^n))$, the ideal of trace class operators. $\mathscr{B}_1$ is a Banach space under the norm $\|c\|_1 = \mathrm{tr}(|c|) = \mathrm{tr}(c^*c)^{1/2}$ and any element of $\mathscr{B}_1$ can be written as the product FG of two Hilbert–Schmidt operators $F.G$.

Let $A(\mathbb{C}^n)$ denote the space of function f on $\mathbb{C}^n$ whose Weyl transforms $W(f)$ are in B_1. Define

$$\|f\|_A = \|W(f)\|_1 \quad f\in A(\mathbb{C}^n).$$

Then $A(\mathbb{C}^n)$ is an algebra with the multiplication operation given by twisted convolution. Since any element of $\mathscr{B}_1$ is a product of two Hilbert–Schmidt operators and since any Hilbert–Schmidt operator is the Weyl transform of an L^2 function, $A(\mathbb{C}^n)$ contains precisely functions of the form $f\times g$ where f and g are from L^2. Thus $A(\mathbb{C}^n)$ is a subspace of $L^2(\mathbb{C}^n)$. It is easy to see that it is complete with the norm defined above. Thus $A(\mathbb{C}^n)$ is a Banach algebra under $\|.\|_A$, which also shows that Weyl transform is an isometric isomorphism of $A(\mathbb{C}^n)$ onto $\mathscr{B}_1$. We define $P(\mathbb{C}^n)$ to be the dual space of $A(\mathbb{C}^n)$. Then the adjoint of W, W^* will map $\mathscr{B}(L^2(\mathbb{R}^n))$ onto $P(\mathbb{C}^n)$. We call the elements of $P(\mathbb{C}^n)$, pseudo measures. Now, for $\sigma\in P(\mathbb{C}^n)$, $W(\sigma)$ is defined to be the unique element of $\mathscr{B}(L^2(\mathbb{R}^n))$ so that $W^*(W(\sigma)) = \sigma$. Thus, we have the following.

Theorem 3.1. *The Weyl transform $\sigma\mapsto W(\sigma)$ is an isometric isomorphism of $P(\mathbb{C}^n)$ onto $\mathscr{B}(L^2(\mathbb{R}^n))$.*

Let σ_1, $\sigma_2\in P(\mathbb{C}^n)$. We define $\sigma_1\times\sigma_2$ to be that pseudo measure for which $W(\sigma_1\times\sigma_2) = W(\sigma_1)W(\sigma_2)$. This definition makes sense by the above theorem.

Theorem 3.2. *Let M be an L^p-multiplier for the Weyl transform. Then there exists a pseudo measure σ such that $T_M f = \sigma\times f$ for every $f\in L^1\cap L^p(\mathbb{C}^n)$.*

Proof. As $M \in \mathscr{B}$, by theorem 3.1, there exists an element $\sigma \in P(\mathbb{C}^n)$ such that $W(\sigma) = M$. If h is a function in $L^p(\mathbb{C}^n)$, define, for each $g \in A(\mathbb{C}^n)$, $h(g) = \mathrm{tr}(W(h)W(g))$. Then

$$|h(g)| \leqslant \|W(h)\| \, \|W(g)\|_1 = \|W(h)\| \, \|g\|_A,$$

which shows that h can be considered as an element of $P(\mathbb{C}^n)$. If $f \in L^1 \cap L^p(\mathbb{C}^n)$, then the function $T_M f$ can be regarded as a pseudo measure for which the Weyl transform $W(T_M f)$ is defined as earlier. Now we claim that $W(T_M f)$, Weyl transform of the pseudo measure $T_M f$ coincide with the Weyl transform $W(T_M f)$ of the function $T_M f$. Let $P \in \mathscr{B}_1$. Then there exists a $g \in A(\mathbb{C}^n)$ such that $W(g) = P$. Thus we have

$$\begin{aligned} \langle P, W(T_M f)\rangle(\text{pseudo measure}) &= \langle W(g), W(T_M f)\rangle \\ &= \langle W^* W(g), T_M f\rangle \\ &= \langle g, T_M f\rangle \\ &= (T_M f)(g) \\ &= \mathrm{tr}\,(W(T_M f)\,W(g)) \\ &= \mathrm{tr}\,(W(T_M f)P) \\ &= \langle P, W(T_M f)\rangle(\text{function}), \end{aligned}$$

as $W(T_M f)$ (function) belongs to M.

