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LOCAL UNCERTAINTY INEQUALITIES FOR COMPACT GROUPS

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ABSTRACT. Conditions are established on $\alpha, \beta \in \mathbf{R}$ for there to exist a constant $K = K(\alpha, \beta)$ such that

$$\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}(\gamma)^* \hat{f}(\gamma)) \leq K \left(\sum_{\gamma \in E} d(\gamma)^2 \right)^{\alpha} \|w^{\beta} f\|_2$$

for all $f \in L^1(G)$ and $E \subseteq \hat{G}$ where G is a compact metric group, $\hat{G}$ is its dual, $\hat{f}$ is the Fourier transform of f and $w\colon G \to \mathbf{R}^+$ is the function taking $x \in G$ to the area of the ball in G with centre e and x on its boundary. This is followed by a partial analogy for compact riemannian manifolds.

1. Introduction. The following is a special case of a result in [**6**] for multiple Fourier series: given $\alpha, \beta \in \mathbf{R}$ and $k \in \mathbf{Z}^+ = \{1, 2, \dots\}$, there exists a constant K such that

$$\left(\sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^{\alpha} \| |x|^{k\beta} f \|_2 \tag{1.1}$$

for all $f \in L^1(\mathbf{T}^k)$ and all finite $E \subseteq \mathbf{Z}^k$ if and only if α, β satisfy

$$\beta < 1/2, \quad \alpha \geq 0 \quad \text{and} \quad \alpha \geq \beta. \tag{1.2}$$

($|E|$ denotes the cardinality of E and the function $|x|$ is defined on $\mathbf{T}^k$ by identifying this group with $(-\frac{1}{2}, \frac{1}{2}]^k$.) This is a local uncertainty inequality in the sense that concentration of f limits the localization of $\hat{f}$ on any given set. The main result below, Theorem 2.4, is a direct analogue valid for all compact metrizable groups. We then give a somewhat less complete version for compact analytic manifolds. Local uncertainty inequalities for certain noncompact Lie groups are given in [**7**] and for $\mathbf{R}^d$ in [**1**, **4**, **5**].

2. Compact metric groups. Throughout this section G will be a compact nonfinite metric group equipped with normalized Haar measure $d\mu$ and $\hat{G}$ will be its unitary dual, that is, $\hat{G}$ is a maximal set of pairwise inequivalent unitary irreducible continuous representations of G. Denote by $\mathcal{H}_\gamma$ the (finite-dimensional) Hilbert space on which $\gamma \in \hat{G}$ acts. As usual, the Fourier series of $f \in L^1(G)$ is written as

$$f \sim \sum_{\gamma \in \hat{G}} d(\gamma) \operatorname{tr}(\hat{f}(\gamma)\gamma(\cdot)),$$

where $d(\gamma)$ is the dimension of $\mathcal{H}_\gamma$ and $\hat{f}(\gamma) = \int_G f(x)\gamma(x^{-1})\, d\mu(x)$. Our first concern is to introduce a function which plays the role of $|x|$ when $G = \mathbf{T}^k$.

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Let $d(\cdot,\cdot)$ be a metric on G which describes its topology. Without loss of generality assume that the metric is normalized so that $\sup\{d(e,x)\colon x \in G\} = 1$. Since G is compact, this supremum is actually attained. Denote $[0,1]$ by I and define a nondecreasing measurable function $A\colon I \to I$ by

$$A(r) = \mu(B_r) \quad \text{where } B_r = \{x \in G\colon d(x,e) \le r\}.$$

Since G is nonfinite and compact, $A(0) = 0$. Also the fact that if $r_n \searrow r$ as $n \to \infty$ for some sequence (r_n), then $B_r = \bigcap_n B_{r_n}$, shows that A is continuous on the right.

Define a continuous map $\phi\colon G \to I$ by $\phi(x) = d(x,e)$. Also let B_r' denote the complement of B_r in G.

2.1 LEMMA. *For any $\varepsilon > 0$, $\mu\{x \in G\colon A(\phi(x)) < \varepsilon\} \le \varepsilon$.*

REMARK. A is right continuous, ϕ is continuous and so $A \circ \phi$ is measurable. Hence the set in Lemma 2.1 is measurable.

