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ON IDEALS AND DUALS OF C^*-ALGEBRAS

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The following result, together with some of its consequences, is established :
Let I be a closed ideal in a C^*-algebra. Then, any $\phi \in I^*$ extends uniquely to a $\tilde{\phi} \in A^*$ such that $\|\tilde{\phi}\| = \|\phi\|$. Further, if $\psi \in A^*$ satisfies $\psi(I) = 0$, then $\|\tilde{\phi} + \psi\| = \|\phi\| + \|\psi\|$. In particular, if $\pi : A \to B$ is a surjective *-homomorphism of C^*-algebras, with ker $\pi = I$, then there is a canonical isometric isomorphism of Banach spaces : $A^* \cong I^* \oplus_{l1} B^*$.

We give an elementary proof of the following result :

Let I be a closed ideal in a C^*-algebra A. Then, any $\phi \in I^*$ extends uniquely to a $\tilde{\phi} \in A^*$ such that $\|\tilde{\phi}\| = \|\phi\|$. Further, if $\psi \in A^*$ satisfies $\psi(I) = 0$, then $\|\tilde{\phi} + \psi\| = \|\phi\| + \|\psi\|$. In particular, if $\pi : A \to B$ is a*-epimorphism of C^*-algebras, there is a canonical isometric isomorphism of Banach spaces :

$$A^* \cong I^* \oplus_{l1} B^*.$$

The usual proof of the above result (cf. Takesaki 1979) appeals to the universal representation of the C^*-algebra and applies techniques from the theory of von Neumann algebras (such as the polar decomposition for linear functions) to the enveloping von Neumann algebra of A. This proof is presented here, since it uses only a few basic facts from C^*-algebra theory, in the hope that the result may be amenable to one who is not a specialist in von Neumann algebras, such as a Banach space-theorist, who may like to know more examples of situations when the Hahn-Banach extension is unique.

Notation : Throughout this short note, the symbols H and $L(H)$ will denote, respectively, a complex Hilbert space, and the C^*-algebra of bounded linear operators on H. For a Banach space $\bar{X}$, the symbol X^* will denote the Banach space of bounded linear functionals on x. For Banach spaces X and Y, the symbol $X \oplus_{l1} Y$ will denote the Banach space $\{(x, y) : x \in X, y \in Y\}$ with coordinatewise vector operations and norm $\|(x, y)\|_{l1} = \|x\| + \|y\|$.

The symbols A, B will always denote C^*-algebras with identity 1, while the symbol I will invariably denote a closed two-sided ideal of a C^*-algebra. The results extend easily to C^*- algebras without identity. The assumed existence of identity is largely just a matter of convenience; for instance, as in Lemma 1, we may make statements such as 'let $0 \leqslant x \leqslant 1$'.

Lemma 1—Let $P \in L(H)$ satisfy $0 \leqslant P \leqslant 1_H$. Let Q be the operator on $H \oplus H$ defined by the matrix.

$$\begin{bmatrix} 1_H - P & P \\ P & 1_H - P \end{bmatrix}.$$

Then

$$\|Q\| \leqslant 1.$$

PROOF : *Case* (*i*) : dim $H = 1$—In this case $Q = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \in L(\mathbb{C}^2)$ where $0 \leqslant p \leqslant 1$. Observe that $Q = (1-p)\, 1_{\mathscr{C}^2} + p \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ expresses Q as a convex combination of unitary operators, and hence, $\|Q\| \leqslant 1$.

Case (*ii*) : P has pure point spectrum—Thus, there exists an orthonormal basis $\{\phi_i\}$ of H such that $P\phi_i = p_i\phi_i$, where $0 \leqslant p_i \leqslant 1$ for i. Then, Q is unitarily equivalent to the operator $\bigoplus_i \begin{bmatrix} 1-p_i & p_i \\ p_i & 1-p_i \end{bmatrix}$, and so, by Case (*i*), it follows that $\|Q\| \leqslant 1$.