Thus $(W(T_M f))$, the Weyl transform of the pseudo measure $T_M f$ coincides with ordinary $W(T_M f)$ (Weyl transform of the function $T_M f$), which is precisely $MW(f)$. Again, as f can be considered as a pseudo measure, $\sigma \times f$ makes sense and $W(\sigma \times f) = W(\sigma)\,W(f)$. But $W(\sigma) = M$, from which it follows that $W(\sigma \times f) = W(T_M f)$, which in turn implies that $T_M f = \sigma \times f$. □

4. Laguerre Sobolev spaces

Let m be a positive integer. The Sobolev spaces $W_L^{m,p}(\mathbb{C}^n)$ are defined using certain vector on $\mathbb{C}^n$.

The special Hermite operator L can be written as

$$L = -\frac{1}{2}\sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j),$$

where the vector fields Z_j and $\bar{Z}_j$ on $\mathbb{C}^n$ are given by

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} z_j.$$

For $m \geqslant 1$, an integer, we define $W_L^{m,p}(\mathbb{C}^n)$ to be the collection of those functions f in $L^p(\mathbb{C}^n)$ for which $Z^\alpha \bar{Z}^\beta \in L^p(\mathbb{C}^n)$, $|\alpha| + |\beta| \leqslant m$. Here

$$Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} \cdots Z_n^{\alpha_n}, \quad \bar{Z}^\beta = \bar{Z}_1^{\beta_1} \bar{Z}_2^{\beta_2} \cdots \bar{Z}_n^{\beta_n},$$

$\alpha.\beta \in \mathbb{N}^n$. When m is an even integer it follows that $L^{m/2} f \in L^p(\mathbb{C}^n)$ whenever $f \in W_L^{m,p}(\mathbb{C}^n)$. Now if we define

$$\|f\|_{W_L^{m,p}} = \sum_{|\alpha|+|\beta| \leqslant m} \|Z^\alpha \bar{Z}^\beta f\|_p,$$

then $W_L^{m,p}$ turns out to be a Banach space under $\|.\|_{W_L^{m,p}}$.

Let $D(W_L^{1,p})(\mathbb{C}^n)$ denote the collection of functions of the form

$$f=f_0+\sum_{j=1}^{n} Z_jf_j+\sum_{j=1}^{n} \bar{Z}_jg_j,\ldots,$$

where $f_0, f_j\cdot g_j \in W_L^{1,p}(\mathbb{C}^n)$ for $j=1,2,\ldots,n$. Then $D(W_L^{1,p})(\mathbb{C}^n)$ becomes a Banach space if we define the norm $\|.\|_D$ as follows:

$$\|f\|_D=\inf\left\{\max_{j=1\text{ to }n}(\|f_0\|_{W_L^{1,p}}, \|f_j\|_{W_L^{1,p}}, \|g_j\|_{W_L^{1,p}})\right\},$$

where the infimum is taken over all representations of f in the above form. Clearly $W_L^{1,p}(\mathbb{C}^n)$ is contained in $D(W_L^{1,p})(\mathbb{C}^n)$, which in turn is contained in $L^p(\mathbb{C}^n)$. In Proposition 4.1 we will actually show that $D(W_L^{1,p})(\mathbb{C}^n)=L^p(\mathbb{C}^n)$.