PROOF OF 2.1. Given $\varepsilon > 0$, let $Y = \{x \in G\colon A(\phi(x)) < \varepsilon\}$. Since always $y \in B_{\phi(y)}$, $Y \subseteq \bigcup\{B_{\phi(x)}\colon x \in Y\}$. On the other hand, suppose $y \in \bigcup\{B_{\phi(x)}\colon x \in Y\}$, that is, $y \in B_{\phi(x)}$ for some $x \in Y$. Hence $\phi(y) \le \phi(x)$ and so $A(\phi(y)) \le A(\phi(x)) < \varepsilon$, with the conclusion that $y \in Y$. This has established the fact that

$$Y = \bigcup\{B_{\phi(x)}\colon x \in Y\},$$

from which the conclusion in Lemma 2.1 is a straightforward consequence.

2.2 LEMMA. *The function $w = A \circ \phi\colon G \to I$ is measurable and satisfies*

$$\int_{B_r} w^{-\theta}\, d\mu \le \frac{A(r)^{1-\theta}}{1-\theta} \quad \textit{for } 0 \le \theta < 1$$

and

$$\|w^{-\theta} 1_{B_r'}\|_\infty \le A(r)^{-\theta} \quad \textit{for } \theta \ge 0$$

for each $r > 0$.

Further, for $\theta \le 0$, $w^{-\theta}$ is continuous and hence bounded since G is compact. Consequently

$$\int_G w^{-\theta}\, d\mu < \infty \quad \textit{for } \theta \le 0.$$

PROOF. We just give a sketch of the proof of the first inequality. Define $G_t = \{x \in G\colon w(x)^{-1} > t\}$ for $t \ge 0$. By the change of variable formula [**3**, (21.72)],

$$\int_{B_r} w^{-\theta}\, d\mu = \int_0^\infty \theta t^{\theta-1} \mu(G_t \cap B_r)\, dt. \tag{2.1}$$

First consider the integral $I_1 = \int_0^{1/A(r)} \theta t^{\theta-1}\mu(G_t \cap B_r)\, dt$. Since $\mu(G_t \cap B_r) \le \mu(B_r) = A(r)$, we have

$$I_1 \le A(r) \int_0^{1/A(r)} \theta t^{\theta-1}\, dt = A(r)^{1-\theta}. \tag{2.2}$$

Now consider $I_2 = \int_{1/A(r)}^\infty \theta t^{\theta-1}\mu(G_t \cap B_r)\, dt$. Whenever $t > 1/A(r)$, $G_t \subseteq B_r$. (Let $x \in G_t$; then $A(\phi(x)) < t^{-1} < A(r)$ and so $\phi(x) \le r$, that is, $x \in B_r$.) Hence

$$I_2 = \int_{1/A(r)}^\infty \theta t^{\theta-1} \mu(G_t)\, dt.$$

Lemma 2.1 shows that $\mu(G_t) \leq 1/t$ and so

$$I_2 \leq \int_{1/A(r)}^{\infty} \theta t^{\theta-1} t^{-1}\, dt = \frac{A(r)^{1-\theta}\theta}{1-\theta}. \tag{2.3}$$

Combining (2.2) and (2.3) and substituting in (2.1) gives the required inequality.

REMARK. In the above, notice that $A(r) > 0$ for $r > 0$. This is because B_r is a neighbourhood of e for each $r > 0$ and so has positive Haar measure.

2.3 ASSUMPTION. To obtain a more complete analogy with the result (1.1) for $\mathbf{T}^k$ we will need G and its metric to satisfy the following: there exists $\lambda > 0$ such that for all $s \in [0,1]$ there exists $r \in [0,1]$ with

$$s \leq A(r) \leq \lambda s. \tag{2.4}$$

A wide class of groups, including the connected compact Lie groups, can be equipped with compatible metrics so that this condition is satisfied.

Whenever $E \subseteq \hat{G}$, define $|E|_2 = (\sum_{\gamma\in E} d(\gamma)^2)^{1/2}$.

2.4 THEOREM. *Let G be a compact metric group and suppose $\alpha, \beta \in \mathbf{R}$. Consider the following inequality: there exists a constant $K = K(\alpha,\beta)$ such that*

$$\left(\sum_{\gamma\in E} d(\gamma)\mathrm{tr}(\hat{f}(\gamma)\hat{f}(\gamma)^*)\right)^{1/2} \leq K|E|_2^{2\alpha}||w^\beta f||_2 \tag{2.5}$$

for all finite $E \subseteq \hat{G}$ and all $f \in L^1(G)$.

