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DISTANCE BETWEEN NORMAL OPERATORS

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ABSTRACT. Lidskii and Wielandt have proved independently that if A and B are selfadjoint operators on an n-dimensional space H, with eigenvalues $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ respectively (counting multiplicity), then,

$$\|A - B\| \geqslant \min_{\sigma \in S_n} \|\operatorname{diag}(\alpha_k - \beta_{\sigma(k)})\|$$

for any unitarily invariant norm on $L(H)$. In this note an example is given to show that this result is no longer true if A and B are only required to be normal (even unitary). It is also shown that the above inequality holds in the operator norm, if A is selfadjoint and B is skew-self-adjoint.

Introduction. In [2], Weyl proved the following: if A and B are Hermitian operators on a finite-dimensional Hilbert space, and if $\operatorname{sp} A = \{\alpha_k\}_{k=1}^n$, $\operatorname{sp} B = \{\beta_k\}_{k=1}^n$ (counting multiplicity), then

$$(*) \qquad \|A - B\| \geqslant \min_{\sigma \in S_n} \|\operatorname{diag}(\alpha_k - \beta_{\sigma(k)})\|.$$

(Here, $\operatorname{sp} A$ denotes the spectrum of A, and $\operatorname{diag}(\gamma_k)$ denotes any normal operator with spectrum $\{\gamma_k\}_{k=1}^n$.) The norm used above is the operator norm. It was later shown by Lidskii [1] and Wielandt [3] that $(*)$ is valid in any norm on $L(H)$ that is unitarily invariant. (Throughout this note, H denotes a finite-dimensional Hilbert space.)

In this short note, we first give an example to show that the analogue of Lidskii's theorem for general normal operators is false. In the example given, A is Hermitian and unitary, while B is skew-Hermitian and unitary. We show, however (cf. Theorem 2) that Weyl's theorem remains valid if A is Hermitian and B is skew-Hermitian.

EXAMPLE 1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then A is Hermitian, B is skew-Hermitian and $\operatorname{sp} A = \{\alpha_1, \alpha_2\}$, $\operatorname{sp} B = \{\beta_1, \beta_2\}$ where $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1 = i$, $\beta_2 = -i$. (Note that A and B are unitary.) If $\|\cdot\|_p$ is the Schatten p-norm, then it is easily checked that

$$\|A - B\|_p = 2, \qquad \|\operatorname{diag}(\alpha_k - \beta_{\sigma(k)})\|_p = 2^{1/2+1/p},$$

for any σ in S_2. Thus, if $1 \leqslant p < 2$,

$$\|A - B\|_p < \min_{\sigma \in S_2} \|\operatorname{diag}(\alpha_k - \beta_{\sigma(k)})\|_p. \quad \square$$

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In the above example, the operator norm of $A - B$ is larger than $\min_{\sigma \in S_2} \|\text{diag}(\alpha_k - \beta_{\sigma(k)})\|$. That this is not a coincidence is the content of the next theorem.

THEOREM 2. *Let $A = A^*$, $B = B^* \in L(H)$. Let* $\text{sp}\, A = \{\alpha_k\}_{k=1}^n$, $\text{sp}\, B = \{\beta_k\}_{k=1}^n$. *Assume the numbering to satisfy* $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$, $|\beta_1| \leqslant \cdots \leqslant |\beta_n|$. *Then,*

$$\|A + iB\| \geqslant \|\text{diag}(\alpha_k + i\beta_k)\| = \max_k |\alpha_k + i\beta_k|.$$

PROOF. Let $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ be orthonormal bases of H such that $Ae_k = \alpha_k e_k$ and $Bf_k = \beta_k f_k$ for all k. Let $M_k = \bigvee\{e_l\colon l \leqslant k\}$ and $N_k = \bigvee\{f_l\colon l \geqslant k\}$. Then, since $\dim M_k = k$ and $\dim N_k = n - k + 1$, it follows that $\dim(M_k \cap N_k) \geqslant 1$. Let x_k be a unit vector in $M_k \cap N_k$. The ordering of the α_k's and the β_k's ensures that $\|Ax_k\| \geqslant |\alpha_k|$, $\|Bx_k\| \geqslant |\beta_k|$. Hence,

$$\begin{aligned} |\alpha_k + i\beta_k|^2 &= |\alpha_k|^2 + |\beta_k|^2 \leqslant \|Ax_k\|^2 + \|Bx_k\|^2 \\ &= \tfrac{1}{2}\left[\|(A + iB)x_k\|^2 + \|(A - iB)x_k\|^2\right] \\ &\leqslant \tfrac{1}{2}\left[\|A + iB\|^2 + \|A - iB\|^2\right] = \|A + iB\|^2. \end{aligned}$$

(The first equality follows from the parallelogram identity, while the last equality is a consequence of the fact that $A - iB = (A + iB)^*$.) Hence,

$$\|A + iB\| \geqslant \max_k |\alpha_k + i\beta_k|. \quad \square$$

The following is just a reformulation of Theorem 2: if $T \in L(H)$, then there exists a normal operator N on H such that (i) the real and imaginary parts of N are, respectively, unitarily equivalent to the real and imaginary parts of T, and (ii) $\|N\| \leqslant \|T\|$.

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