Given a bounded operator M on $L^2(\mathbb{R}^n)$, we can define an operator T_M on $L^2\cap(W_L^{1,p})(\mathbb{C}^n)$ by $W(T_Mf)=MW(f)$. We say that M is a left $W_L^{1,p}$ multiplier for the Weyl transform if T_M extends to a bounded operator on $(W_L^{1,p})(\mathbb{C}^n)$.

We first prove the following result.

Theorem 4.1. *Let $1\leqslant p<\infty$. Then we have the following:*

$$M_W(W_L^{1,p})=M_W(D(W_L^{1,p}))=M_W(D(W_L^{1,p}),L^p).$$

Proof. As $D(W_L^{1,p})\subset L^p$, we get

$$M_W(D(W_L^{1,p}),D(W_L^{1,p}))\subset M_W(D(W_L^{1,p}),L^p). \tag{4.1}$$

Suppose $M\in M_W(D(W_L^{1,p}),L^p)$. Let $f\in W_L^{1,p}$. Define T_M on $W_L^{1,p}\cap L^2(\mathbb{C}^n)$ by $W(T_Mf)=MW(f)$. As $W_L^{1,p}\subset D(W_L^{1,p})$, $T_Mf\in L^p$. Let $A_j=-(\partial/\partial x_j)+x_j$, $A_j^*=(\partial/\partial x_j)+x_j$. Then

$$W(Z_jT_Mf)=iW(T_Mf)A_j=MiW(f)A_j=W(T_MZ_jf) \tag{4.2}$$

and

$$W(\bar{Z}_jT_Mf)=iW(T_Mf)A_j^*=W(T_M\bar{Z}_jf). \tag{4.3}$$

Further, as $f\in W_L^{1,p}$, Z_jf, $\bar{Z}_jf\in D(W_L^{1,p})$, and so T_MZ_jf, $T_M\bar{Z}_jf\in L^p$. Thus it follows from (4.2) and (4.3), that $Z_jT_Mf, \bar{Z}_jT_Mf\in L^p$, which will then imply that $T_Mf\in W_L^{1,p}$. By definition,

$$\|T_Mf\|_{W_L^{1,p}}=\|T_Mf\|_p+\sum_{j=1}^{n}\|Z_jT_Mf\|_p+\sum_{j=1}^{n}\|\bar{Z}_jT_Mf\|_p.$$

But

$$\|T_Mf\|_p\leqslant C_M\|f\|_D\leqslant C_M\|f\|_{W_L^{1,p}},$$

$$\|Z_jT_Mf\|_p=\|T_MZ_jf\|_p\leqslant C_M\|Z_jf\|_D\leqslant C_M\|f\|_{W_L^{1,p}}$$

and

$$\|\bar{Z}_jT_Mf\|_p\leqslant C_M\|f\|_{W_L^{1,p}}.$$

Thus

$$\|T_Mf\|_p\leqslant(2n+1)C_M\|f\|_{W_L^{1,p}},$$

which shows that T_M is a bounded operator on $W_L^{1,p}(\mathbb{C}^n)$. Hence

$$M_W(D(W_L^{1,p}),L^p)\subset M_W(W_L^{1,p}). \tag{4.4}$$

Now let $M \in M_W(W_L^{1,p})$. For $f \in D(W_L^{1,p})$, we define

$$\tilde{T}_M f = T_M f_0 + \sum_{j=1}^{n} Z_j T_M f_j + \sum_{j=1}^{n} \bar{Z}_j T_M g_j.$$

To prove $\tilde{T}_M$ is well defined, assume that $f \in D(W_L^{1,p})$ is a representation of 0, viz

$$f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j = 0, \quad f_0, f_j, g_j \in W_L^{1,p}, 1 \leqslant j \leqslant n.$$

Consider

$$\begin{aligned} W(\tilde{T}_M f) &= W(T_M f_0) + \sum_{j=1}^{n} W(Z_j T_M f_j) + \sum_{j=1}^{n} W(\bar{Z}_j T_M g_j) \\ &= M W(f_0) + \sum_{j=1}^{n} iMW(f_j) A_j + \sum_{j=1}^{n} iMW(g_j) A_j^* \\ &= M W(f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j) \\ &= 0. \end{aligned}$$

