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STOCHASTIC INTEGRATION IN FOCK SPACE

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In this paper, using purely Hilbert space-theoretic methods, an analogue of the Itô integral is constructed in the symmetric Fock space of a direct integral $\mathfrak{H}$ of Hilbert spaces over the real line. The classical Itô integral is the special case when $\mathfrak{H} = L^2[0, \infty)$. An explicit formula is obtained for the projection onto the space of 'non-anticipating functionals', which is then used to prove that simple non-anticipating functionals are dense in the space of all non-anticipating functionals. After defining the analogue of the Itô integral, its isometric nature is established. Finally, the range of this 'integral' is identified; this last result is essentially the Kunita-Watanabe theorem on square-integrable martingales.

Preliminaries. (a) *Symmetric Fock space*: If $\mathfrak{H}$ is a (complex) Hilbert space, the symbol $\mathfrak{H}^{(s)n}$ will denote the Hilbert space of symmetric tensors of rank n; alternatively, $\mathfrak{H}^{(s)n}$ is the closed subspace of $\otimes^n \mathfrak{H}$ spanned by $\{x \otimes \cdots \otimes x \colon x \in \mathfrak{H}\}$. (In the sequel, the symbol $\operatorname{sp} S$ will denote the closed subspace spanned by the set S of vectors.) By convention, $\mathfrak{H}^{(s)0} = \mathbf{C}$. We shall also write $\otimes^n x$ for $x \otimes \cdots \otimes x$, with the convention that $\otimes^0 x = 1$.

The symmetric Fock space over $\mathfrak{H}$, is by definition, the Hilbert direct sum

$$\Gamma(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{(s)n}.$$

If $x \in \mathfrak{H}$, then $\Gamma(x)$ will denote the 'exponential' vector in $\Gamma(\mathfrak{H})$ defined by

$$\Gamma(x) = \left(1, x, \frac{\otimes^2 x}{\sqrt{2!}}, \ldots, \frac{\otimes^n x}{\sqrt{n!}}, \ldots\right).$$

The following are easily verified:

$$\text{(1)}\quad \begin{aligned} &\text{(i)} && \Gamma(\mathfrak{H}) = \operatorname{sp}\{\Gamma(x) \colon x \in \mathfrak{H}\}; \\ &\text{and} && \\ &\text{(ii)} && \langle \Gamma(x), \Gamma(y) \rangle = \exp\langle x, y \rangle, \qquad x, y \in \mathfrak{H}. \end{aligned}$$

The symbol Ω is reserved for the 'vacuum' vector: $\Omega = \Gamma(0) = (1, 0, 0, \ldots)$.

If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces, it follows from (1) that the correspondence

$$\Gamma((x_1, x_2)) \leftrightarrow \Gamma(x_1) \otimes \Gamma(x_2)$$

extends to a canonical unitary isomorphism of Hilbert spaces:

$$\Gamma(\mathfrak{H}_1 \oplus \mathfrak{H}_2) \cong \Gamma(\mathfrak{H}_1) \otimes \Gamma(\mathfrak{H}_2).$$

If A is a contraction on $\mathfrak{H}$ (i.e., A is an operator with $\|A\| \leq 1$), there exists a unique contraction $\Gamma(A)$ on $\Gamma(\mathfrak{H})$ such that $\Gamma(A)\Gamma(x) = \Gamma(Ax)$ for all x in $\mathfrak{H}$. (In fact, $\Gamma(A) = \oplus_{n=0}^{\infty}(\otimes^n A)$). If A and B are contractions on $\mathfrak{H}$, it is clear that

$$\Gamma(AB) = \Gamma(A)\Gamma(B); \qquad \Gamma(A)^* = \Gamma(A^*). \tag{2}$$

In particular, if A is a projection, so also is $\Gamma(A)$.

