Schrödinger Operators with Fairly Arbitrary Spectral Features

M. Krishna
V.S. Sunder

Schrödinger operators with fairly arbitrary spectral features

M. Krishna and V.S. Sunder

The Institute of Mathematical Sciences
Madras 600113
INDIA

October 2, 2001

Abstract

It is shown, using methods of inverse-spectral theory, that there exist Schrödinger operators on the line with fairly general spectral features. Thus, for instance, it follows from the main theorem, that if $0 < \alpha < 1$ is arbitrary, and if Σ is any perfect subset of $(-\infty, 0]$ with Hausdorff dimension α, then there exist potentials $q_j, j = 1, 2$ such that the associated Schrödinger operators H_j are self-adjoint and satisfy : $\sigma(H_j) = \Sigma \cup [0, \infty), \sigma_{ac}(H_j) = [0, \infty), \sigma_{pp}(H_1) = \sigma_{sc}(H_2) = \Sigma$. The main result also implies existence of states with interesting transport properties.

1   Introduction

Inverse spectral theory for Jacobi/Schrödinger operators has a long history. However almost all the effort had gone in understanding the operators with absolutely continuous spectra and recover to some extent detailed information about the potential. We refer to the monographs of Newton [26], Chadan-Sabatier [6], Marchenko [25], Levitan [24] and Carmona-Lacroix [5]. In addition the recent advances in the theory can be found in the works of Kotani [19] (who developed an inverse spectral theory for random Schrödinger operators), Kotani-Krishna [20], Craig [8] and Gesztesy-Simon [16, 17], Gesztesy-Teschl [18] for Schrödinger operators and [2, 30] for Jacobi operators. In particular the recent lectures of Simon [28] and the article of Gesztesy [14] give an overview of the recent advances.

In contrast very little is known regarding inverse spectral theory for singular spectra and in fact even the direct spectral problem for operators with singular continuous spectra is gaining attention only recently in a substantial way for example in the works of Del Rio-Makaraov-Simon[10], Del Rio-Jitomirskaya-Last-Simon[9] and Simon[29]. We refer to [28, 29, 9] for the collection of results in this direction.

This paper is motivated partly to understand the structure of inverse theories for singular spectra and partly by the work of Last [21] and Del Rio-Jitomirskaya-Last-Simon [9] where the operators with singular continuous spectra in the context of rank one perturbations is analyzed, for the geometry of spectrum, the transport properties of the physical systems corresponding to such operators.

In the case of Jacobi operators, Gesztesy-Krishna-Teschl [15] worked out the iso-spectral set for reflection less bounded Jacobi operators with the singular spectrum given by a countable set. Recently we came to know of the preprint of Last-Jitomirskaya [22] constructing explicit Jacobi matrices with α dimensional singular spectra.

The aim of this paper is to prove the following result.

THEOREM 1 *Let μ_0 be a compactly supported probability measure with supp μ_0 = $\Sigma \subset [-M, M]$, say, and such that μ_0 is singular with respect to Lebesgue measure. Then there exists a function $q : \mathbb{R} \to \mathbb{R}$ such that (i) q is continuous away from 0, and (ii) q has at most a jump discontinuity at 0, and such that the symmetric operator $(\frac{-d^2}{dx^2} + q(x))$ (defined on the space*

2

$C_0^\infty(I\!R)$ of compactly supported smooth functions) is essentially self-adjoint, and its unique self-adjoint extension H in $L^2(I\!R)$ has the following spectral properties:

(a) $\sigma(H) = \Sigma \cup [0,\infty)$; and

(b) there exists a vector v in the domain of H such that the associated 'spectral measure' (defined by $\nu(E) = \langle 1_E(H)v, v \rangle$) is mutually absolutely continuous with the measure $\mu_0 + m$, where m denotes - here and throughout this paper - the measure defined by $m(E) = \int_{E \cap [0,\infty)} \frac{1}{2\pi\sqrt{\lambda}} \, d\lambda$).

Further, if it is the case that $\Sigma \subset (-\infty, 0]$, then, $\sigma_s(H) = \Sigma$, where $\sigma_s(H)$ denotes the singular spectrum of H.

$\square$

The proof of the theorem relies on inverse-spectral theory *à la* Gelfand-Levitan.

The paper is broken up into two sections, as follows. The first section is devoted to gathering together (for the convenience of the reader) various facts - mostly known, and an occasional fact which might, perhaps, not occur in existing literature in quite the form we have stated - concerning (i) Herglotz functions, (ii) symmetric operators, (iii) Schrödinger operators, and (iv) the Gelfand-Levitan theory. The second section contains the proof of the theorem, and concludes with a discussion of some examples which justify the assertions in the abstract concerning perfect sets of arbitrary Hausdorff measure.

In the course of the proof, we also find a curious criterion for the essential self-adjointness (on $C_c^\infty(I\!R)$) of the Schrödinger operator on the line - see Proposition 13.

We make a few remarks to justify the assertions made in the abstract.

REMARK 2 *If Σ is any compact set, then there exists an atomic measure μ - i.e., $\mu = \mu_{pp}$ - such that $\operatorname{supp} \mu = \Sigma$. This shows that if we had chosen Σ to be a subset of $(-\infty, 0]$, then, by Theorem 1, there does indeed exist a potential q on $I\!R$ such that the associated Schrödinger operator H satisfies $\sigma(H) = \Sigma \cup [0,\infty)$, and $\sigma_{pp}(H) = \Sigma$.*

REMARK 3 *If Σ is a perfect set - i.e., Σ is compact and has no isolated points - then there exists a singular continuous measure μ - i.e., $\mu = \mu_{sc}$*

3

- such that $\Sigma = supp\ \mu$. (Reason : if Σ has empty interior, this can be done by adapting the construction of the Cantor function, to come up with a continuous monotonic function which 'rises' only on Σ and is consequently the distribution function of the desired measure μ. If Σ has interior, we may construct several probability measures supported on perfect nowhere dense sets whose union is dense in Σ and take an appropriate weighted average of the associated countable family of probability measures.)

It follows now from Theorem 1 that if Σ is a perfect set in $(-\infty, 0]$, then we can (start with a singular continuous probability measure μ_0 with support given by Σ) find a potential q on $\mathbb{R}$ such that the associated Schrödinger operator H has $\sigma(H) = \Sigma \cup [0, \infty)$ and $\sigma_{sc}(H) = \Sigma$.

Finally, for any $0 < \alpha < 1$, there do exist - see [12] - perfect sets Σ (contained in $(-\infty, 0]$) with Hausdorff dimension exactly equal to α. This justifies the assertions in the abstract.