Thus $\tilde{T}_M$ is well defined and $W(\tilde{T}_M f) = M W(f)$. The proof will be complete if we could show that $\tilde{T}_M$ is bounded. Let $f \in D(W_L^{1,p})$. Consider

$$\begin{aligned} \| \tilde{T}_M f \|_D &\leqslant \max_{1 \leqslant j \leqslant n} (\| T_M f_0 \|_{W_L^{1,p}}, \| T_M f_j \|_{W_L^{1,p}}, \| T_M g_j \|_{W_L^{1,p}}) \\ &\leqslant C_M \max_{1 \leqslant j \leqslant n} (\| f_0 \|_{W_L^{1,p}}, \| f_j \|_{W_L^{1,p}}, \| g_j \|_{W_L^{1,p}}) \end{aligned}$$

which is true for any representation $f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j$ of f. Hence it follows that $\| \tilde{T}_M f \|_D \leqslant C_M \| f \|_D$, which shows that $\tilde{T}_M$ is a bounded operator on $D(W_L^{1,p})$. Thus

$$M_W(W_L^{1,p}) \subset M_W(D(W_L^{1,p})). \tag{4.5}$$

From (4.1), (4.4) and (4.5) we get the required result. □

For $m \in \mathbb{N}$, $D(W_L^{m,p})$ is defined as earlier, viz

$$D(W_L^{m,p}) = \left\{ f = f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j \right\},$$

where $f_0, f_j, g_j \in W_L^{m,p}(\mathbb{C}^n)$, $j = 1, 2, \ldots, n$. With this definition, we have the following.

Theorem 4.2. *Let $1 \leqslant p < \infty$ and m, an integer $\geqslant 1$. Then we have the following*

$$M_W(W_L^{m,p}) = M_W(D(W_L^{m,p})) = M_W(D(W_L^{m,p}), W_L^{m-1,p}).$$

PROPOSITION 4.1

Let $1 < p < \infty$. Then $D(W_L^{1,p}) = L^p$.

Proof. Let $f \in L^p$. Write $f = LL^{-1} f$, viz

$$f = \sum_{j=1}^{n} Z_j \left(\frac{1}{2} \bar{Z}_j L^{-1} f \right) + \sum_{j=1}^{n} \bar{Z}_j \left(\frac{1}{2} Z_j L^{-1} f \right).$$

We claim that $\bar{Z}_j L^{-1} f$ and $Z_j L^{-1} f$ are in $W_L^{1,p}(\mathbb{C}^n)$. In theorem 2.2.2 of [11] it has been proved that $\bar{Z}_j L^{-1/2}$ and $Z_j L^{-1/2}$ are bounded operators on $L^p(\mathbb{C}^n)$, $1 < p < \infty$. The same argument shows that $\bar{Z}_j L^{-1}$ and $Z_j L^{-1}$ are also bounded on $L^p(\mathbb{C}^n)$ (see the reasoning below).

Let now $S_j f = Z_j \bar{Z}_j L^{-1} f$, $S_j^* f = \bar{Z}_j Z_j L^{-1} f$. We claim that $S_j f$, $S_j^* f \in L^p$. In view of Theorem 2.2.1 of [11], we have to show that S_j and S_j^* are twisted convolution operators with Calderon–Zygmund kernels and they are bounded on L^2. Consider the operator S_j. We can write

$$S_j f = -(2\pi)^{-n/2} \sum_{\mu} \left(\frac{2\mu_j + 2}{2|\mu| + n} \right) f \times \phi_{\mu\mu}, \tag{4.6}$$