(b) *Continuous tensor products*: If $(X, \mathscr{B}, \mu)$ is a measure space and $\mathfrak{H} = \int_X^{\oplus} \mathfrak{H}(t)\,\mu(dt)$ is a direct integral of Hilbert spaces over X (cf. **[2]** for definition and basic facts about direct integrals), then, for each M in $\mathscr{B}$, the operator of multiplication by χ_M will be denoted by $P(M)$. Thus, $M \to P(M)$ is the canonical spectral measure in $\mathfrak{H}$. If $\mathscr{H} = \Gamma(\mathfrak{H})$, we shall use the symbol $E(M)$ for $\Gamma(P(M))$. By the last remark in (a), each $E(M)$ is a projection; further, (2) implies that if $M \subseteq N$, then $E(M) \leq E(N)$. Further, we shall write $\mathfrak{H}(M) = P(M)\mathfrak{H}$ and $\mathscr{H}(M) = E(M)\mathscr{H}$. Then, $\mathscr{H}(M)$ can be naturally identified with $\Gamma(\mathfrak{H}(M))$, and it is easy to see that $\mathscr{H}(M) = \{(f_n)_{n=0}^{\infty} \in \mathscr{H}: f_n \in \mathfrak{H}(M)^{(s)n} \text{ for all } n\}$.

If M and N are disjoint sets in X, then $\mathfrak{H}(M \cup N) \cong \mathfrak{H}(M) \oplus \mathfrak{H}(N)$, and so, there exists a canonical unitary operator (cf. (a))

$$U_{M,N}: \mathscr{H}(M) \otimes \mathscr{H}(N) \to \mathscr{H}(M \cup N).$$

(If $x \in \mathfrak{H}(M)$, $y \in \mathfrak{H}(N)$, $U_{M,N}(\Gamma(x) \otimes \Gamma(y)) = \Gamma(x + y)$.) The following properties of the $U_{M,N}$'s are easily established (by verifying them on exponential vectors).

PROPOSITION (U). (i) *If L, M and N are disjoint Borel sets in X, the following diagram of Hilbert spaces and unitary operators is commutative*:

$$\begin{tikzcd}
\mathscr{H}(L) \otimes \mathscr{H}(M) \otimes \mathscr{H}(N) \arrow[r, "1_{\mathscr{H}(L)} \otimes U_{M,N}"] \arrow[d, "U_{L,M} \otimes 1_{\mathscr{H}(N)}"] & \mathscr{H}(L) \otimes \mathscr{H}(M \cup N) \arrow[d, "U_{L,M \cup N}"] \\
\mathscr{H}(L \cup M) \otimes \mathscr{H}(N) \arrow[r, "U_{L \cup M,N}"] & \mathscr{H}(L \cup M \cup N)
\end{tikzcd}$$

(ii) *If* $M \subseteqq N$, *then* $\mathscr{H}(M) \subseteqq \mathscr{H}(N)$ *and*

$$U_{M,N\setminus M}(f \otimes \Omega) = f, \qquad f \in \mathscr{H}(M).$$

(*Note that* $\Omega \in \mathscr{H}(L)$ *for all* $L \in \mathscr{B}$.)

Briefly, $\mathscr{H}$ has a continuous tensor product structure over X (cf. [**1**] and [**6**]).

In case $X = [0, \infty)$, μ is Lebesgue measure and $\mathfrak{H} = L^2[0, \infty)$, it is known that $\mathscr{H} = \Gamma(\mathfrak{H})$ can be identified with $L^2(\mathscr{C}, P)$, where $\mathscr{C} = \{f \in C[0, \infty)\colon f(0) = 0\}$ and P is the Wiener (probability) measure defined on the σ-algebra generated by point-evaluations. Explicitly, the correspondence is given by

$$\Gamma(\phi) \leftrightarrow \exp\left(\int \phi(t)\, dw(t) - \frac{1}{2}\int \phi(t)^2\, dt\right),$$

where $\phi \in L^2[0, \infty)$ and the first integral on the right is the Wiener integral (cf. [**6**]).

