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A THEOREM OF CRAMÉR AND WOLD REVISITED

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ABSTRACT. Let $H = \{(x, y); x > 0\} \subseteq \mathbf{R}^2$ and let E be a Borel subset of H of positive Lebesgue measure. We prove that if μ and υ are two probability measures on $\mathbf{R}^2$ such that $\mu(\sigma(E)) = \upsilon(\sigma(E))$ for all rigid motions σ of $\mathbf{R}^2$, then $\mu = \upsilon$ This generalizes a well-known theorem of Cramér and Wold.

1. Introduction. A celebrated theorem of H. Cramér and H. Wold particularly well known to probabilists (see [3]) asserts: If μ and υ are probability measures on $\mathbf{R}^2$ such that they agree on all half planes, then $\mu = \upsilon$. This can be reformulated in the following way: Let $H = \{(x, y) \in \mathbf{R}^2; x \geqslant 0\}$. If μ and υ are probability measures such that $\mu(\sigma(H)) = \upsilon(\sigma(H))$ for all rigid motions σ of $\mathbf{R}^2$, then $\mu = \upsilon$. The aim of this note is to generalize this result to an arbitrary Borel set E of positive Lebesgue measure contained in H. The results in this paper are valid for $\mathbf{R}^n$, $n \geqslant 2$, but for notational simplicity we consider $\mathbf{R}^2$—the same proofs go through for any $n \geqslant 2$. A special case of our result—for a restricted class of Borel sets E, i.e. those that "pave" the half space H—appears in [4, §I.2.4]. However, the methods in [4] are different, where the Radon transform is used.

2. Notation and terminology. For any unexplained notation or terminology please see [5].

By a rigid motion of $\mathbf{R}^2$ we mean a homeomorphism of $\mathbf{R}^2$ of the form $(x, y) \to T(x, y) + (x_0, y_0)$, where (x_0, y_0) is a fixed vector in $\mathbf{R}^2$ and T is a special orthogonal linear transformation of $\mathbf{R}^2$ (i.e. T is a matrix of the form

$$\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right).$$

Throughout this paper λ denotes the Lebesgue measure on $\mathbf{R}^2$. Let C denote the class of all (finite) complex measures on $\mathbf{R}^2$. If T is a tempered distribution (in the sense of Schwartz), then $\hat{T}$ denotes the Fourier transform of T (which is again a distribution) and $\operatorname{Supp} T$ denotes the (closed) support of T. For standard facts regarding distributions, Fourier transforms etc., see [5]. If g is a bounded Borel function on $\mathbf{R}^2$, then g defines a tempered distribution and $\hat{g}$ will denote the (distributional) Fourier transform of g. If μ is a finite complex measure, then $\mu * g$ is the bounded Borel function defined by

$$(\mu * g)(x) = \int_{\mathbf{R}^2} g(x - y)\, d\mu(y).$$

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Finally we note that for a complex measure or an L^1-function the usual notion of Fourier transform coincides with the notion of distributional Fourier transform.

If $E \subset \mathbf{R}^2$, let 1_E denote the indicator function of E, i.e. $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$.

H will always stand for the subset of $\mathbf{R}^2$ defined by $H = \{(x, y) \in \mathbf{R}^2; x \geqslant 0\}$.

We end this section by quoting a result that will be needed in the next section.

PROPOSITION. *Let f be a bounded measurable function on $\mathbf{R}^2$ and $\mu \in C$. If $\mu * f$ vanishes identically, then $\hat{\mu}$ vanishes on* $\operatorname{Supp} \hat{f}$.

(*Note*. For a proof of this theorem, we refer to p. 232 of [**2**]. In [**2**] μ is taken to be an L^1-function but by convolving μ with an L^1-function whose Fourier transform is nowhere vanishing (e.g. the Gaussian) we can get the theorem quoted above. Note also that $\operatorname{Supp} \hat{f}$ is called "spectrum of f" in [**2**].)

3. The main result. We start with a proposition which combined with the proposition quoted in §2 yields the main result.

PROPOSITION 3.1. *Let h be a nonnegative bounded Borel measurable function on $\mathbf{R}^2$ such that h is positive on a set of positive Lebesgue measure and such that* $\operatorname{Supp} h \subset H$. *Then* $\mathbf{R} \times \{0\} \subset \operatorname{Supp} \hat{h}$.