REMARK 4 *One final remark concerns the transport properties of the states associated with the singular spectra of the type we constructed. The results of Last [21] on the lower bounds on transport will be valid for the Shcrödinger operators we constructed with the α dimensional singular spectra, since in this case the operator $\chi_E(H)\chi_{[0,R]}(|x|)$ can be shown to be Hilbert-Schmidt (where $\chi_E(.)$ is the characteristic function of the Borel set E).*

2   Preliminaries

Throughout this paper, the word 'measure' will mean a σ-finite positive measure defined on the σ-algebra $\mathcal{B}_{\mathbb{R}}$ of Borel sets in the real line $\mathbb{R}$. For any measure, we shall write

$$\mu = \mu_{ac} + \mu_{pp} + \mu_{sc} \tag{2.1}$$

for the usual Lebesgue decomposition (where the subscripts are, of course, acronyms for 'absolutely continuous', 'pure point' and 'singular continuous', respectively). We shall also use the notation $\mu_s = \mu_{pp} + \mu_{sc}$.

Given a self-adjoint operator H in a Hlbert space $\mathcal{H}$, we shall use the symbols $B_b(H)$ (resp., $C_c(H)$) to denote the algebra given by $\{\phi(H) : \phi$ is a bounded Borel measurable function on $\mathbb{R}\}$ (resp., $\{\phi(H) : \phi$ is a continuous function on $\mathbb{R}$ which vanishes at $\infty\}$. For $x \in \mathcal{H}$, the unique measure

μ_x satisfying $\langle \phi(H)x, x \rangle = \int \phi d\mu_x \; \forall \phi(H) \in C_c(H)$ will be referred to as the 'spectral measure of H associated with the vector x'.

A subset $S \subset \mathcal{H}$ is said to be 'cyclic' for the self-adjoint operator H if the set $\{Tx : T \in C_b(H), x \in S\}$ is total in $\mathcal{H}$ - meaning that the linear subspace generated by the latter set is dense in $\mathcal{H}$. By the multiplicity of H is meant the number $min \{ \text{card } S : S$ is cyclic for $H\}$. If H is of finite multiplicity, and if S is any finite set which is cyclic for H, then any measure which is mutually equivalent to $\sum_{x \in X} \mu_x$ will be called a total spectral measure for H. In more conventional terminology, we have: μ is a total spectral measure for H if and only if the projection-valued measure $E \mapsto 1_E(H)$ and μ have the same class of null sets.

Finally, if H is as above, and if μ is a total spectral measure for H, then we write $\sigma(H)$ for the spectrum of H, and we write

$$\sigma_{ac}(H) = supp \; \mu_{ac}, \; \sigma_{pp}(H) = supp \; \mu_{pp}, \; \sigma_{sc}(H) = supp \; \mu_{sc}$$

where we write $supp \; \mu$ to denote the closed support of the measure μ.

2.1 Herglotz functions

Recall that a *Herglotz* function is, by definition, an analytic map f of the upper half-plane Π^+ into itself. For the sake of convenient reference, we state below the standard representation of such functions as well as some other simple properties, as a proposition.

PROPOSITION 5 *(a) Let μ be a positive measure (not identically equal to zero) such that $\int_{I\!R} \frac{1}{1+\lambda^2} d\mu(\lambda) < \infty$; then the equation*

$$F(z) = \int_{I\!R} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\mu(\lambda) \; ,$$

defines a Herglotz function F;

(b) Conversely if F is a Herglotz function, there exist unique real constants a, b with $b \geq 0$, and a unique positive measure μ which satisfies the condition $\int_{I\!R} \frac{1}{1+\lambda^2} d\mu(\lambda) < \infty$, such that

$$F(z) = a + bz + \int_{I\!R} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\mu(\lambda) \; ; \qquad (2.2)$$

5

in particular, $Im\ F(\lambda + i0)\ =\ \lim_{\epsilon\downarrow 0} Im\ F(\lambda + i\epsilon)$ *exists in* $[0, +\infty]$ *for all* $\lambda \in I\!\!R$;

(c) If μ *and F are as in (b) above, then, we have:*

(i) $supp\ \mu_{ac}\ =\ \{\lambda \in I\!\!R : 0 < Im\ F(\lambda + i0) < \infty\}^{-}$, *where the supserscript* $^{-}$ *denotes closure; and*

$$d\mu_{ac}\ =\ \frac{1}{\pi}\ Im\ F(\lambda + i0)\ d\lambda\ ;$$

(ii) $\mu(\{\lambda\})\ =\ \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \epsilon F(\lambda + i\epsilon)$, *and consequently* $supp\ \mu_{pp}\ =\ \{\lambda \in I\!\!R :\ \lim_{\epsilon\downarrow 0} \epsilon F(\lambda + i\epsilon) \neq 0\}^{-}$;

(iii) $supp\ \mu_{s}\ =\ \{\lambda \in I\!\!R : Im\ F(\lambda + i0)\ =\ \infty\}^{-}$; *and consequently,*

(iv) $\lambda_0 \notin supp\ \mu\ \Leftrightarrow\ Im\ F(\lambda+i0) = 0$ *for all* λ *in some neighbourhood of* λ_0.

$\square$

REMARK 6 *If F and* μ *are related as in Proposition 5 (a), then F is called the* Borel transform *of* μ. *If F and* μ *are related as in Proposition 5 (b), then we shall write* $\mu\ =\ \mu_F$.

2.2   Symmetric operators

We start with some preliminary facts about symmetric operators and their symmetric, resp., self-adjoint, extensions. For all these facts, we refer the reader to the book [11] by Dunford and Schwartz.

Suppose T is a closed symmetric operator with (dense) domain contained in the Hilbert space $\mathcal{H}$; thus, $T \subset T^*$. Let us write $\mathcal{D}\ =\ dom\ T^*$. It is then true that an extension S of T is symmetric if and only if $S \subset T^*$. Define the subspaces

$$\mathcal{D}_{\pm}\ =\ \{x \in \mathcal{D}\ :\ T^*x = \pm ix\}\ , \tag{2.3}$$

where, of course, $i\ =\ \sqrt{-1}$.

The numbers $n_{\pm}\ =\ dim\ \mathcal{D}_{\pm}$ are called the deficiency indices of the operator T.

Since T^* is a closed operator, the space $\mathcal{D}$ becomes a Hilbert space with respect to the inner-product defined by

$$(x, y)_*\ =\ \langle x, y\rangle\ +\ \langle T^*x, T^*y\rangle\ .$$

It is then a fact that, with respect to the inner-product $(\cdot,\cdot)_*$, the Hilbert space $\mathcal{D}$ admits the orthogonal decomposition $\mathcal{D} = dom\ T \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$.

A crucial fact is that T admits self-adjoint extensions if and only if $n_+ = n_-$, and that self-adjoint extensions of T are in bijective correspondence with unitary operators from $\mathcal{D}_+$ onto $\mathcal{D}_-$. More precisely, if $V : \mathcal{D}_+ \to \mathcal{D}_-$ is unitary, then the operator given by

$$S \subset T^* \ , \ dom\ S = \{x \in \mathcal{D} : (x, y - Vy)_* = 0 \ \forall \ y \in \mathcal{D}_+\} \qquad (2.4)$$

is self-adjoint, and conversely, every self-adjoint extension of T arises in this fashion.

By a *boundary value for T* is meant a bounded linear functional on the Hilbert space $\mathcal{D}$ which vanishes on the closed subspace $dom\ T$. Thus, in view of the last paragraph, a boundary value for T is just a map of the form $\mathcal{D} \ni x \mapsto (x, y)_*$, where $y \in \mathcal{D}_+ \oplus \mathcal{D}_-$.