The text. In the sequel, the notation and terminology will be exactly as in (b) above. We shall further restrict ourselves to the case where

(a) $X = \mathbf{R}$

(b) $\mathscr{B}$ is the σ-algebra of Borel sets in $\mathbf{R}$; and

(c) μ is a non-atomic, positive, σ-finite measure defined on $\mathscr{B}$. Thus,

$$\mathfrak{H} = \int_{\mathbf{R}}^{\oplus} \mathfrak{H}(t)\mu(dt); \qquad \mathscr{H} = \Gamma(\mathfrak{H}).$$

For any t in $\mathbf{R}$, we shall use the abbreviations P_t, E_t, $\mathfrak{H}_t$ and $\mathscr{H}_t$ respectively for $P(-\infty, t]$, $E(-\infty, t]$, $P_t\mathfrak{H}$ and $E_t\mathscr{H}$. The non-atomicity of μ ensures that inclusion or exclusion of one or both end-points of intervals is irrelevant. (Thus, $P_t = P(-\infty, t)$.) Further, the non-atomicity of μ implies that $\{P_t\}$ and $\{E_t\}$ are strongly continuous one-parameter families of projections.

The symbol W will be reserved for the natural (isometric) inclusion of $\mathfrak{H}$ in $\mathscr{H}$:

$$Wx = (0, x, 0, 0, \dots). \tag{3}$$

The map W clearly satisfies

$$\begin{aligned} &W(\mathfrak{H}(M)) \subseteqq \mathscr{H}(M), \qquad M \in \mathscr{B}, \quad \text{and} \\ &\langle Wx, \Omega\rangle = 0, \qquad x \in \mathfrak{H}. \end{aligned} \tag{4}$$

In case $\mathfrak{H} = L^2[0, \infty)$ and $\mathscr{H} = L^2(\mathscr{C}, P)$, it can be verified that W is just the Wiener integral: $W\phi = \int \phi(t)\, dw(t)$. In order to define the analogue of the Itô integral, we begin with the following:

DEFINITION 1. A non-anticipating tensor (abbreviated to n.a.t. in the sequel) is an element of the closed subspace $\mathfrak{N}$ of $\mathscr{H} \otimes \mathfrak{H}$ defined by

$$\mathfrak{N} = \{\phi \in \mathscr{H} \otimes \mathfrak{H}: (1_{\mathscr{H}} \otimes P_t)\phi = (E_t \otimes P_t)\phi\ \forall t\}$$

EXAMPLE 2. Let $a \in \mathbf{R}, f \in \mathscr{H}(-\infty, a], x \in \mathfrak{H}(a, \infty)$, and let $\phi = f \otimes x$. Then ϕ is a n.a.t., since $(1_{\mathscr{H}} \otimes P_t)\phi = (E_t \otimes P_t)\phi = 0$ if $t \leq a$, while if $t > a$,

$$(E_t \otimes P_t)\phi = E_t f \otimes P_t x = f \otimes P_t x = (1_{\mathscr{H}} \otimes P_t)\phi.$$

DEFINITION 3. A n.a.t. of the sort described in Example 2 will be called an elementary n.a.t.; a finite linear combination of elementary n.a.t.s will be called a simple n.a.t.

The following elementary result is recorded here for later use.

PROPOSITION 4. *If $\phi \in \mathfrak{N}$ and $-\infty < a \leq b < \infty$, then*

$$(1_{\mathscr{H}} \otimes P(a, b])\phi = (E_b \otimes P(a, b])\phi.$$

Proof.

$$\begin{aligned}(1_{\mathscr{H}} \otimes P(a, b])\phi &= (1_{\mathscr{H}} \otimes P_b - 1_{\mathscr{H}} \otimes P_a)\phi \\ &= (E_b \otimes P_b - E_a \otimes P_a)\phi, \quad \text{since } \phi \in \mathfrak{N}.\end{aligned}$$

Hence

$$\begin{aligned}(E_b \otimes P(a, b])\phi &= (E_b \otimes 1_{\mathfrak{H}})(E_b \otimes P_b - E_a \otimes P_a)\phi \\ &= (E_b \otimes P_b - E_b E_a \otimes P_a)\phi \\ &= (E_b \otimes P_b - E_a \otimes P_a)\phi = (1_{\mathscr{H}} \otimes P(a, b])\phi\end{aligned}$$

by the previous equality, and the proof is complete.

We now wish to obtain a formula for the projection of $\mathscr{H} \otimes \mathfrak{H}$ onto $\mathfrak{N}$, which will henceforth be denoted by Q. However, some notation should be established first.