as $Z_j(\phi_{\mu\nu}) = i(2\nu_j)^{1/2} \phi_{\mu,\nu-\varepsilon_j}$, $\bar{Z}_j(\phi_{\mu\nu}) = i(2\nu_j + 2)^{1/2} \phi_{\mu,\nu+\varepsilon_j}$ and $L(\phi_{\mu\nu}) = (2|\nu| + n)\phi_{\mu\nu}$. From (4.6), it is clear that S_j is bounded on L^2. And S_j is given by $S_j f = f \times k_j$ where

$$k_j = Z_j \bar{Z}_j \int_0^\infty k_t(z) \mathrm{d}t$$

and $k_t(z)$ is the kernel of $\exp(-tL)$ given by

$$k_t(z) = (\sinh 2t)^{-n} \exp(-\coth t |z|^2).$$

We can show that k_j satisfies

$$|k_j(z)| \leqslant c|z|^{-2n},$$
$$|\nabla k_j(z)| \leqslant c|z|^{-2n-1}.$$

Thus, from theorem 2.2.1 of [11], we conclude that S_j is bounded on L^p. Similarly we can show that S_j^* is bounded on L^p. Then, it follows that $\bar{Z}_j L^{-1} f, Z_j L^{-1} f \in W_L^{1,p}$, which shows that $f \in D(W_L^{1,p})$.

We can also prove the following.

PROPOSITION 4.2

Let $1 < p < \infty$. Then $D(W_L^{m,p}) = W_L^{m-1,p}$ for $m \geqslant 1$ any integer.

Putting the above facts together, we obtain the following.

Theorem 4.3. *Let $1 < p < \infty$ and m, any integer $\geqslant 1$. Then the space of Weyl multipliers for the Laguerre Sobolev space $W_L^{m,p}(\mathbb{C}^n)$ coincide with the space of Weyl multipliers for $L^p(\mathbb{C}^n)$.*

This, combined with the theorem 3.2, leads to the following Corollary.

COROLLARY 4.1

Let $1 < p < \infty$ and m, any integer $\geqslant 1$. Let M be a Weyl multiplier for the Laguerre Sobolev space $W_L^{m,p}(\mathbb{C}^n)$. Then there exists a pseudo measure σ such that $T_M f = \sigma \times f \quad \forall f \in C_c^\infty(\mathbb{C}^n)$.

5. The space $M_W(W_L^{m,1}(\mathbb{C}^n))$

We first remark that $M_W(L^1(\mathbb{C}^n)) \subset M_W(W_L^{1,1}(\mathbb{C}^n))$. For, suppose $M \in M_W(L^1(\mathbb{C}^n))$, define T_M on $L^2 \cap (W_L^{1,1}(\mathbb{C}^n))$ by $W(T_M f) = MW(f)$. Let $f \in W_L^{1,1}$. Then $Z_j f$,

$\bar{Z}_j f \in L^1$. Therefore, it is easy to verify that $W(T_M Z_j f) = W(Z_j T_M f)$ and $W(T_M \bar{Z}_j f) = W(\bar{Z}_j T_M f)$. Then, as $M \in M_W(L^1, L^1)$, it follows that $T_M f \in W_L^{1,1}$ and $\|T_M f\|_{W_L^{1,1}} \leqslant C_M \|f\|_{W_L^{1,1}}$ for every $f \in W_L^{1,1}$.

Let S denote the collection of elements f of the form $f = \sum f_i \times g_i$, where $f_i \in D(W_L^{1,1})$, $g_i \in C_0(\mathbb{C}^n)$, $\sum \|f_i\|_D \|g_i\|_\infty < \infty$. Then S is a Banach space under the norm

$$\|f\|_s = \inf\left(\sum \|f_i\|_D \|g_i\|_\infty\right),$$

where the infimum is taken over all representations of f in the above form. Then we prove the following theorem.