Thus, every self-adjoint extension of T arises by restricting T^* to a subdomain which is characterised by $n = n_+ = n_-$ 'boundary conditions'. The reason for the use of the expressions 'boundary values' and 'boundary conditions' stems from the case of differential operators.

2.3 Schrödinger operators

Let τ denote the differential expression $-\frac{d^2}{dx^2} + q(x)$, where q is a real-valued function on some interval $I = (a,b)$, $-\infty \le a < b \le \infty$. (Later, we will impose more stringent conditions on q, when needed.) We shall write T_0 and T_1 on $L^2(I)$ with domains given, respectively, by

$$
\begin{aligned}
\mathcal{D}_0 &= C_c^\infty(I) & (2.5)\\
\mathcal{D} &= \{u \in L^2(I) : u, u' \text{ absolutely continuous, } \tau u \in L^2(I)\} & (2.6)
\end{aligned}
$$

Then, it is well-known - see [27] Proposition 2, Appendix to Chapter X.1, for instance - that T_0 is a symmetric operator and that $T_0^* = T_1$. We shall need to apply the preceding general analysis to the closed symmetric operator $T = T_0^-$, i.e., the closure of T_0 or equivalently the operator defined by $T \subset T_1$ where $dom\ T$ consists of those $f \in \mathcal{D}$ for which there exists a sequence $\{f_n\} \subset \mathcal{D}_0$ such that both $\{f_n\}$ and $\{\tau f_n\}$ are Cauchy sequences in $L^2(I)$.

The deficiency subspaces $\mathcal{D}_\pm$ are at most 2-dimensional, since a linear differential equation of second order can have at most two linearly independent solutions; further, since q is real. it follows that $f \in \mathcal{D}_+ \Leftrightarrow \overline{f} \in \mathcal{D}_-$, and consequently $n_+ = n_-$, whence T has self-adjoint extensions. Since $0 \leq n_\pm \leq 2$, there are three possibilities.

Case (i): $n_\pm = 0$.
In this case, it follows from $\mathcal{D}_\pm = \{0\}$ that $T = T_1$ and that consequently (T_0 is essentially self-adjoint and) $T = T_1$.

Case (ii): $n_\pm = 1$.
In this case, there exists a unique solution $u_\pm$ of the equation $\tau u = \pm iu$ which satisfies $u \in L^2(I)$.

Case (iii): $n_\pm = 2$.
In this case, every solution $u_\pm$ of the equation $\tau u = \pm iu$ satisfies $u \in L^2(I)$.

We pause to mention here - for ease of reference - the very useful Green's formula, which states, in the above notation, (and follows from twice integrating by parts) that if $f, g \in \mathcal{D}$, then

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle \;=\; \lim_{x \uparrow b,\; y \downarrow a} \{[f, \overline{g}](x) - [f, \overline{g}](y)\} \;\;, \tag{2.7}$$

(where $[f, g](x) = f(x)g'(x) - f'(x)g(x)$ denotes the Wronskian). For typographical convenience, we shall simply write $[f, g]_a^b$ to denote the expression $lim_{x \uparrow b,\; y \downarrow a}\{[f, g](x) - [f, g](y)\}$.

The following lemmas will be needed in the sequel.

LEMMA 7 *If T_0, $\mathcal{D}$, $\mathcal{D}_\pm$ are as above, then, for arbitrary $f \in \mathcal{D}$ and $u \in \mathcal{D}_+ \oplus \mathcal{D}_-$, we have*

$$(f, u)_* \;=\; [f, T^*\overline{u}]_a^b \;\;. \tag{2.8}$$

Proof: We may, without loss of generality, assume that $u \in \mathcal{D}_\pm$. Then,

$$\begin{aligned}(f, u)_* \;&=\; \langle f, u \rangle \;+\; \langle T^*f, T^*u \rangle \\ &=\; \langle f, u \rangle \;+\; \langle T^*f, \pm iu \rangle\end{aligned}$$

$$
\begin{aligned}
&= \ \langle f, u \rangle \ \mp \ i[f, \overline{u}]_a^b \ \mp \ i\langle f, T^*u \rangle \\
&= \ \langle f, u \rangle \ \mp \ i[f, \overline{u}]_a^b \ \mp \ i\langle f, \pm iu \rangle \\
&= \ \langle f, u \rangle \ \mp \ i[f, \overline{u}]_a^b \ - \ \langle f, u \rangle \\
&= \ \mp \, i[f, \overline{u}]_a^b \\
&= \ [f, T^*\overline{u}]_a^b \ ,
\end{aligned}
$$

as desired.

$\square$

LEMMA 8 *Let* $-\infty < a < b \leq \infty$. *Suppose* q *is a real-valued continuous function on* (a, b) *such that* $q \in L^2(a, a + \epsilon)$. *Let* T_0 *be the symmetric operator on* $L^2(a, b)$ *defined by* $\mathrm{dom}\, T_0 = C_c^\infty(a, b)$, $T_0u = -u'' + qu$, *and let* $\mathcal{D} = \mathrm{dom}\, T^*$ *and* $\mathcal{D}_\pm$ *be as above.*

(i) $u \in \mathcal{D} \ \Rightarrow \ \mathcal{D} \ni f \mapsto [f, u]_a^b$ *defines a continuous linear functional on* $\mathcal{D}$ *(with respect to the norm coming from* $(\cdot, \cdot)_*$ *) which vanishes on* $\mathrm{dom}\, \overline{T_0}$, *and is consequently a 'boundary value' in the sense discussed earlier (in the context of general symmetric operators).*

(ii) If $f \in \mathcal{D}$ *is arbitrary, then both* f *and* f' *have limits (from the right) at* a *and further* $\mathcal{D} \ni f \mapsto f(a)$ *and* $\mathcal{D} \ni f \mapsto f'(a)$ *(which make sense by the preceding assertion) are 'boundary values' in the sense of (i) above.*

(iii) If $u \in \mathcal{D}$, *then* $\mathcal{D} \ni f \mapsto \lim_{x \uparrow b}[f, u](x)$ *defines a boundary value. (In the sequel, we shall simply write* $[f, u](b)$ *to denote the preceding limit.)*

Proof: (i) This is an immediate consequence of Green's formula 2.7.

(ii) Suppose $\phi \in C_c^\infty(a, b)$ be such that ϕ is identically equal to 1 in a neighbourhood of a and identically equal to 0 in a neighbourhood of b. Then, clearly $\phi \in \mathcal{D}$, by the assumed hypothesis on q. If $f \in \mathcal{D}$, note that $[f, \phi]_a^b = \lim_{x \downarrow a}\{f(x)\phi'(x) - f'(x)\phi(x)\}$; since $(\phi(x), \phi'(x)) = (1, 0)$ in a neighbourhood of a, it follows from (i) $\lim_{x \downarrow a} f(x)$ does indeed exist and the limit defines a boundary value in the sense of (i).