(i) *The inequality is valid for $\{(\alpha,\beta)\colon\ \alpha \geq 0,\ \beta < 1/2$ and $\beta < \alpha\}$ and $\{(\alpha,\beta)\colon\ \alpha = 0,\ \beta \leq 0\}$.*

(ii) *If G also satisfies Assumption 2.3 the inequality continues to hold when $0 < \alpha = \beta < 1/2$.*

2.5 REMARK. When $G = \mathbf{T}^k$ the function $w = A \circ \phi$ can be chosen to equal $|x|^k$. Furthermore, in this case $|E|_2$ reduces to $|E|$.

PROOF (OF THEOREM 2.4). We first introduce spaces which are nonabelian analogues of $l^p(\mathbf{Z})$. Full details are available in Hewitt and Ross [**2**]. Let $\mathfrak{E}$ be the set of functions ψ on $\hat{G}$ with $\psi(\gamma) \in \mathcal{B}(\mathcal{H}_\gamma)$ for $\gamma \in \hat{G}$, where $\mathcal{B}(\mathcal{H}_\gamma)$ is the space of bounded linear operators on $\mathcal{H}_\gamma$. For $1 \leq p \leq \infty$, let $\mathfrak{E}_p$ be the normed subspace of $\mathfrak{E}$ as in [**2**, (28.24)]: denote the corresponding norm by $||\cdot||_p$. In particular, $||\psi||_2 = (\sum_{\gamma\in\hat{G}} d(\gamma)\mathrm{tr}(\psi(\gamma)^*\psi(\gamma)))^{1/2}$ and $||\psi||_\infty = \sup_{\gamma\in\hat{G}} ||\psi(\gamma)||$, where $||\psi(\gamma)||$ is the operator norm of $\psi(\gamma)$. Let E be a finite subset of $\hat{G}$.

(i) Throughout the proof of part (i) we assume that $\alpha, \beta \in \mathbf{R}$ satisfy $\alpha \geq 0$ and $\beta < 1/2$. Define $\psi_E \in \mathfrak{E}$ by $\psi_E(\gamma) = I_{d(\gamma)}$, the identity operator in $\mathcal{B}(\mathcal{H}_\gamma)$, when $\gamma \in E$ and 0 otherwise. For $p \in [1,\infty]$ define p' and $p^\#$ by $p' = p(p-1)^{-1}$ and $p^\# = 2p(p-2)^{-1}$.

Given $f \in L^1$, the following sequence of inequalities follows from (28.33) and (31.22) of [**2**] and Hölder's inequality:

$$\begin{aligned}\left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}(\gamma)\hat{f}(\gamma)^*)\right)^{1/2} &= \|\psi_E\hat{f}\|_2 \\ &\le \|\psi_E\|_{p^\#}\|\hat{f}\|_p \quad (\text{where } 2\le p\le\infty) \\ &\le \|\psi_E\|_{p^\#}\|f\|_{p'} \\ &\le \left(\sum_{\gamma\in E} d(\gamma)^2\right)^{1/p^\#} \|(A\circ\phi)^{-\beta}\|_{p^\#}\|(A\circ\phi)^{\beta}f\|_2.\end{aligned}$$

Let $\alpha = 1/p^\#$. By Lemma 2.2, $\|(A\circ\phi)^{-\beta}\|_{p^\#}$ is finite when $\beta p^\# < 1$ (that is, when $\beta < \alpha$) and $\alpha = 1/p^\# > 0$, and when $\beta \le 0$ and $\alpha = 0$.

In the preceding argument we required $2 \le p \le \infty$ which implies $0 \le \alpha \le 1/2$. Hence $\beta < 1/2$ is also required. Thus for the pairs $\{(\alpha,\beta)\colon 0\le\alpha\le 1/2,\ \beta < 1/2,\ \beta<\alpha\}$ and $\{(0,\beta)\colon \beta\le 0\}$ we have the required inequality with the constant $K = \|(A\circ\phi)^{-\beta}\|_{1/\alpha}$. Since $|E|_2 \ge 1$ for nonempty E, $|E|_2^{2\alpha} \le |E|_2^{2\alpha'}$ whenever $\alpha\le\alpha'$. This completes part (i) because the validity of the inequality (2.5) for a pair (α,β) implies its validity for all pairs (α',β), with $\alpha'\ge\alpha$, with the same constant.