Let $J = \{(t_0, t_2, \ldots, t_n): -\infty < t_0 < t_1 < \cdots < t_n < \infty,\ n = 1, 2, \ldots\}$.The set J is a directed set with respect to the partial order defined by

$$(t_0, \ldots, t_n) \leq (s_0, \ldots, s_n) \quad \text{iff} \quad \{t_0, \ldots, t_n\} \subseteqq \{s_0, \ldots, s_m\}.$$

If $\Delta = (t_0, \dots, t_n) \in J$, define

$$(5) \qquad Q_\Delta = \sum_{i=1}^{n} E(-\infty, t_{i-1}] \otimes P(t_{i-1}, t_i]$$

Since the projections $\{P(t_{i-1}, t_i]: i = 1, \dots, n\}$ are mutually orthogonal it follows that Q_Δ, being a sum of mutually orthogonal projections, is itself a projection with

$$(6) \qquad \operatorname{ran} Q_\Delta = \bigoplus_{i=1}^{n} \mathscr{H}(-\infty, t_{i-1}] \otimes \mathfrak{H}(t_{i-1}, t_i].$$

LEMMA 5. *$\{Q_\Delta: \Delta \in J\}$ is a monotone net of projections; i.e., if Δ, $\Delta' \in J$ and $\Delta \leq \Delta'$, then $Q_\Delta \leq Q_{\Delta'}$.*

Proof. It clearly suffices to prove the following: If

$$\Delta = (a, b) \quad \text{and} \quad \Delta' = (s_0, \dots, s_n)$$

where $a = s_0 < s_1 < \cdots < s_n = b$, then $Q_\Delta \leq \Delta_{\Delta'}$. In this case, however,

$$\begin{aligned} Q_\Delta &= E(-\infty, a] \otimes P(a, b] \\ &= \sum_{i=1}^{n} E(-\infty, a] \otimes P(s_{i-1}, s_i] \\ &\leq \sum_{i=1}^{n} E(-\infty, s_{i-1}] \otimes P(s_{i-1}, s_i] = Q_{\Delta'}. \end{aligned}$$

PROPOSITION 6. *$Q = \lim_{\Delta \in J} Q_\Delta$, in the strong operator topology.*

Proof. Example 2 shows that every product vector in $\mathscr{H}(-\infty, a] \otimes \mathfrak{H}(a, b]$ is a n.a.t. It follows that (cf. (6)) $\operatorname{ran} Q_\Delta \subseteqq \operatorname{ran} Q$ for all Δ in J; i.e., $Q_\Delta \leq Q$ for all Δ in J.

Since Q and each Q_Δ are projections, it suffices to show that $Q_\Delta \to Q$ weakly. Further, since $Q_\Delta \leq Q$ for all Δ, and since the Q_Δ's are uniformly bounded, it is enough to show that $\langle Q_\Delta \Psi, \phi \rangle \to \langle Q\Psi, \phi \rangle$ for all ϕ in $\mathfrak{N}$ and for all Ψ belonging to some total set of vectors in $\mathscr{H} \otimes \mathfrak{H}$.

Observe that $\{f \otimes x: f \in \mathscr{H}(-T, T], \ x \in \mathfrak{H}(-T, T], \ T > 0\}$ is a total set of vectors in $\mathscr{H} \otimes \mathfrak{H}$. What we shall prove is that $\langle Q_\Delta(f \otimes x), \phi \rangle \to \langle Q(f \otimes x), \phi \rangle$ for all ϕ in $\mathfrak{N}$, where $f \in \mathscr{H}(-T, T]$ and $x \in \mathfrak{H}(-T, T]$ for some $T > 0$.