Case (*iii*) : P arbitrary—There exists a sequence $\{P_n\}$ of operators on H such that $\|P_n - P\| \to 0$, and further, each P_n satisfies (*a*) $0 \leqslant P_n \leqslant 1_H$, and (*b*) P_n has pure point spectrum. In $Q_n = \begin{bmatrix} 1_H - P_n & P_n \\ P_n & 1_H - P_n \end{bmatrix}$, then $Q_n \to Q$. Since, by Case (*ii*), $\|Q_n\| \leqslant 1$, it follows that $\|Q\| \leqslant 1$.

Lemma 2—Let A be a C^*-algebra. Let $p \in A$ satisfy $0 \leqslant p \leqslant 1$. Then, for any x, y in A,

$$\|(1-p)\, x\, (1-p) + pyp\| \leqslant \max\{\|x\|, \|y\|\}.$$

PROOF : In veiw of Gelfand-Naimark's theorem, we may assume that $A \subseteq L(H)$. Define Q, T in $L(H \oplus H)$ by

$$Q = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}, T = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$

Then,

$$\|QTQ\| \leqslant \|T\| \, \|Q\|^2$$
$$\leqslant \|T\| \text{ (by Lemma 1)}$$
$$= \max \{\|x\|, \|y\|\}.$$

It suffices now to observe that the (1, 1)-entry QTQ is $(1-p)\,x\,(1-p) + pyp$.

Proposition 3—Let I be a closed (two-sided) ideal in a C^*-algebra A. Let $\{e_\alpha\}_{\alpha \in \Lambda}$ be an approximate identity for I (cf., for instance Arveson 1976). Let $\phi \in I^*$. Then, $\tilde{\phi}(x) = \lim_\alpha \phi(xe_\alpha)$ exists and defines an element $\tilde{\phi} \in A^*$ such that $\tilde{\phi}/I = \phi$. Further, if $\{f_\beta\}$ is any other approximate identity for I, then, $\tilde{\phi}(x) = \lim_\beta \phi(xf_\beta) = \lim_\beta \phi(f_\beta x) = \lim_\beta \phi(f_\beta\, x\, f_\beta)$, for all x in A.

PROOF: *Case* (*i*): $\phi \geqslant 0$—The *GNS*-construction yields (cf. Arveson 1976, Sakai 1971) a representation $w : I \to L(H)$ for some H, and a vector $\xi \in H$ such that $H = (\pi(I)\,\xi)^-$ and $\phi(x) = \langle \pi(x)\,\xi, \xi \rangle$ for all x in I. The non-degeneracy of the representation implies that (*a*) $\pi(f_\beta) \to 1_H$ in the strong topology, whenever $\{f_\beta\}$ is an approximate identity for I, and (*b*) there exists a unique representation $\tilde{\pi} : A \to L(H)$ such that $\tilde{\pi}/I = \pi$. Now, if one defines $\tilde{\phi}(x) = \langle \tilde{\pi}(x)\,\xi, \xi \rangle$ for all x in A, it is clear that $\tilde{\phi}/I = \phi$, and that, with $\{f_\beta\}$ as above, $\tilde{\phi}(x) = \lim \phi(xf_\beta) = \lim \phi(f_\beta\, x) = \lim \phi(f_\beta\, x\, f_\beta)$ for any x in A.

Case (*ii*): ϕ arbitrary—It is possible (cf. Sakai 1971) to express ϕ as $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$, where $\phi_j \geqslant 0$. The result follows by applying Case (*i*) to each ϕ_j.

Corollary 4—With I, A ϕ, $\tilde{\phi}$ as in Proposition 3, define $\sigma : I^* \to A^*$ by $\sigma(\phi) = \tilde{\phi}$. Then, σ is a linear isometric map.

PROOF: Linearity of σ is obvious.

If $\{e_\alpha\}$ is an approximate identity for I, then, for any x in A, it is clear that

$$|\,\phi(xe_\alpha)\,| \leqslant \|\phi\|\,\|xe_\alpha\| \leqslant \|\phi\|\,\|x\|$$

since $\|e_\alpha\| \leqslant 1$. Passage to limits yields $\|\tilde{\phi}\| \leqslant \|\phi\|$. The reverse inequality is a consequence of $\tilde{\phi}/I = \phi$.