(ii) Up until the last step, the proof of part (ii) follows that of Theorem 1′ of [**4**] or Theorem 1.1 of [**7**]. We then invoke (2.4). Given $r\in(0,1)$, let $f_1 = f1_{B_r}$ and $f_2 = f - f_1$. Then

$$\left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}(\gamma)^*\hat{f}(\gamma))\right)^{1/2} \le I_1 + I_2,$$

where

$$\begin{aligned}I_1 &= \left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}_1(\gamma)^*\hat{f}_1(\gamma))\right)^{1/2} \\ &\le |E|_2\|\hat{f}_1\|_\infty \le |E|_2\|f_1\|_1 \le |E|_2\|w^{-\beta}\|_2\|w^\beta f_1\|_2 \\ &\le |E|_2\frac{A(r)^{-\beta+1/2}}{(1-2\beta)^{1/2}}\|w^\beta f_1\|_2\end{aligned}$$

and

$$\begin{aligned}I_2 &= \left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}_2(\gamma)^*f_2(\gamma))\right)^{1/2} \\ &\le \|f_2\|_2 \le \|w^{-\beta}1_{B_r'}\|_\infty\|w^\beta f_2\|_2 \le A(r)^{-\beta}\|w^\beta f_2\|_2.\end{aligned}$$

(In both cases the final inequality follows from Lemma 2.2.) Hence

$$\left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}(\gamma)^*\hat{f}(\gamma))\right)^{1/2} \le A(r)^{-\beta}\left(\frac{A(r)^{1/2}|E|_2}{(1-2\beta)^{1/2}}+1\right)\|w^\beta f\|_2, \tag{2.6}$$

using the fact that $||w^\beta f_1||_2, ||w^\beta f_2||_2 \leq ||w^\beta f||_2$.

The proof is completed by applying inequality (2.4) (that is, Assumption 2.3) with $s = |E_2|^{-2}$. If E is nonempty (which we can of course assume), then $|E|_2 \geq 1$ and so $s = |E_2|^{-2} \leq 1$. Thus by inequality (2.4), there exists r such that $A(r) \leq \lambda |E|_2^{-2}$ and $A(r) \geq |E|_2^{-2}$. Thus $A(r)^{1/2} \leq \lambda^{1/2}|E|_2^{-1}$ and $A(r)^{-\beta} \leq |E|_2^{2\beta}$ which, upon substitution into (2.6), gives

$$\left(\sum_{\gamma\in E} d(\gamma)\operatorname{tr}(\hat{f}(\gamma)\hat{f}(\gamma)^*)\right)^{1/2} \leq |E|_2^{2\beta}\left(\frac{\lambda^{1/2}}{(1-2\beta)^{1/2}}+1\right)||w^\beta f||_2,$$

as required.

3. Compact manifolds. In this section X will denote a compact oriented riemannian manifold. A suitable reference is Warner [**10**]. Let d denote the metric on X induced by the given riemannian structure on X. Fix $x_0 \in X$ and write $\phi(x) = d(x, x_0)$ for $x \in X$.

Denote the Laplace-Beltrami operator (with respect to the given riemannian structure) on $C^\infty(X)$, the space of infinitely differentiable functions on X, by Δ. The spectrum Λ of Δ is of the form $\Lambda = \{\lambda_1, \lambda_2, \dots\}$, where $0 \leq \lambda_1 < \lambda_2 < \cdots$. Let $\mathcal{H}_\lambda$ be the eigenspace corresponding to $\lambda \in \Lambda$. Then $d(\lambda) = \dim \mathcal{H}_\lambda < \infty$ and

$$L^2(X) = \bigoplus \sum_{\lambda\in\Lambda} \mathcal{H}_\lambda.$$

Fix an orthonormal basis $\phi_\lambda^{(1)}, \dots, \phi_\lambda^{d(\lambda)}$ for each $\mathcal{H}_\lambda$ and define $c(\lambda)$ by

$$c(\lambda) = \max\{||\phi_\lambda^{(j)}||_\infty \colon j \in \{1, \dots, d(\lambda)\}\}.$$

(For simplicity we suppress the fact that $c(\lambda)$ depends upon the chosen basis.)

For each subset $E \subseteq \Lambda$ denote the orthogonal projection of $L^2(X)$ onto $\bigoplus\sum_{\lambda\in E}\mathcal{H}_\lambda$ by P_E. (When E is a singleton $\{\lambda\}$, denote P_E by P_λ.) Suppose $0 \leq \theta < 1/2$ and define K_θ by

$$K_\theta = ||(A\circ\phi)^{-\theta}||_2,$$

where $A(r)$ is the volume (in the canonical riemannian measure induced by the riemannian structure) of the set $\{x \in X\colon d(x, x_0) \leq r\}$. As in Lemma 2.2, $K_\theta < \infty$ since $0 \leq \theta < 1/2$.