PROOF. First assume $h \in L^1(\mathbf{R}^2)$. Let $f(x) = \int_{\mathbf{R}} h(x, y)\,dy$. The hypotheses on h easily imply that f is a nontrivial, nonnegative L^1-function on $\mathbf{R}$ which is supported in $\mathbf{R}^+$. Now if $\hat{f}$ is the one-dimensional Fourier transform of f, then $\hat{f}$ can be extended to a bounded function g in the region $S = \{z \in \mathbf{C}; \operatorname{Im} z \leqslant 0\}$. g will be analytic in $S_0 = \{z \in \mathbf{C}; \operatorname{Im} z < 0\}$ and continuous in S. Thus $\hat{f}$ is the "boundary-value" of a bounded analytic function in S_0 and consequently $\hat{f}$ cannot vanish identically on any nonempty open subset of $\mathbf{R}$, i.e. $\operatorname{Supp} \hat{f} = \mathbf{R}$. Now observe that if $\hat{h}$ is the (two dimensional) Fourier transform of h, then

$$\hat{h}(\lambda, 0) = \int_{\mathbf{R}^2} h(x, y) e^{-i\lambda x}\,dx\,dy = \hat{f}(\lambda).$$

Thus $\operatorname{Supp} \hat{h} \supset \mathbf{R} \times \{0\}$ because $\operatorname{Supp} \hat{f} = \mathbf{R}$.

Now we drop the assumption that $h \in L^1(\mathbf{R}^2)$. To prove the proposition let us assume $\hat{h}$ vanishes in a neighborhood U (in $\mathbf{R}^2$) of a point $(\lambda_0, 0) \in \mathbf{R} \times \{0\}$. Choose ε sufficiently small such that the open ball of radius 2ε with centre at $(\lambda_0, 0)$ is contained in U. Let $0 \neq h_1$ be a nonnegative function in $L^1(\mathbf{R}^2)$ such that $\hat{h}_1$ is a C^∞-function and $\operatorname{Supp} \hat{h}_1$ is contained in the ball of radius ε around 0. (It is always possible to do this.) Then $\operatorname{Supp}(hh_1)\hat{} = \operatorname{Supp}(\hat{h} * \hat{h}_1) \subset \operatorname{Supp} \hat{h} + \operatorname{Supp} \hat{h}_1$. So if $U' = \{(x, y) \in \mathbf{R}^2; \sqrt{(x - \lambda_0)^2 + y^2} < \varepsilon\}$, then $\operatorname{Supp}(hh_1)\hat{} \cap U' = \varnothing$. However hh_1 is a nonnegative L^1-function with $\operatorname{Supp} hh_1 \subseteq H$ and by the first part of our proof hh_1 must be zero almost everywhere on $\mathbf{R}^2$. Since $\hat{h}_1$ is a C^∞-function of compact support, h_1 is the restriction of an entire function to $\mathbf{R}^2$ and hence $h_1(x) \neq 0$ a.e. on $\mathbf{R}^2$. Thus h is zero a.e. which gives us a contradiction and the proof of our proposition is complete.

Proposition 3.1 and the Proposition in §2 easily imply the following generalization of the Cramér-Wold theorem.

THEOREM 3.2. *Let E be a Borel subset of H such that $\lambda(E) > 0$. Let $\mu, \nu \in C$ such that $\mu(\sigma(E)) = \nu(\sigma(E))$ for all rigid motions σ of $\mathbf{R}^2$. Then $\mu = \nu$.*

PROOF. Let l be any line through $(0,0)$ in $\mathbf{R}^2$. We will prove $\hat{\mu} = \hat{\nu}$ on l. By Proposition 3.1, Supp $\hat{1}_E \supseteq \mathbf{R} \times \{0\}$. This implies that there exists a rotation T of $\mathbf{R}^2$ such that Supp $\hat{1}_{TE} \supseteq l$. Now $\mu(\sigma(E)) = \nu(\sigma(E))$ for all rigid motions σ, implies that $\check{\mu} * 1_{TE} = \check{\nu} * 1_{TE}$ (where $\check{\mu}(A) = \mu(-A)$), for every rotation T of $\mathbf{R}^2$. Thus by the Proposition in §2 it follows that $(\check{\mu})\hat{} = (\check{\nu})\hat{}$ on l. Since l is arbitrary this implies $(\check{\mu})\hat{} = (\check{\nu})\hat{}$, i.e. $\check{\mu} = \check{\nu}$, i.e. $\mu = \nu$, and the proof of our theorem is complete.

REMARK. The technique used in this paper is essentially that of [**1**]—this paper could be considered a continuation of [**1**].

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