Let $\varepsilon > 0$ be given. Since $t \mapsto \|E_t f\|^2$ is monotone and uniformly continuous (recall that μ is non-atomic, and so the above function is continuous and constant in each of the intervals $(-\infty, T]$ and (T, ∞)), there exists $\Delta_0 = (s_0, \dots, s_N)$ in J such that

(i) $s_0 = -T$; $s_N = T$, and

(ii) $\|E_t f\|^2 - \|E_{t'} f\|^2 < \varepsilon^2/\|x\|^2\|\phi\|$ whenever

$$s_{i-1} \leq t' < t \leq s_i, \qquad \text{for } i = 1, \ldots, N.$$

Claim. $\Delta_0 \leq \Delta \Rightarrow |\langle (Q_\Delta - Q)(f \otimes x), \phi\rangle| < \varepsilon.$

Suppose $\Delta = (t_0, \ldots, t_n) \geq \Delta_0$. Then $t_0 \leq s_0 = -T$ and $t_n \geq s_N = T$ and hence,

$$x = P(t_0, t_n]x = \sum_{i=1}^{n} P(t_{i-1}, t_i]x;$$

thus,

$$\begin{aligned}
\langle Q(f \otimes x), \phi\rangle &= \langle f \otimes x, \phi\rangle \quad (\text{since } \phi \in \mathfrak{N}) \\
&= \sum_{i=1}^{n} \langle f \otimes P(t_{i-1}, t_i]x, \phi\rangle \\
&= \sum_{i=1}^{n} \langle (1_{\mathscr{H}} \otimes P(t_{i-1}, t_i])(f \otimes x), \phi\rangle \\
&= \sum_{i=1}^{n} \langle (f \otimes x), (1_{\mathscr{H}} \otimes P(t_{i-1}, t_i])\phi\rangle \\
&= \sum_{i=1}^{n} \Big\langle \big(f \otimes x, E_{t_i} \otimes P(t_{i-1}, t_i]\big)\phi\Big\rangle \quad (\text{by Proposition 4}) \\
&= \sum_{i=1}^{n} \Big\langle \big(E_{t_i} \otimes P(t_{i-1}, t_i]\big)(f \otimes x), \phi\Big\rangle,
\end{aligned}$$

while, by definition,

$$\langle Q_\Delta(f \otimes x), \phi\rangle = \sum_{i=1}^{n} \Big\langle \big(E_{t_{i-1}} \otimes P(t_{i-1}, t_i]\big)(f \otimes x), \phi\Big\rangle.$$

Hence

$$\begin{aligned}
&|\langle Q(f \otimes x), \phi\rangle - \langle Q_\Delta(f \otimes x), \phi\rangle| \\
&\quad = \left| \sum_{i=1}^{n} \Big\langle \Big(\big(E_{t_i} - E_{T_{i-1}}\big) \otimes P(t_{i-1}, t_i]\Big), (f \otimes x), \phi\Big\rangle \right| \\
&\quad = \left| \sum_{i=1}^{n} \Big\langle \Big(\big(E_{t_i} - E_{t_{i-1}}\big) \otimes P(t_{i-1}, t_i]\Big)(f \otimes x), \right. \\
&\qquad\qquad\qquad \left. \Big\langle (1_{\mathscr{H}} \otimes P(t_{i-1}, t_i])\phi\Big\rangle \right|
\end{aligned}$$

(continues)

$$\leq \sum_{i=1}^{n} \left\| \left((E_{t_i} - E_{t_{i-1}}) \otimes P(t_{i-1}, t_i] \right)(f \otimes x) \right\| \cdot \left\| (1_{\mathscr{H}} \otimes P(t_{i-1}, t_i]) \phi \right\|$$

$$\leq \left[\sum_{i=1}^{n} \left\| (E_{t_i} - E_{t_{i-1}}) f \otimes P(t_{i-1}, t_i] x \right\|^2 \right]^{1/2} \cdot \left[\sum_{i=1}^{n} \left\| (1_{\mathscr{H}} \otimes P(t_{i-1}, t_i]) \phi \right\|^2 \right]^{1/2}.$$

$$\text{The first term} = \left[\sum_{i=1}^{n} \left\| (E_{t_i} - E_{t_{i-1}}) f \right\|^2 \left\| P(t_{i-1}, t_i] x \right\|^2 \right]^{1/2}$$

$$\leq \left[\frac{\varepsilon^2}{\|x\|^2 \|\phi\|^2} \sum_{i=1}^{n} \left\| P(t_{i-1}, t_i] x \right\|^2 \right]^{1/2} = \varepsilon \|\phi\|^{-1},$$

the first inequality being a consequence of the choice of Δ_0, the inequality $\Delta \geq \Delta_0$ and the assumption $f \in \mathscr{H}(-T, T]$, while the last equality follows from $x \in \mathfrak{H}[-T, T]$.