Let $f \in L^2(X)$. Then

$$\begin{aligned} ||P_\lambda f||_2^2 &= \sum_{j=1}^{d(\lambda)} \left(\int_X \overline{f\phi_\lambda^{(j)}}\right)^2 \\ &\leq c(\lambda)^2 d(\lambda)\left(\int_X |f|\right)^2 \leq K_\theta^2 c(\lambda)^2 d(\lambda)||(A\circ\phi)^\theta f||_2^2. \end{aligned}$$

Hence, whenever $E \subseteq \Lambda$,

$$\begin{aligned} ||P_E f||_2^2 &\leq K_\theta^2 \sum_{\lambda\in E} c(\lambda)^2 d(\lambda)||(A\circ\phi)^\theta f||_2^2 \\ &= K_\theta^2 \mu(E)^2 ||(A\circ\phi)^\theta f||_2^2, \end{aligned}$$

where $\mu(E) = (\sum_{\lambda\in E} d(\lambda)c(\lambda)^2)^{1/2}$.

In summary, with notation as above,

$$\|P_E f\|_2 \leq K_\theta \mu(E) \|(A \circ \phi)^\theta f\|_2 \tag{3.1}$$

for $f \in L^2(X)$, $E \subseteq \Lambda$ and $0 \leq \theta < 1/2$ where $K_\theta < \infty$, a local uncertainty inequality directly analogous to (1.1) and Theorem 2.4.

3.1 THE TWO-DIMENSIONAL SPHERE. Suppose $X = S^2$, the two-dimensional sphere, with the usual riemannian structure. In spherical coordinates

$$S^2 = \{(\alpha, \beta)\colon\ 0 \leq \alpha < 2\pi,\ 0 \leq \beta \leq \pi\}$$

with the usual identifications. The eigenvalues of the Laplace-Beltrami operator are $n(n+1)$ with $n \in \mathbf{N} = \{0\} \cup \mathbf{Z}^+$ and the corresponding eigenspaces $\mathcal{H}_{n(n+1)}$ have dimension $2n+1$ [8]. Let $\{Y_n^m\colon\ -n \leq m \leq n,\ m \in \mathbf{Z}\}$ be the associated spherical functions: they form a column of entry functions for the usual description of the $(2n+1)$-dimensional representation of $SU(2)$ [9] and thus satisfy $\|Y_n^m\|_\infty \leq 1$.

The functions $\{(2n+1)^{1/2} Y_n^m\colon\ -n \leq m \leq n,\ m \in \mathbf{Z}\}$ make up an orthonormal basis for $\mathcal{H}_{n(n+1)}$ and so, with respect to this basis, $c_n = c(n(n+1)) \leq (2n+1)^{1/2}$. The metric $d(\cdot,\cdot)$ with respect to the usual riemannian structure on S^2 is just the euclidean distance along great circles. Define ϕ on S^2 by $\phi(\alpha,\beta) = \beta$, that is, the geodesic distance between the pole $(0,0)$ and (α,β). Then

$$(A \circ \phi)(\alpha, \beta) = 2\pi(1 - \cos\beta),$$

the surface area of the cap $\{(\alpha, \beta')\colon\ 0 \leq \alpha < 2\pi,\ 0 \leq \beta' \leq \beta\}$. Suppose $E \subseteq N$; with the above notation, (3.1) becomes

$$\|P_E f\|_2 \leq K_\theta \left(\sum_{n \in E} (2n+1)^2 \right)^{1/2} \int_{S^2} (2\pi(1 - \cos\beta))^{2\beta} |f(\alpha,\beta)|^2 \, d\mu$$

for $f \in L^2(S^2)$ and $0 \leq \theta < 1/2$ where

$$\begin{aligned} K &= \left(\int_{S^2} |A \circ \phi|^{-2\theta} \right)^{1/2} \\ &= \int_0^{2\pi} \int_0^{\pi} (2\pi(1 - \cos\beta))^{-2\theta} \sin\beta \, d\beta \, d\alpha \\ &= 2(2\pi)^{-2\theta}(1 - 2\theta)^{-1} \end{aligned}$$

and $d\mu$ is the (riemannian) measure given by $d\mu = \sin\beta \, d\beta \, d\alpha$.

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