Now let $\psi \in C_c^\infty[a, b)$ be any function such that $\psi(x) = x - a$ in a neighbourhood of a. Again, $\psi \in \mathcal{D}$, and we find that if $f \in \mathcal{D}$, then, $[f, \psi]_a^b = \lim_{x \downarrow a} f'(x)$.

(iii) This is an immediate consequence of (i) and (ii).

$\square$

LEMMA 9 *Suppose* q *is a continuous function on* $\mathbb{R}^+ = (0, \infty)$ *which is square-integrable in a neighbourhood of 0. Let* T_0 *be the symmetric operator on* $L^2(\mathbb{R}^+)$ *defined by* $dom\ T_0 = C_c^\infty(\mathbb{R}^+)$, $T_0 u = -u'' + qu$, *and let* $\mathcal{D} = dom\ T^*$ *as above. Also, as before, let* $\mathcal{D}_\pm = ker(T^* \mp i)$, *and let* $n = dim\ ker\ (T^* \pm i)$. *Suppose* H *is a self-adjoint extension of* T, *and suppose it is true that* $dom\ H \subset \{u \in \mathcal{D} : u(0) = 0\}$. *Then,*

(a) $n \neq 0$;

(b) *if* $n = 1$, *then* $dom\ H = \{u \in \mathcal{D} : u(0) = 0\}$; *and*

(c) *if* $n = 2$, *then there exists* $u_1 \in \mathcal{D}_+ \oplus \mathcal{D}_-$ *such that* $dom\ H = \{u \in \mathcal{D} : u(0) = 0, [u, u_1](\infty) = [\lim_{R \to \infty}[u, u_1](R) = 0\}$.

Proof (a) If $n = 0$, then $\overline{T_0} = T^*$ - see Case (i) just before Green's formula 2.7. This, together with the assumed self-adjointness of H and the fact that $H \subset T^*$ would imply that $H = T^*$, which contradicts our assumption about $dom\ H$.

(b) If $n = 1$, then it follows from equation 2.4 (and the remark following it) that there exists a non-zero vector $v_0 \in \mathcal{D}$ such that $dom\ H = \{f \in \mathcal{D} : (f, v_0)_* = 0\}$. On the other hand, we know from Lemma 8 (ii) (and the Riesz representation theorem applied to the Hilbert space $\mathcal{D}$) that there exists a vector $u_0 \in \mathcal{D}$ such that $(f, u_0)_* = f(0)$. Our assumption on $dom\ H$ then translates into the statememt that $f \in \mathcal{D}, (f, v_0)_* = 0 \Rightarrow (f, u_0)_* = 0$. Since u_0 is clearly non-zero, this implies that u_0 and v_0 are non-zero multiples of one another, which yields the desired conclusion.

(c) If $n = 2$, appeal to equation 2.4 to conclude that there exists a two-dimensional subspace $\mathcal{M}$ of $\mathcal{D}_+ \oplus \mathcal{D}_-$ such that $dom\ H = \{f \in \mathcal{D} : (f, u)_* = 0\ \forall\ u \in \mathcal{M}\}$. The assumption on $dom\ H$ implies that $u_0 \in \mathcal{M}$, where u_0 is as in the proof of (b) above. Pick $v_1 \in \mathcal{M}$ such that $(v_1, u_0)_* = 0$, $\|v_1\|_* = 1$. Then $v_1(0) = (v_1, v_0)_* = 0$, and $dom\ H = \{f \in \mathcal{D} : f(0) = (f, v_1)_* = 0\}$. Now, appeal to equation 2.8, set $u_1 = T^*\overline{v_1}$, and conclude that $f \in dom\ H \Rightarrow (f, v_1)_* = [f, u_1]_0^\infty = [f, u_1](\infty)$, where we have used the fact that $u_1(0) = 0$ as well as the fact - see Lemma 8(ii) - that f and f' have finite right-limits at 0, to conclude that $[f, u_1](0) = 0$. □

In this paper, we will have to deal with the Schrödinger operator $H =$

$-\frac{d^2}{dx^2} + q$, where the potential q has the form

$$q(x) = \begin{cases} q^+(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ q^-(-x) & \text{if } x < 0 \end{cases} \qquad (2.9)$$

where $q^\pm$ are continuous functions on $(0, \infty)$ with finite limits at zero - i.e., the limits $q^\pm(0+) = \lim_{x \downarrow 0} q^\pm(x)$ exist and are finite. Throughout this section, this is the potential we shall be working with.

We shall encounter the following situation; if we write T_0, (resp., $T_0^\pm$) for the operator on $L^2(\mathbb{R})$ (resp., $L^2(\mathbb{R}^\pm)$) with domain given by $C_c^\infty(\mathbb{R})$ (resp., $C_c^\infty(\mathbb{R}^\pm)$ and defined by the differential operator given by $\tau = -\frac{d^2}{dx^2} + q(x)$, it will be the case that both operators $T_0^\pm$ will have deficiency indices (1,1), and will give rise to self-adjoint operators $H_\pm$ on $L^2(\mathbb{R}^\pm)$ given by $H_\pm \subset T_\pm^*$ and

$$\mathcal{D}(H_\pm) = \{u \in L^2(\mathbb{R}^\pm) : -u'' + qu \in L^2(\mathbb{R}^\pm), u(0) = 0\} \ . \qquad (2.10)$$

It will, further be the case that the Schrödinger operator we shall be interested in will be given by $H = T_0^*$; i.e.,

$$\mathcal{D}(H) = \{u \in L^2(\mathbb{R}) : -u'' + qu \in L^2(\mathbb{R})\} \ . \qquad (2.11)$$

Temporarily fix $z \in \mathbb{C}$ with $Im\ z \neq 0$ and consider the differential equation

$$-u''(x) + q(x)u(x) = zu(x) \ . \qquad (2.12)$$

This second-order differential equation has a two-dimensional linear space of solutions; let $\phi(\cdot, z), \psi(\cdot, z)$ denote the unique solutions satisfying $\phi(0, z) = \psi'(0, z) = 0$, $\phi'(0, z) = \psi(0, z) = 1$, where we write $\phi'(0, z)$ to denote $\frac{\partial \phi}{\partial x}(0, z)$.

The fact that $T_0^\pm$ have deficiency indices (1,1) implies that there exist unique scalars $m_\pm(z)$ such that if

$$u^\pm(x, z) = \psi(x, z) \pm m_\pm(z)\phi(x, z) \ , \qquad (2.13)$$

then $u^\pm(\cdot, z) \in L^2(\mathbb{R}^\pm)$.