Theorem 5.1. *There is an isometric isomorphism of $M_W(W_L^{1,1})$ onto the dual space S^* of S.*

Proof. By theorem 4.1, we have $M_W(W_L^{1,1}) = M_W(D(W_L^{1,1}), L^1)$. Suppose $M \in M_W(D(W_L^{1,1}), L^1)$. For $f = \sum f_i \times g_i \in S$, define $\beta_M(f) = \sum_i T_M f_i \times g_i(0)$. As $T_M f_i \in L^1$ and $g_i \in C_0$, $T_M f_i \times g_i \in C_0$ and $T_M f_i \times g_i(0)$ is meaningful. To prove β_M is well defined, let $f = \sum f_i \times g_i$ be a representation of 0. Choose an approximate identity $\{e_\alpha\} \subset C_c^\infty(\mathbb{C}^n)$ for $D(W_L^{1,1})$ such that $\|e_\alpha\|_1 \leqslant 1$. As

$$\begin{aligned}\|T_M(e_\alpha \times f_i) \times g_i - T_M f_i \times g_i\|_\infty &\leqslant \|T_M(e_\alpha \times f_i) - T_M f_i\|_1 \|g_i\|_\infty \\ &\leqslant \|T_M\| \|e_\alpha \times f_i - f_i\|_D \|g_i\|_\infty,\end{aligned}$$

left hand side tends to 0 as $\alpha \to \infty$. Further,

$$\begin{aligned}\|\sum_i T_M(e_\alpha \times f_i) \times g_i(0)\| &\leqslant \|T_M\| \sum_i \|e_\alpha \times f_i\|_D \|g_i\|_\infty \\ &\leqslant \|T_M\| \sum_i \|f_i\|_D \|g_i\|_\infty \text{ (as } \|e_\alpha\|_1 \leqslant 1),\end{aligned}$$

which shows that $\sum_i T_M(e_\alpha \times f_i) \times g_i(0)$ converges to $\sum_i T_M f_i \times g_i(0)$. Now for each α,

$$\sum_i T_M(e_\alpha \times f_i) \times g_i(0) = (T_M e_\alpha \times \sum_i f_i \times g_i)(0) = 0.$$

Thus $\sum_i T_M f_i \times g_i(0) = 0$, proving that β_M is well defined. β_M satisfies

$$|\beta_M(f)| \leqslant \|T_M\| \sum_i \|f_i\|_D \|g_i\|_\infty,$$

which is true for every representation $\sum f_i \times g_i$ of f, showing that $|\beta_M(f)| \leqslant \|T_M\| \|f\|_S$ or

$$\|\beta_M\|_{S^*} \leqslant \|T_M\|. \tag{5.1}$$

On the other hand,

$$\|T_M\| = \sup_{\|f\|_D \leqslant 1} \|T_M f\|, \quad (f \in D(W_L^{1,1}))$$

and

$$\|T_M f\|_1 = \|T_M f\|_{C_0^*} = \sup_{\|g\|_\infty \leqslant 1} |T_M f(g)| \quad (g \in C_0).$$

But

$$|T_M f(g)| = |T_M f \times g(0)| = |\beta_M(f \times g)| \leqslant \|\beta_M\|_{S^*} \|f\|_D \|g\|_\infty,$$

from which it follows that

$$\|T_M\| \leqslant \|\beta_M\|_{S^*}. \tag{5.2}$$

From (5.1) and (5.2), we see that $T_M \mapsto \beta_{T_M}$ is an isometry. To prove the mapping is surjective, let us assume that $\beta \in S^*$. Fix $f \in D(W_L^{1,1})$, define for each $g \in C_0$, $F_f(g) = \beta(f \times g)$. Then

$$|F_f(g)| = |\beta(f \times g)| \leqslant \|\beta\| S^* \|f\|_D \|g\|_\infty,$$

which shows that F_f is a continuous linear functional on C_0. Hence there exists a unique $\mu_f \in M(\mathbb{C}^n)$ such that