The second term is dominated by $\|\phi\|$ since $\{1_{\mathscr{H}} \otimes P(t_{i-1}, t_i]: i = 1, \ldots, n\}$ is a set of mutually orthogonal projections; hence, the proof of the claim, and consequently, the proof of the proposition, is complete.

The next result is an easy consequence of the last proposition.

PROPOSITION 7. *Simple n.a.t.s (cf. Definition 3) are dense in* $\mathfrak{N}$.

Proof. It is to be proved that $\mathfrak{N} = \mathfrak{N}_0$, where $\mathfrak{N}_0$ is the closure of the set of simple n.a.t.s.

To start with, note that if $f \in \mathscr{H}$ and $x \in \mathfrak{H}$, then $Q_\Delta(f \otimes x)$ is a simple n.a.t. for every Δ in J, and so, by Proposition 6, it follows that $Q(f \otimes x) \in \mathfrak{N}_0$.

Since $\mathscr{H} \otimes \mathfrak{H} = \mathrm{sp}\{f \otimes x: f \in \mathscr{H}, x \in \mathfrak{H}\}$ it follows (from the linearity and continuity of Q) that

$$\mathfrak{N} = Q(\mathscr{H} \otimes \mathfrak{H}) = \mathrm{sp}\{Q(f \otimes x): f \in \mathscr{H}, x \in \mathfrak{H}\} \subseteqq \mathfrak{N}_0$$

the last inclusion following from the previous paragraph. Since, clearly, $\mathfrak{N}_0 \subseteqq \mathfrak{N}$, the proof is complete.

Observe that $\Omega \otimes x \in \mathfrak{N}$ for any x in $\mathfrak{H}$, since $E_t \Omega = \Omega$ for all t.

THEOREM 8. *There exists a unique isometic operator* $\mathscr{I}\colon \mathfrak{R} \to \mathscr{H}$ *such that*

(i) $\mathscr{I}(\Omega \otimes x) = Wx$ *for all* x *in* $\mathfrak{H}$; *and more generally,*

(ii) *if* $a \in \mathbf{R}$, $f \in \mathscr{H}(-\infty, a]$, $x \in \mathfrak{H}(a, \infty)$ *and* $\phi = f \otimes x$, *then* $\mathscr{I}\phi = U_{(-\infty,a],(a,\infty)}(f \otimes Wx)$.

(*Note*: This is the analogue of the Itô integral and it is tempting to write $\mathscr{I}\phi = \int \phi\, dW$.)

Proof. Since elementary n.a.t.s span $\mathfrak{R}$, it is clear that (ii) forces uniqueness of $\mathscr{I}$, so it suffices to prove existence.

For typographical economy, let us write U_a for $U_{(-\infty,a],(a,\infty)}$ and $U_{a,b}$ for $U_{(-\infty,a],(a,b]}$ when $a \le b$, where the $U_{L,M}$'s are as defined in (b) of Preliminaries.

If a, f, x and ϕ are as in (ii) above, then $Wx \in \mathscr{H}(a,\infty)$ (cf. (4)) and so, it makes sense to define $\mathscr{I}\phi = U_a(f \otimes Wx)$. That $\mathscr{I}\phi$ is unambiguously defined (in the sense that $\mathscr{I}\phi$ depends only on ϕ, and not on a, f or x) is a consequence of the consistency properties of the $U_{L,M}$'s stated in Proposition (U).