Recall that the Green's function for H is given by

$$g(z; x, y) = \frac{1}{[u^+(x, z), u^-(x, z)]} \begin{cases} u^-(x, z)u^+(y, z) & \text{if } x \leq y \\ u^-(y, z)u^+(x, z) & \text{if } x \geq y \end{cases} \qquad (2.14)$$

11

It is a fact that the Weyl m-functions $m_\pm(z)$, when restricted to Π^+, are Herglotz functions, and that they are related to the Green's function by

$$g(z; 0, 0) \;=\; \frac{-1}{m_+(z) + m_-(z)} \qquad (2.15)$$

and

$$\lim_{0<x<y\downarrow 0} \frac{\partial^2}{\partial x \partial y} g(z; x, y) \;=\; \frac{m^+(z) m_-(z)}{m_+(z) + m_-(z)} \quad . \qquad (2.16)$$

Further, it is true that half-space Green's functions - which are given by

$$g_\pm(z; x, y) \;=\; \frac{\pm 1}{[u^\pm(x, z), \phi(x, z)]} \begin{cases} \phi(x, z) u^\pm(y, z) & if \;\; \pm x \le \pm y \\ \phi(y, z) u^\pm(x, z) & if \;\; \pm x \ge \pm y \end{cases}$$

- are related to the Weyl m-functions by

$$\lim_{0<x<y\downarrow 0} \frac{\partial^2}{\partial x \partial y} g^\pm(z; x, y) \;=\; m^\pm(z) \quad . \qquad (2.17)$$

The following known fact - which we single out as a proposition, for convenience of reference - is a consequence of equation 2.17.

PROPOSITION 10 *With the foregoing notation, define*

$$F \;=\; \frac{-1}{m^+ + m^-} \;\;,\;\; G \;=\; \frac{m^+ m^-}{m^+ + m^-} \;\;; \qquad (2.18)$$

then

(i) F and G are Herglotz functions;

(ii) there exists vectors $x_\pm \in L^2(\mathbb{R}^+)$ such that the singleton set $\{x_\pm\}$ is cyclic for $H^\pm$, and the associated spectral measure for $H^\pm$ (which, in view of the cyclicity of $x_\pm$, is a total spectral measure for $H^\pm$) is equivalent to the measure given by $\mu^\pm \;=\; \mu_{m_\pm}$; and

(iii) there exists vectors $x_1, x_2 \in L^2(\mathbb{R})$ such that the two-element set $\{x_1, x_2\}$ is cyclic for H, and the associated spectral measures are equivalent to the measures given by $\mu_1 \;=\; \mu_{x_1} \;=\; \mu_F$ and $\mu_2 \;=\; \mu_{x_2} \;=\; \mu_G$; and consequently $\mu_1 + \mu_2$ is a total spectral measure for H.

Sketch of Proof: (i) This is because $m^{\pm}$ are Herglotz functions.

(ii) In view of our stated conditions on $H_{\pm}$, it follows that what we have called $T_0^{\pm}$ is 'in the limit-point case at $\pm\infty$'; the desideratum follows from [7] Chapter 9, Theorem 3.1 and Chapter 9, Problems 12 and 14. (The roles of ϕ and ψ in our discussion and in the discussion in [5] are reversed because of the boundary condition at 0 that we work with.)

(iii) See [5] (top of page 108).

$\square$

2.4 Gelfand-Levitan theory

The Gelfand-Levitan theory starts with appropriate measures μ and proves the existence of a Schrödinger operator on $L^2(I\!R^+)$ with continuous potential, and with total spectral measure given by μ. We briefly outline the facts of this theory that we shall need. The conditions on the measure vary according to the boundary conditions imposed in the domain of the Schrödinger operator.

We present only the case of the Dirichlet boundary condition at 0.

The result we shall use - see [13] and [23] - is the following - which we state much more elaborately than in the original, in order to facilitate later reference.

THEOREM 11 (**Gelfand-Levitan**) *Let ν be the measure with distribution function given by*

$$\nu(-\infty, x]) \;=\; \left\{ \begin{array}{ll} 0 & \text{if } x \leq 0 \\ \frac{2}{3\pi} x^{\frac{3}{2}} & \text{if } x > 0 \end{array} \right.$$

Suppose μ is a measure satisfying the following conditions:

(i) $\int_{-\infty}^{0} e^{a\sqrt{-\lambda}} d\mu(\lambda) \;<\; \infty \;\; \forall\, a > 0$;

(ii) put $\sigma \;=\; \mu - \nu$; suppose the function defined by

$$a(x) \;=\; \int_{1}^{\infty} \frac{\cos(\sqrt{\lambda}x)}{\lambda^2} \, d\sigma(\lambda) \tag{2.19}$$

(where the integral is interpreted as an improper integral) satisfies $a \in C^4(0, \infty)$.

Then,

(a) Define

$$f(x, y) \;=\; \int_{\mathbb{R}} \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\frac{\sin\sqrt{\lambda}y}{\sqrt{\lambda}} \; d\sigma(\lambda) \;,\qquad (2.20)$$

where the integral is again interpreted as an improper integral; then $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$, and the integral equation

$$K(x, y) \;+\; f(x, y) \;+\; \int_0^x K(x, s)f(y, s)ds \;=\; 0 \;, 0 < y \le x \qquad (2.21)$$

has a unique solution $K(x, y)$; further, $K \in C^1(\{(x, y) \;:\; 0 \le y \le x\}$.

(b) Define $q(x) \;=\; 2\frac{d}{dx}K(x, x)$, and

$$\phi(x, \lambda) \;=\; \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \;+\; \int_0^x K(x, y)\frac{\sin\sqrt{\lambda}y}{\sqrt{\lambda}} \; dy \;\;, x > 0, \lambda \in \mathbb{R} \;; \qquad (2.22)$$

then ϕ satisfies the eigenvalue equation 2.12 for $x > 0$, with the boundary conditions $\phi(0, \lambda) = 0$, $\phi'(0, \lambda) = 1$.

(c) Finally, the equation

$$(Ug)(\lambda) \;=\; \int_0^\infty g(x)\phi(x, \lambda) \; dx \qquad (2.23)$$

(with the right-hand side interpreted as a limit, as $R \to \infty$, in $L^2(\mu)$ of the functions defined by $(U_R g)(\lambda) \;=\; \int_0^R g(x)\phi(x, \lambda) \; dx$) defines a unitary operator $U : L^2(\mathbb{R}^+, dx) \;\rightarrow\; L^2(\mathbb{R}, \mu)$.

The inverse (or adjoint) of the operator U is given by

$$(U^*g)(x) \;=\; \int_{\mathbb{R}} g(\lambda)\phi(x, \lambda)d\mu(\lambda) \qquad (2.24)$$

where this equation is also interpreted in the improper L^2-sense as was equation 2.23

Before proceeding to discuss consequences of this theorem, we wish to point out that the version of their theorem which we have stated here corresponds to what they call the case $h = \infty$, and that although they do not

14

quite state their theorem in this fashion, the proof of their theorem establishes this version.

We state the consequences of this theorem that we shall need as a proposition.

PROPOSITION 12 *Suppose μ is a measure satisfying conditions (i) and (ii) of Theorem 11. Suppose, also, that the following conditions are satisfied:*

(i) μ is not purely atomic;

(ii) $\int_{\mathbb{R}} \frac{1}{(1+|\lambda|)^2} \, d\mu < \infty$;

(iii) supp $\mu \subset [c, \infty)$, for some $c > -\infty$; and

(iv) the function K given by equation 2.21 is bounded in $\{(x, y) : 0 \leq y \leq x \leq 1\}$;

(v) the potential q constructed as in Theorem 11(b) has a finite right-limit at 0.