$$F_f(g) = \beta(f \times g) = \mu_f(g) = \mu_f \times g(0). \tag{5.3}$$

Choose an approximate identity $\{e_\alpha\} \subset C_c^\infty(\mathbb{C}^n)$ for $D(W_L^{1,1})$. Then corresponding to each e_α, we have a unique μ_{e_α} in $M(\mathbb{C}^n)$ satisfying (5.3). Since $\mu_{e_\alpha} \in M(\mathbb{C}^n) = M_W(L^1(\mathbb{C}^n))$, μ_{e_α} is identified with $M_{e_e} = M_\alpha \in M_W(L^1(\mathbb{C}^n))$ such that

$$T_{M_\alpha} f = \mu_{e_\alpha} \times f \quad \forall f \in L^1(\mathbb{C}^n). \tag{5.4}$$

As $\{e_\alpha\}$ is an approximate identity for $D(W_L^{1,1})$, we get $\|e_\alpha \times h \times g - h \times g\|_D \to 0$ as $\alpha \to \infty$. Then as $\beta \in S^*$, we have

$$\lim_\alpha \beta(e_\alpha \times h \times g) = \beta(h \times g).$$

Thus, it follows from (5.3) that $\lim_\alpha \mu_{e_\alpha}(h \times g)$ exists for every $h \in C_c^\infty(\mathbb{C}^n)$ and $g \in C_0(\mathbb{C}^n)$. As the collection of elements $h \times g$, $h \in C_c^\infty(\mathbb{C}^n)$, $g \in C_0(\mathbb{C}^n)$, is dense in $C_0(\mathbb{C}^n)$, we have $\lim_\alpha \mu_{e_\alpha}(g)$ exists for every $g \in C_0(\mathbb{C}^n)$. There exists a $\mu \in M(\mathbb{C}^n)$ such that μ_{e_α} converges to μ in the weak * topology. Also $\|\mu\| = \lim_\alpha \|\mu_{e_\alpha}\|$. As $\mu \in M(\mathbb{C}^n)$, μ is identified with an operator $M \in M_W(L^1(\mathbb{C}^n))$ such that $T_M f = \mu \times f$ for every $f \in L^1(\mathbb{C}^n)$. As μ_{e_α} converges to μ, it follows from (5.4) that $\lim_\alpha T_{M_\alpha} f = T_M f$ for every $f \in L^1(\mathbb{C}^n)$. By the remark mentioned earlier, we get $M \in M_W(W_L^{1,1}) = M_W(D(W_L^{1,1}).L^1)$. The proof will be complete if we could show that β_M coincides with β on S. For $h \in C_c^\infty(\mathbb{C}^n)$, $g \in C_0(\mathbb{C}^n)$, we have

$$\begin{aligned}
\beta_M(h \times g) &= \lim_\alpha T_{M_\alpha} h \times g(0) \\
&= \lim_\alpha \mu_{e_\alpha} \times h \times g(0) \\
&= \lim_\alpha \beta(e_\alpha \times h \times g) \\
&= \beta(h \times g).
\end{aligned}$$

Using the density argument, we can show that $\beta_M(f \times g) = \beta(f \times g)$ for every $f \in D(W_L^{1,1})$, $g \in C_0$, thus proving our assertion.

If we define S_m ($m \in \mathbb{N}$) to be the collection of elements f of the form $f = \sum f_i \times g_i$, where $f_i \in D(W_L^{m,1})$, $g_i \in C_0$, $\sum \|f_i\|_D \|g_i\|_\infty < \infty$. Then S_m becomes a Banach space. Using the facts that $\|f\|_1 \leqslant \|f\|_D$ for every $f \in D$, $M_W(L^1) \subset M_W(W_L^{m,1})$ and by making use of the proof of Theorem 5.1,we obtain the following.

Theorem 5.2. *Let m be an integer such that $m \geqslant 1$. Then there is a continuous isomorphism of $M_W(W_L^{m,1})$ onto the dual space S_m^* of S_m.*

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