Next suppose $a, b \in \mathbf{R}$, $f \in \mathscr{H}(-\infty, a]$, $x \in \mathfrak{H}(a,\infty)$, $\phi = f \otimes x$, and $g \in \mathscr{H}(-\infty, b]$, $y \in \mathfrak{H}(b,\infty)$, $\Psi = g \otimes y$. Assume (without loss of generality) that $a \le b$. Then, observe that

$$\begin{aligned}\mathscr{I}\phi &= \mathscr{I}(f \otimes x) = U_a(f \otimes Wx)\\ &= U_a(f \otimes W(P(a,b]x + P(b,\infty)x))\\ &= U_a(f \otimes WP(a,b]x) + U_a(f \otimes WP(b,\infty)x).\end{aligned}$$

Notice that $U_a(f \otimes WP(a,b]x) \in \mathscr{H}(-\infty, b]$ and so,

$$U_a(f \otimes WP(a,b]x) = U_b(U_{a,b}(f \otimes WP(a,b]x) \otimes \Omega).$$

Similarly

$$U_a(f \otimes WP(b,\infty)x) = U_b(U_{a,b}(f \otimes \Omega) \otimes WP(b,\infty)x).$$

On the other hand, by definition,

$$\mathscr{I}\Psi = \mathscr{I}(g \otimes y) = U_b(g \otimes Wy).$$

Since U_b is unitary, conclude that

$$\begin{aligned}\langle \mathscr{I}\phi, \mathscr{I}\Psi\rangle &= \langle U_{a,b}(f \otimes WP(a,b]x) \otimes \Omega,\ g \otimes Wy\rangle\\ &\quad + \langle U_{a,b}(f \otimes \Omega) \otimes WP(b,\infty)x, g \otimes Wy\rangle.\end{aligned}$$

The first term on the right is zero since $\langle \Omega, Wy \rangle = 0$ (cf. (4)), and so, since W is isometric,

$$\begin{aligned}\langle \mathscr{I}\phi, \mathscr{I}\Psi \rangle &= \langle U_{a,b}(f \otimes \Omega), g \rangle \langle P(b,\infty)x, y \rangle \\ &= \langle f, g \rangle \langle x, P(b,\infty)y \rangle \quad (\text{since } U_{a,b}(f \otimes \Omega) = f) \\ &= \langle f, g \rangle \langle x, y \rangle \quad (\text{since } y \in \mathscr{H}(b,\infty)) \\ &= \langle f \otimes x, g \otimes y \rangle = \langle \phi, \Psi \rangle .\end{aligned}$$

So, the equation (ii) (in the statement of the theorem) unambiguously defines a vector $\mathscr{I}\phi$ in $\mathscr{H}$ for every elementary n.a.t. ϕ; further, if ϕ and Ψ are elementary n.a.t.s, then $\langle \mathscr{I}\phi, \mathscr{I}\Psi \rangle = \langle \phi, \Psi \rangle$. Since elementary n.a.t.s generate $\mathfrak{N}$ (by Proposition 7), it is clear that $\mathscr{I}$ extends to a unique isometric operator from $\mathfrak{N}$ into $\mathscr{H}$.

Finally, we identify the range of $\mathscr{I}$, and this result is essentially the Kunita-Watanabe Theorem.

THEOREM 9. $\mathscr{I}(\mathfrak{N}) = \{\Omega\}^{\perp} = \mathscr{H} \ominus \mathbf{C}\Omega$.

Proof. Since $\{\Omega\}^{\perp} = \text{sp}\{\Gamma(x) - \Omega \colon x \in \mathfrak{H}\}$, and since $\mathscr{I}$ (being isometric) has closed range, it suffices to prove the following:

Claim. $\Gamma(x) - \Omega = \mathscr{I}(Q(\Gamma(x) \otimes x))$ for all x in $\mathfrak{H}$. Since $\mathscr{H} = \text{sp}\{\Gamma(y) \colon y \in \mathfrak{H}\}$, it is enough to establish that

$$\langle \mathscr{I}(Q(\Gamma(x) \otimes x)), \Gamma(y) \rangle = \langle \Gamma(x) - \Omega, \Gamma(y) \rangle = \exp\langle x, y \rangle - 1.$$

In view of Proposition 6 (and the continuity of $\mathscr{I}$), it is enough to prove that

$$\lim_{\Delta \in J} \langle \mathscr{I}(Q_\Delta(\Gamma(x) \otimes x)), \Gamma(y) \rangle = \exp\langle x, y \rangle - 1.$$