Theorem 5—Let I be a closed two-sided ideal of a C^*-algebra A. Then, the map $I^* \oplus_1 I^\perp \to A^*$ defined by $(\phi_1, \phi_2) \to \sigma(\phi_1) + \phi_2$ (with σ as above) is an isometric isomorphism of Banach spaces. The inverse of the above map is given by $\phi \to (\phi/I, \phi - \sigma(\phi/I))$.

Proof: It is clear that the map in question is a bijection with inverse given as above. Only the statement about the norms is to be proved, viz. $\|\sigma(\phi_1) + \phi_2\| = \|\phi_1\| + \|\phi_2\|$ whenever $\phi_1 \in I^*$ and $\phi_2 \in I^\perp$. (Recall that $I^\perp = \{\phi \in A^* : I \subseteq \ker \phi\}$.)

So, suppose $\phi_1 \in I^*$, $\phi_2 \in I^\perp$. Let $\epsilon > 0$ be given. Pick unit vectors $x \in A$ and $y \in I$ such that

$$\text{Re}\, \phi_2(x) > \|\phi_2\| - \epsilon$$

and

$$\text{Re}\, \phi_1(y) > \|\phi_1\| - \epsilon.$$

Let $\{e_\alpha : \alpha \in \Lambda\}$ be an approximate identity for I; for each α, define

$$x_\alpha = (1 - e_\alpha)\, x\, (1 - e_\alpha) + y$$

and

$$z_\alpha = (1 - e_\alpha)\, x\, (1 - e_\alpha) + e_\alpha\, y\, e_\alpha.$$

It follows from Lemma 2 that $\|z_\alpha\| \leqslant 1$ while $\lim_\alpha \|x_\alpha - z_\alpha\| = \lim_\alpha \|y - e_\alpha\, y\, e_\alpha\| = 0$, since $y \in I$. Hence, there exists $\alpha_1 \in \Lambda$ such that $\|x_\alpha\| < 1 + \epsilon$ for $\alpha \geqslant \alpha_1$.

Observe that

$$x_\alpha = x + (e_\alpha\, x\, e_\alpha - e_\alpha\, x - x e_\alpha + y)$$

where the term in parentheses belongs to I. Hence

$$\text{Re}\, \phi_2(x_\alpha) = \text{Re}\, \phi_2(x) > \|\phi_2\| - \epsilon.$$

On the other hand,

$$\sigma(\phi_1)(x_\alpha) = \sigma(\phi_1)(x) + \phi_1(e_\alpha\, x\, e_\alpha) - \phi_1(e_\alpha\, x) - \phi_1(x e_\alpha) + \phi_1(y).$$

It follows from Proposition 3 that

$$\lim_\alpha \sigma(\phi_1)(x_\alpha) = \phi_1(y).$$

So, there exists $\alpha_2 \in \Lambda$ such that

$$\text{Re}\, \sigma(\phi_1)(x_\alpha) > \|\phi_1\| - \epsilon \text{ for } \alpha \geqslant \alpha_2.$$

Now, if $\alpha \geqslant \alpha_1$ and $\alpha \geqslant \alpha_2$, it follows that

$$\begin{aligned} \|\sigma(\phi_1) + \phi_2\| (1 + \epsilon) &> \|\sigma(\phi_1) + \phi_2\| \, \|x_\alpha\| \\ &\geqslant |(\sigma(\phi_1) + \phi_2)(x_\alpha)| \\ &\geqslant \text{Re}\, \sigma(\phi_1)(x_\alpha) + \text{Re}\, \phi_2(x_\alpha) \\ &> \|\phi_1\| - \epsilon + \|\phi_2\| - \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$ yields

$$\|\sigma(\phi_1) + \phi_2\| \geqslant \|\phi_1\| + \|\phi_2\|.$$

The reverse inequality follows from the isometric nature of σ (and the triangle inequality !).