Then, the operator T on $L^2(\mathbb{R}^+)$ defined by dom $T = \{u \in L^2(\mathbb{R}^+) : u, u'$ absolutely continuous, $\tau u \in L^2(\mathbb{R}^+)$, $u(0) = 0\}$, $(Tu) = \tau u$ (where $\tau u = (-\frac{d^2}{dx^2} + q) \, u$) is self-adjoint and μ is a total spectral measure for T. (In fact, τ has deficiency indices (1,1) and is in the limit-point case at ∞.)

Proof: We break the proof into a sequence of steps.

Step 1: Let T_0 be the operator given by $T_0 f = \tau f$, dom $T_0 = C_c^\infty(\mathbb{R}^+)$. If U is as in Theorem 11(c), then

$$(UT_0U^*f)(\lambda) = \lambda f(\lambda) \ for \ \mu \ a.a.\lambda \ . \qquad (2.25)$$

This follows from the definition of U and the properties of the function ϕ stated in Theorem 11(b).

Step 2: Let $T = U^*M_\lambda U$, where M_λ denotes the (self-adjoint) operator of multiplication by the independent variable. Then,

$$dom \ T \ \subset \ \{f \in \ dom \ T_0^* \ : \ \lim_{x \downarrow 0} \ f(x) \ = \ 0\}.$$

Reason: With c as in assumption (iii) of the proposition, it is a consequence of assumption (iv) and equation 2.22 that

$$\sup_{0 \le x \le 1, c \le \lambda < \infty} |\phi(x, \lambda)| < \infty . \tag{2.26}$$

Also, note that assumption (iv) shows that $\phi(x, \lambda) \to 0$ as $x \to 0$, for all $\lambda \in I\!\!R$. An application of equation 2.24 (and the Cauchy-Schwarz inequality and bounded convergence theorem) shows that

$$
\begin{aligned}
f \in \ dom \ T \ &\Rightarrow \ g \ = \ Uf \ \in \ dom \ M_\lambda \\
&\Rightarrow \ \int_c^\infty (1 + |\lambda|)^2 |g(\lambda)|^2 \ d\mu(\lambda) \ < \ \infty \\
&\Rightarrow \ |f(x)| \le (\int (1 + |\lambda|)^2 |g(\lambda)|^2 \ d\mu(\lambda) \ )^{\frac{1}{2}} (\int (1 + |\lambda|)^{-2} |\phi(x, \lambda)|^2 \ d\mu(\lambda) \ )^{\frac{1}{2}}
\end{aligned}
$$

which establishes the desired limit assertion, since the first integral is finite, and the second integral converges to 0.

Step 3: τ has deficiency indices (1,1) and is in the limit-point case at ∞.

Reason: In view of Lemma 9 - which is applicable because continuity of q is guaranteed by Theorem 11, and because of assumption (v) and the already established Step 2 above - we only need to show that τ does not have deficiency indices (2,2). However, [31] Theorem 10.19 states that if τ had deficiency indices (2,2), then the operator T should have only point spectrum, and this is ruled out by assumption (i). The assumption (v) guarantees that τ is in the limit-circle case at 0, and hence the fact that the deficiency indices of τ are (1,1) imply that τ is necessarily in the limit-point case at ∞.

Step 4: $dom \ T \ = \ \{f \in \ dom \ T_0^* \ : \ \lim_{x \downarrow 0} \ f(x) \ = \ 0\}$.

This follows from Lemma 9 and the proof of the proposition is complete.
$\square$

Befor concluding this section, we would like to point out the following interesting criterion for essential self-adjointness of the one-dimensional Schrödinger operator, which can be proved as in the proof of Step 3 above.

16

PROPOSITION 13 *Let $q \in C(\mathbb{R})$ be a continuous potential. Suppose the differential operators T_0, T_0^+ and T_- defined by $u \mapsto -u'' + qu$, with domains $C_c^\infty(\mathbb{R}), C_c^\infty((-\infty, a))$ and $C_c^\infty((a, \infty))$ have self-adjoint extensions H, H_+, and H_- respectively. Assume that neither H_+ nor H_- has pure point spectrum. Then T_0 is essentially self-adjoint.*

3   Proof of Theorem 1

Before going into details, we quickly outline the structure of the proof.

With μ_0, m as in the statement of Theorem 1, consider the function

$$F(z) \;=\; \frac{-1}{2\sqrt{-z}} \;+\; \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d(\mu_0)(\lambda) \;,$$

(where we have chosen that branch of the square root which is positive on the negative axis), and note that $\mu_F = m + \mu_0$. Define (the Herglotz function) $f(z) = \frac{-1}{F(z)}$. Then we have, by Theorem 5(b),

$$f(z) \;=\; a + bz + \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\mu_f(\lambda) \;, \qquad (3.27)$$

for some uniquely determined constants a, b. (We have used the notational convention described in Remark 6, and we shall continue, in the sequel, to use this convention.)

It is easy to see that $(\mu_f)_{ac} \cong m$; define $\mu_+ = \frac{1}{2}(\mu_f)_{ac}$, $\mu_- = \mu_f - \mu_+$ and note that $\mu_\pm$ are both non-zero positive measures and satisfy $(\mu_+)_{ac} = (\mu_+)_{ac} \cong m$. Now define

$$m_+ \;=\; \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\mu_+(\lambda) \qquad (3.28)$$

$$m_- \;=\; a + bz + \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\mu_-(\lambda) \qquad (3.29)$$

and note that $b = 0$ and $m_\pm$ are Herglotz functions such that $f = m_+ + m_-$.

We then verify that the measures $\mu_\pm$ satisfy the assumptions of Proposition 12, and deduce (from that proposition) the existence of potentials $q_\pm$ on $\mathbb{R}^+$ such that the associated Schrödinger operators with Dirichlet boundary

condition at 0 are self-adjoint (and that $q_\pm$ are in the limit-point case at ∞.)

We finally put these two potentials together to construct a potential q on $\mathbb{R}$ as in equation 2.9 and then verify that the associated Schrödinger operator H satisfies the conclusions of the theorem.

Step 0: We first verify that the measures $\mu_\pm$ satisfy conditions (i) and (ii) of Theorem 11.

We only verify condition (ii) since condition (i) of Theorem 11 is easily seen to follow from Condition (iii) of Proposition 12, which we will verify later in Step 3.