Let $\Delta = (t_0, \dots, t_n) \in J$. Writing U_t for $U_{(-\infty,t],(t,\infty)}$ (as in the proof of Theorem 8), we see that

$$\begin{aligned}\mathscr{I}(Q_\Delta(\Gamma(x) \otimes x)) &= \mathscr{I}\left(\sum_{i=1}^{n} E_{t_{i-1}}(\Gamma(x)) \otimes P(t_{i-1}, t_i]x \right) \\ &= \mathscr{I}\left(\sum_{i=1}^{n} \Gamma(P_{t_{i-1}})\Gamma(x) \otimes P(t_{i-1}, t_i]x \right) \\ &= \mathscr{I}\left(\sum_{i=1}^{n} \Gamma(P_{t_{i-1}}x) \otimes P(t_{i-1}, t_i]x \right) \\ &= \sum_{i=1}^{n} U_{t_{i-1}}\left(\Gamma(P_{t_{i-1}}x) \otimes WP(t_{i-1}, t_i]x\right).\end{aligned}$$

On the other hand, for any t in $\mathbf{R}$,

$$\Gamma(y) = U_t(\Gamma(P_t y) \otimes \Gamma(P(t,\infty)y)).$$

Hence,

$$\begin{aligned}\langle \mathscr{I}(Q_\Delta(\Gamma(x) \otimes x)), \Gamma(y)\rangle &= \sum_{i=1}^{n} \langle \Gamma(P_{t_{i-1}}x), \Gamma(P_{t_{i-1}}y)\rangle \\ &\qquad \cdot \langle WP(t_{i-1}, t_i]x, \Gamma(P(t_{i-1}, \infty)y)\rangle \\ &= \sum_{i=1}^{n} (\exp\langle P_{t_{i-1}}x, P_{t_{i-1}}y\rangle)\langle P(t_{i-1}, t_i]x, P(t_{i-1}, \infty)y\rangle \\ &= \sum_{i=1}^{n} (\exp\langle P_{t_{i-1}}x, y\rangle)\langle P(t_{i-1}, t_i]x, y\rangle \\ &= \sum_{i=1}^{n} (\exp \alpha(t_{i-1})) \cdot (\alpha(t_i) - \alpha(t_{i-1})),\end{aligned}$$

where $\alpha(t) = \langle P_t x, y\rangle$.

Hence $\langle \mathscr{I}(Q_\Delta(\Gamma(x) \otimes x)), \Gamma(y)\rangle$ is a typical Riemann sum (considering the left end point) corresponding to the partition Δ, in the evaluation of the Riemann-Stieltje's integral $\int_{-\infty}^{\infty}(\exp \alpha(t))\, d\alpha(t)$. (Note that $\alpha(t)$ is a function of finite total variation.) Taking limits as the partition is indefinitely refined, we get, by Proposition 6,

$$\begin{aligned}\langle \mathscr{I}(Q(\Gamma(x) \otimes x)), \Gamma(y)\rangle &= \int_{-\infty}^{\infty} e^{\alpha(t)}\, d\alpha(t) \\ &= e^{\alpha(t)}\Big|_{-\infty}^{\infty} = e^{\langle x, y\rangle} - 1, \quad \text{as desired.}\end{aligned}$$

The Kunita-Watanabe theorem (cf. **[4]**) is stated in terms of martingales. To make contact with that formulation, one can define a martingale (in this setting) as a curve $\{\phi(t)\colon t \in \mathbf{R}\}$ in $\mathscr{H}$ such that $E_s\phi(t) = \phi(s)$ for $s \leq t$. It can easily be verified, that $\phi(t) = E_t\mathscr{I}(\phi)$ defines a martingale with 'mean zero' for any ϕ in $\mathfrak{N}$ (i.e., $\langle E_t\mathscr{I}(\phi), \Omega\rangle = 0$). It can now be deduced from Theorem 9 that if $\{\phi(t)\colon t \in \mathbf{R}\}$ is a martingale such that (i) $\langle \phi(t), \Omega\rangle 0$ for all t, and (ii) $\sup_t\|\phi(t)\| < \infty$, then there exists ϕ in $\mathfrak{N}$ such that $\phi(t) = E_t\mathscr{I}(\phi) = \mathscr{I}((1_{\mathscr{H}} \otimes P_t)\phi)$. The verification of the above details is fairly painless and we shall be content to stop here.

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