Corollary 6—Let I and A be as above. Then, for any $\phi \in I^*$, there exists a unique $\tilde{\phi} \in A^*$ such that $\tilde{\phi} \mid I = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$. (i.e., the Hahn-Banach extension is unique.)

PROOF : $\tilde{\phi} = \sigma(\phi)$ is an extension with the same norm. If ψ is any other extension of ϕ, it follows from Theorem 5 that

$$\begin{aligned} \|\psi\| &= \|\sigma(\psi \mid I\| + \|\psi - \sigma(\psi/I)\| \\ &= \|\sigma(\phi)\| + \|\psi - \sigma(\phi)\| \\ &= \|\phi\| + \|\psi - \sigma(\phi)\| \end{aligned}$$

and so,

$$\|\psi\| = \|\phi\| \text{ iff } \psi = \sigma(\phi).$$

An alternative formulation of Theorem 5 is as follows : If $\pi : A \to B$ is a surjective *-homomorphism of C^*-algebras, then, in a natural way, $A^* \cong (\ker \pi)^* \oplus_{l_1} B^*$.

Definition 7—If $\pi : A \to B$ is a surjective *-homomorphism of C^*-algebra, let $\pi_* : A^* \to B^*$ be the projection map induced by the above decomposition.

A more precise definition of π_* is as follows : Let $I = \ker \pi$. Then π induces an isometric *-isomorphism $\tilde{\pi} : A/I \to B$ (cf., for instance Arveson 1976). It follows that $\pi^* : B^* \to A^*$ (defined by $\pi^*(\psi) = \psi \circ \pi$) is isometric. Finally, if $\sigma : I^* \to A^*$

is as in Proposition 3, then, for any $\phi \in A^*$, it is seen that $\phi - \sigma(\phi/I) \in$ range π^*; define $\pi_*(\phi) = \pi^{*-1}(\phi - \sigma(\phi/I))$.

Corollary 8—Let $\pi : A \to B$ be a surjective *-algebra homomorphism, and assume $B \neq (0)$. Then,

(i) $\pi^* \circ \pi_*$ is a norm one projection of A^* onto $\pi^*(B^*)$.

(ii) π_* is linear and $\|\pi_*\| = 1$.

(iii) $\|\phi - \pi^* \circ \pi_*(\phi) + \pi^*(\psi)\| = \|\phi - \pi^* \circ \pi_*(\phi)\| + \|\psi\| \ \forall\ \psi \in B^*$.

(iv) $\|\phi - \pi^* \circ \pi_*(\phi)\| \leqslant \|\phi - \pi^*(\psi)\| \ \forall\ \psi \in B^*$; further, the above is an equality if and only if $\psi = \pi_*(\phi)$. (In other words, if $\phi \in A^*$, $\pi^* \circ \pi_*(\phi)$ is the unique best approximant to ϕ from $\pi^*(B^*)$.)

(v) $\pi \to \pi_*$ is functorial; in other words,

(a) if $\pi = 1_A : A \to A$, $\pi_* = 1_{A*}$ (more generally, if $\pi : A \to B$ is *a* *-isomorphism, then $\pi_* = (\pi^{-1})^*$); and

(b) if $\pi_1 : A \to B$ and $\pi_2 : B \to C$ are surjective *-homomorphisms of C^*-algebras, then $(\pi_2 \circ \pi_1)_* = \pi_{2*} \circ \pi_{1*}$.

PROOF : Let $I = \ker \pi$, and let $\sigma : I^* \to A^*$ as in Proposition 3.

(i) $\pi^* \circ \pi_*(\phi) = \phi - \sigma(\phi/I)$ by definition. So, by Theorem 5, if follows that $\pi^* \circ \pi_*$ is a projection on A^*, with norm $\leqslant 1$. Since the range of this projection is $\pi^*(B^*) \neq (0)$ (since $B \neq (0)$), it follows that $\|\pi^* \circ \pi_*\| = 1$.

(ii) follows from $\pi_* = (\pi^{*-1} \mid \pi^*(B^*)) \circ (\pi^* \circ \pi_*)$ and (i).