We first assert that there exist (real) constants A, B and a function $e(\lambda)$, and a constant $R > 0$ such that

$$Im\ f(\lambda + i0)\ =\ 2\lambda^{\frac{1}{2}}\ +\ A\lambda^{\frac{-1}{2}}\ +\ e(\lambda)\ ,\ \forall\ \lambda > R \tag{3.30}$$

and

$$|e(\lambda)|\ \leq\ B\lambda^{\frac{-3}{2}}\ ,\ \forall\ \lambda > R\ . \tag{3.31}$$

For this, we first define

$$C_0(\lambda)\ =\ \int_{\mathbb{R}} \frac{1}{x - \lambda}\ d\mu_0(x)\ \ \forall\ \lambda > K \tag{3.32}$$

and note that since $supp\ \mu_0\ \subset\ [-M, M]$, it follows that $C_0(\lambda)$ is real, where defined. Note further that

$$C_0(\lambda)\ =\ -\lambda^{-1}\ -\ (\int_{\mathbb{R}} x d\mu_0(x))\ \lambda^{-2}\ -\ (\int_{\mathbb{R}} x d\mu_0(x))^2\ \lambda^{-3}\ -\ e_1(\lambda)\ , \tag{3.33}$$

which implies that we can find a sufficiently large $R > M$ such that the following inequalities are valid for any $\lambda > R$:

$$2\lambda^{\frac{1}{2}}|C_0(\lambda)|\ <\ \frac{1}{2}\ ,\qquad |C_0(\lambda)|\ <\ \tfrac{2}{\lambda}\ , \tag{3.34}$$

$$(C_0(\lambda))^2\ =\ \lambda^{-2}\ +\ A_1\lambda^{-3}\ +\ O(\lambda^{-4})\ \ |e_1(\lambda)|\ \leq\ B_1\lambda^{-4}\ , \tag{3.35}$$

for some constants A_1, B_1.

18

Note next that since $(\mu_f)_{ac} \cong (\mu_F)_{ac} = m$, it follows that $Im\ f(\lambda+i0)$ is finite and strictly positive on (M, ∞). Now, for $\lambda > R$, compute thus, using the geometric series:

$$f(\lambda + i0) = 2\sqrt{-\lambda} \left(1 - 2\sqrt{-\lambda} \int_{\mathbb{R}} \frac{1}{x - \lambda}\ d\mu_0(x)\ \right)^{-1}$$

$$= \sum_{n=0}^{4} (2\sqrt{-\lambda})^{n+1} C_0(\lambda)^n\ +\ k(\lambda)\ , \qquad (3.36)$$

where $k(\lambda) = (2\sqrt{-\lambda})^6 C_0(\lambda)^5 \left(1 - 2\sqrt{-\lambda} C_0(\lambda)\right)^{-1}$.

The desired asymptotic assertion 3.30 follows upon taking imaginary parts in equation 3.36 and appealing to equation 3.33 and the estimates 3.31.

Since $(\mu_\pm)_{ac} = \frac{1}{2}(\mu_f)_{ac}$, it follows that $(\mu_\pm)_{ac}$ has density $\frac{1}{2\pi} Im\ f(\lambda + i0)$; we now define

$$d\sigma_\pm(\lambda) = d\mu_\pm(\lambda) - \frac{1}{\pi}\chi_{(0,\infty)}(\lambda)\sqrt{\lambda}d\lambda. \qquad (3.37)$$

Then, we write

$$a(x) = \int_1^R \frac{\cos\sqrt{\lambda}x}{\lambda^2}d\sigma_\pm(\lambda)\ and\ b(x) = \int_R^\infty \frac{\cos\sqrt{\lambda}x}{\lambda^2}d\sigma_\pm(\lambda), \qquad (3.38)$$

with R chosen as in equation 3.30, so that

$$\int_1^\infty \frac{\cos\sqrt{\lambda}x}{\lambda^2}d\sigma_\pm(\lambda)\ =\ a(x) + b(x).$$

It is clear that $a(x) \in C^\infty(0, \infty)$ so that to verify condition (ii) of Theorem 11 we need to only verify that $b(x)$ has the required smoothness properties. We note next that in the region (R, ∞), the (signed) measures $\sigma_\pm$ are absolutely continuous and have densities given by

$$2\pi d\sigma_\pm(\lambda)\ =\ \frac{A}{\sqrt{\lambda}}d\lambda\ + e(\lambda)d\lambda \qquad (3.39)$$

which follows from the definition of $\sigma_\pm$ and the equation 3.30. We now write b(x) as a sum of two parts $\frac{1}{2\pi}(b_1(x) + b_2(x))$, where

$$b_1(x)\ =\ A \int_R^\infty \frac{\cos\sqrt{\lambda}x}{\lambda^2}\frac{1}{\sqrt{\lambda}}d\lambda\ and\ b_2(x)\ =\ \int_R^\infty \frac{\cos\sqrt{\lambda}x}{\lambda^2}e(\lambda)d\lambda.$$

19

Then it is clear from the estimate 3.31 on $e(\lambda)$ that $b_2(x)$ satisfies condition (ii) of theorem 11. Finally the fact that $b_1(x)$ verifies condition (ii) of theorem 11 follows from an explicit computation where we first change variables to $k = \sqrt{\lambda}$ and then do integration by parts twice to get, for $x > 0$:

$$b_1(x) \;=\; 2A \left\{ \frac{-\sin \sqrt{R}x}{xR^2} \;+\; \frac{5 \cos \sqrt{R}x}{x^2 \sqrt{R^5}} \;-\; \frac{30}{x^2} \int_{\sqrt{R}}^{\infty} \frac{\cos kx}{k^6} dk \right\}.$$

It is now clear that $b_2(x)$ satisfies the conditions (ii) of theorem 11 for $x > 0$. We now verify the other conditions of the proposition 12 in the following steps.

Step 1: Since $(\mu_{\pm})_{ac} \cong m$, it is clear that $\mu_{\pm}$ satisfy condition (i) of Proposition 12.

Step 2: Since $m_{\pm}$ are Herglotz functions, it follows from Proposition 5(b) that the measures $\mu_{\pm}$ satisfy condition (ii) of Proposition 12.

Step 3: Since the measure μ_0 is supported in $[-M, M]$, note from the definition of F that F is analytic on $\mathbb{C} - [-M, \infty)$, real on $(-\infty, -M]$ and $lim_{\lambda \to -\infty} F(\lambda) = 0$. Also, it is clear by differentiating the expression defining F that the real function $F(x)$ is a strictly increasing function for $x < -M$. Hence $F(x) \neq 0 \ \forall \ x < -M$. Then, a simple computation shows that for $x < -M, \lim_{\epsilon \downarrow 0} Im \ f(x + i\epsilon) = 0$, and consequently, $supp \ \mu_f \subset [-M, \infty)$, by Proposition 5(iv). Since $\mu_{\pm}$ are absolutely continuous with respect to μ_f, it follows that $supp \ \mu_{\pm} \subset [-M, \infty)$.

Step 4: Condition (iv) of Proposition 12 is a direct consequence of the continuity of the kernel $K(x, y)$ - see Theorem 11(a).

Step 5: The smoothness properties of the functions f, K, - stated in Theorem 11 - the definition of $q_{\pm}$ in terms of the kernel - see Theorem 11(b) - and the integral equation 2.21 show (after a routine differentiation with respect to x of the equation obtained by setting $y = x$ in 2.21) that indeed the kernels $q_{\pm}$ have finite right-limits at 0.

To complete the proof, we note that Steps 0-5 permit us to conclude from Proposition 12 that there exist potentials $q_{\pm}$ on $I\!\!R^+$ such that the

associated Schrödinger operators with Dirichlet boundary condition at 0 are self-adjoint, and such that $q_\pm$ are in the limit-point case at ∞. This means that the potential q defined by equation 2.9 is in the limit-point case at both $+\infty$ and $-\infty$. We may now appeal to [27] Theorem X.7 (in Appendix to X.1) to deduce that the differential operator $-\frac{d^2}{dx^2} + q$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$.