(iii) is also an immediate consequence of Theorem 5.

(iv) for any ψ in B^*, use (iii) to write

$$\|\phi - \pi^*(\psi)\| = \|\phi - \pi^* \circ \pi_*(\phi) + \pi^*(\pi_*(\phi) - \psi)\|$$

$$= \|\phi - \pi^* \circ \pi_*(\phi)\| + \|\pi_*(\phi) - \psi\|$$

All the assertions of (iv) follow immediately.

(v) The proof of (a) is trivial.

(b) : In view of the uniqueness assertion of (iv), it suffices to show that

$$\|\phi - (\pi_2 \circ \pi_1)^*(\pi_{2_*} \circ \pi_{1_*}(\phi))\| \leqslant \|\phi - (\pi_2 \circ \pi_1)^*(\psi)\|$$

for all $\psi \in C^*$. For this, observe that

$$\|\phi - (\pi_2 \circ \pi_1)^* (\pi_{2_*} \circ \pi_{1_*} (\phi))\|$$

$$= \|\phi - \pi_1^* (\pi_{1_*} (\phi)) + \pi_1^* (\pi_{1_*} (\phi) - \pi_2^* \circ \pi_{2_*} \circ \pi_{1_*} (\phi))\|$$

$$= \|\phi - \pi_1^* \circ \pi_{1_*} (\phi)\| + \|\pi_{1_*} (\phi) - \pi_2^* \circ \pi_{2_*} \pi_{*} \circ {}_1(\phi)\| \text{ (by (iii))}$$

while, for any ψ in C^*,

$$\|\phi - (\pi_2 \circ \pi_1)^* (\psi)\|$$

$$= \|\phi - \pi_1^* \circ \pi_{1_*} (\phi) + \pi_1^* (\pi_{1_*} (\phi) - \pi_2^* (\psi))\|$$

$$= \|\phi - \pi_1^* \circ \pi_{1_*} (\phi)\| + \|\pi_{1_*} (\phi) - \pi_2^* (\psi)\|$$

(by (iii) applied to π_1)

$$\geqslant \|\phi - \pi_1^* \circ \pi_{1_*} (\phi)\| + \|\pi_{1_*} (\phi) - \pi_2^* \circ \pi_{2_*} (\pi_{1_*} (\phi))\|$$

(by (iv) applied to π_2)

$$= \|\phi - (\pi_2 \circ \pi_1)^* (\pi_{2_*} \circ \pi_{1_*} (\phi))\| ,$$

as desired.

Corollary 9 (Dixmier 1950 or Schatten 1960)—Let H be an infinite-dimensional Hilbert space. Any $\phi \in L(H)^*$ has a unique decomposition $\phi = \phi_1 + \phi_2$, such that

(i) $\phi_2 (x) = 0$ for every compact operator x on H; and

(ii) there exists a trace class operator g on H such that $\phi_1 (x) = tr\, gx$ for all x in $L(H)$.

Further, $\|\phi\| = \|\phi_1\| + \|\phi_2\|$.

PROOF: Let $A = L(H)$ and $I = K(H)$, the closed ideal of compact operators on H. It is known that every $\phi_1 \in K(H)^*$ is induced by a trace-class operator g in the sense of (ii) above. The above result now follows from Theorem 5 and the above identifications.

Finally, we remark that if A is an abelian C^*-algebra, then $A \cong C(X)$ for some compact (assume $1 \in A$) Hausdorff space. A closed ideal I of A is determined by a closed subset F of X in the sense that $I = \{f \in C(x) : f(F) = 0\}$. The Riesz representation theorem identifies A^* with the space $M(X)$ of finite, regular, complex Borel measures on X. For any $\mu \in M(x)$, let μ_1 and μ_2 be the measures defined by $d_{\mu_1} = 1_{X-F}\, d_\mu$, $d_{\mu_2} = 1_F\, d_\mu$. Then, $\mu = \mu_1 + \mu_2$ is the decomposition given by Theorem 5.

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