In view of Proposition 10 (iii) and the discussion leading up to that proposition, the main theorem will be finally proved once we establish that $supp\ \mu_G \subset supp\ \mu_F = supp\ (\mu_0 + m)$, where of course F, G are given as in Proposition 10, with $m_\pm$ having the meanings attributed to them in this section. To do this, (since $supp\ \mu_F \supset [0, \infty)$,) it suffices to establish the following:

(i) if $(a, b) \subset ([-M, 0] - supp\ \mu_F)$, then $\lim_{\epsilon \downarrow 0} Im\ G(x + i\epsilon) = 0, \forall\ x \in (a, b)$;

(ii) if $(a, b) \subset ([0, M] - supp\ \mu_0)$, then $\infty > \lim_{\downarrow 0} Im\ G(x + i\epsilon) > 0, \forall\ x \in (a, b)$; and

(iii) therefore $supp\ (\mu_G)_s \subset \Sigma$.

For (i), note that F is real and strictly increasing in (a, b) and consequently has at most one 0 - say x_0 - in (a, b).

Suppose $a < x \neq x_0 < b$; then $f = \frac{-1}{F}$ is analytic in a neighbourhood of x and $f(x)$ is real; on the other hand, since $supp\ \mu_+ \subset [0, \infty)$, also m_+ is analytic in a neighbourhood of x and $m_+(x)$ is real. This implies that m_- inherits these properties from f and m_+. Hence, so also does $G = -m_+ m_- f$.

Now, consider x_0 - assuming it exists. Note that f has a simple pole at x_0, while m_+ is still analytic in a neighbourhood of x_0 and $m_+(x_0) \in \mathbb{R}$. Since $G = \frac{-m_+}{(\frac{m_+}{m_-})+1}$, it follows that $\lim_{\epsilon \downarrow 0} G(x_0 + i\epsilon) = -m_+(x_0) \in \mathbb{R}$, thus completing the proof of (i) above.

As for (ii), we again see that F has finite positive imaginary part and finite real part in (a, b), since $supp\ \mu_0$ does not intersect (a, b) by assumption. This implies that the imaginary parts of f and $m_\pm$ are positive and finite in $(a.b)$, while the real parts of f, $m_\pm$ are finite in (a, b). This immediately implies that the imaginary part of G is finite in (a, b) since $G = -m_+ m_- f$. This shows that the singular part of μ_G is not supported in (a, b).

References

[1] N. Aronszajn and W. Donoghue, *On the exponential representation of analytic functions in the upper half-plane with positive imaginary part*, J. Analyse Mathematique **5**, 321-388 (1956-57).

[2] A. J. Antony and M. Krishna, *Inverse spectral theory for Jacobi matrices and their almost periodicity*, Proc. Ind. Acad. Sci (Math. Sci.) **104**, 777-818 (1994).

[3] J. M Berezanskii, *Expansions in Eigenfunctions of Self-Adjoint Operators*, Transl. Math. Monographs, Vol. 17, Amer. Math. Soc., Providence, R. I., 1968.

[4] R. Carmona and S. Kotani, *Inverse spectral theory for random Jacobi matrices*, J. Stat. Phys. **46**, 1091-1114 (1989).

[5] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston, 1990.

[6] K. Chadan and P. C. Sabatier, *Inverse scattering problems in Quantum Scattering theory*, Springer-Verlag, New York, 1977.

[7] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Mc-Graw Hill, New York, 1955.

[8] W. Craig, *Trace formulas for Schrödinger operators*, Commun. Math. Phys **126**, 379-407 (1989).

[9] R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations and localization*, Caltech Preprint, 1995.

[10] R. del Rio, N. Makarov and B. Simon, *Operators with singular continuous spectrum II, Rank one perturbations*, to appear in Commun. Math. Phys.

[11] N. Dunford and J.T. Schwartz, *Linear Operators, Part II: Spectral Theorey, Self-adjoint operators in Hilbert space*, Interscience Publ., New York, 1963.

[12] K. J. Falconer, *The geometry of Fractal sets*, Cambridge tracts in Mathematics no **85**, Cambridge University Press, 1985.

[13] S. Gelfand and B.M. Levitan, *On the determination of a differential equation from its spectral function*, Transl., II.Ser., Am.Math.Soc. 1 , 253-304(1955).

[14] F. Gesztesy, *New trace formulas for Schrödinger operators*, Evolution equations, G. Ferreyra, G. Ruiz Goldstein, F. Neubrander (eds.), Marcel Dekker, New York, 201-221, 1995.

[15] F. Gesztesy, M. Krishna and G. Teschl, *On Isospectral sets of Jacobi matrices*, Preprint, 1995.

[16] F. Gesztesy and B. Simon, *The ξ function*, preprint 1994.

[17] F. Gesztesy and B. Simon, *Uniqueness theorems in Inverse spectral theory for one-dimensional Schrödinger operators*, Caltech preprint 1995.

[18] F. Gesztesy and G. Teschl, *On the double commutation method*, Proc. Amer. Math. Soc. (to appear).

[19] S. Kotani, *One dimensional random Schrödinger operators and Herglotz functions*, Proc. Taniguchi Symp,(ed.) S.A. Katata , 443-452 (1985).

[20] S. Kotani and M. Krishna, *Almost periodicity of some random potentials*, J. Func. Anal, **78**, 390-405 (1989).

[21] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, Caltech Preprint 1995.

[22] Y. Last and S. Jitomirskaya, *Dimensional Hausdorff properties of singular continuous spectra*, Erwin Schrödinger Institute Preprint, No 271, 1995.

[23] B. M. Levitan and M. Gasymov, *Determination of a differential equation by two of its spectra*, Russ. Math. Surveys **19**, 1-63 (1964).

[24] B.M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.

[25] V. A. Marchenko, *Strum-Liouville operators and applications* , Operator theory advances and applications, **22**, Birkhäuser, 1986.

[26] R. G. Newton, *Inverse Schrödinger scattering in three dimensions*, Springer-Verlag, New York 1989.

[27] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II : Fourier Analysis, Self-Adjointness*, Academic Press, 1975.

[28] B. Simon, *Spectral analysis of rank one perturbations and applications*, Mathematical Quantum theory II: Schrödinger operators, (eds.,) J. Feldman, R. Froese and L.M. Rosen , CRM Proceedings, American Mathematical Society, Providence, RI (1995).

[29] B. Simon, *Operators with singular continuous spectrum, I: General operators*, Ann. Math,**141**, 131-145(1995).

[30] M. Sodin and P. Yuditskii, *Infinite dimensional Jacobi inversion problem, almost periodic Jacobi matrices with homogeneous spectrum and Hardy classes of character automorphic functions*, preprint (1994).

[31] M.H. Stone, *Linear Transformations in Hilbert space and their Applications to Analysis*, Amer. Math. Soc. Colloquium Publications, Volume XV, New York, 1932.