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The structure of generic subintegrality

LES REID, LESLIE G ROBERTS* and BALWANT SINGH†

Department of Mathematics, Southwest Missouri State University, Springfield, MO 65804, USA

*Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada, K7L 3N6

†School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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Abstract. In order to give an elementwise characterization of a subintegral extension of $\mathbb{Q}$-algebras, a family of generic $\mathbb{Q}$-algebras was introduced in [3]. This family is parametrized by two integral parameters $p \geqslant 0$, $N \geqslant 1$, the member corresponding to p, N being the subalgebra $R = \mathbb{Q}[\{\gamma_n | n \geqslant N\}]$ of the polynomial algebra $\mathbb{Q}[x_1, \ldots, x_p, z]$ in $p+1$ variables, where $\gamma_n = z^n + \sum_{i=1}^p \binom{n}{i} x_i z^{n-i}$. This is graded by weight $(z) = 1$, weight $(x_i) = i$, and it is shown in [2] to be finitely generated. So these algebras provide examples of geometric objects. In this paper we study the structure of these algebras. It is shown first that the ideal of relations among all the γ_n's is generated by quadratic relations. This is used to determine an explicit monomial basis for each homogeneous component of R, thereby obtaining an expression for the Poincaré series of R. It is then proved that R has Krull dimension $p+1$ and embedding dimension $N+2p$, and that in a presentation of R as a graded quotient of the polynomial algebra in $N+2p$ variables the ideal of relations is generated minimally by $\binom{N+p}{2}$ elements. Such a minimal presentation is found explicitly. As corollaries, it is shown that R is always Cohen–Macaulay and that it is Gorenstein if and only if it is a complete intersection if and only if $N+p \leqslant 2$. It is also shown that R is Hilbertian in the sense that for every $n \geqslant 0$ the value of its Hilbert function at n coincides with the value of the Hilbert polynomial corresponding to the congruence class of n.

Keywords. Subintegral extensions; subrings of polynomial rings.

Introduction

Let $A \subseteq B$ be an extension of commutative rings containing the rational numbers $\mathbb{Q}$. In [3] an element $b \in B$ is defined to be subintegral over A if there exist integers $p \geqslant 0$, $N \geqslant 1$ and $c_1, \ldots, c_p \in B$ such that $g_n := b^n + \sum_{i=1}^p \binom{n}{i} c_i b^{n-i} \in A$ for all integers $n \geqslant N$. With this definition the extension $A \subseteq B$ is subintegral in the sense of Swan [7] if and only if every element of B is subintegral over A [3, §4].

In [3] the tuple $(0, p, N; 1, c_1, \ldots, c_p)$ with the above properties was called a system of subintegrality for b over A. There was an extra parameter s which we can take to be 0 in the present discussion, and the 1 represents c_0. In [3] we assumed that

$N \geqslant s+p$. Here (as in [4]) we adopt the conventions that for any element b in a ring, $b^0 = 1$ and $\binom{n}{i} b^{n-i} = 0$ if $i > n$. Then it suffices to assume that $N \geqslant 1$. By [3, proof of (4.2) (iv)$\Rightarrow$(i)] (note also [4, (1.1)]) if b has a system of subintegrality for some $N \geqslant 1$, then b has a system of subintegrality with $N = 1$. Systems with $N > 1$ are still of interest, however, since freedom in the choice of N may result in a simpler system of subintegrality.

Let $x_1, \ldots, x_p, z$ be independent indeterminates over $\mathbb{Q}$, and let $x_0 = 1$. For $n \geqslant 0$ let $\gamma_n = \sum_{i=0}^{p} \binom{n}{i} x_i z^{n-i}$ and let $R^{(N)} := \mathbb{Q}[\{\gamma_n | n \geqslant N\}] \subseteq S := \mathbb{Q}[x_1, \ldots, x_p, z]$. Then z is subintegral over $R^{(N)}$ with system of subintegrality $(0, p, N; 1, x_1, \ldots, x_p)$. Furthermore this setup is universal for subintegral elements together with their systems of subintegrality, in the sense that given any extension of commutative $\mathbb{Q}$-algebras $A \subseteq B$ with $b \in B$ having a system of subintegrality $(0, p, N; 1, c_1, \ldots, c_p)$, the homomorphism $\varphi: S \to B$ given by $\varphi(x_i) = c_i$ and $\varphi(z) = b$ satisfies $\varphi(\gamma_n) = g_n$ and $\varphi(R^{(N)}) \subseteq A$. Such universal extensions played a crucial role in [3].

The rings $R^{(N)}$ have an interesting algebraic structure, which we discuss in the present paper. First of all $R^{(N)}$ and S are graded by weight $(x_i) = i$, weight $(z) = 1$, which imply that weight $(\gamma_n) = n$. In §1 we find relations (1.2) of degree two (but not necessarily homogeneous) among the γ_n, where degree means $\deg(\gamma_n) = 1$ for all $n \geqslant 1$, and is to be distinguished from weight. We show in (2.2) that these quadratic relations generate the ideal of all relations. These quadratic relations include those used in [2] to prove that $R^{(N)}$ is a $\mathbb{Q}$-algebra of finite type, although in [2] we did not find a complete set of relations. In (2.1) we use the quadratic relations to obtain an explicit monomial basis for $R_k^{(N)}$, the weight k part of $R^{(N)}$, from which we obtain in (2.8) the Poincaré series of $R^{(N)}$ for arbitrary p and N (generalizing both [4, (4.4)], which handles the case $N = 1$, and [4, (4.7)], which is the case $p = 1$, N arbitrary).

In §3 we use the quadratic relations to eliminate all but a finite number of the γ_n, obtaining thereby our main result (3.2) which gives a minimal presentation of $R^{(N)}$ as a graded $\mathbb{Q}$-algebra of finite type. Of course, after eliminating these variables, the relations among the remaining variables are no longer all quadratic. From (3.2) we derive several corollaries ((3.3)–(3.7)) on the nature of $R^{(N)}$: (3.5) says that $R^{(N)}$ is always Cohen–Macaulay, which was a surprise to us; (3.6) says that $R^{(N)}$ is Gorenstein if and only if it is a complete intersection if and only if $N + p \leqslant 2$.

In §4 we give an alternative proof of the linear independence of our basis for $R_k^{(N)}$. This method is more complicated but also more precise than the argument of §2.

We conclude the paper by studying in §5 the Hilbert function of $R^{(N)}$. We find the minimal number d of Hilbert polynomials needed to express the Hilbert function of $R^{(N)}$, and show that if $p \geqslant 2$ then $R^{(N)}$ is Hilbertian, meaning that the value of its Hilbert function at n coincides with the value of the Hilbert polynomial corresponding to the congruence class of n modulo d, for every $n \geqslant 0$ (rather than just for $n \gg 0$).

The non-negative integers are denoted by $\mathbb{Z}^+$, and $\lfloor a \rfloor$ is the integral part of the real number a (i.e. the largest integer $\leqslant a$).

1. The quadratic relations

Let $R^{(N)} \subseteq S$ be the universal extension as defined above. Let T be an indeterminate over S, and let $F(T)=1+\sum_{i=1}^{p}\binom{T}{i}x_i z^{-i}$ (so that $\gamma_n = z^n F(n)$). Then we have the following (generalizing [2, (1.2)]).

Theorem 1.1. *Let k be an integer $\geqslant 2p$, and let $0 \leqslant d_1 < d_2 < \cdots < d_{p+1} \leqslant k/2$ be any $p+1$ distinct integers. Let d be any integer $0 \leqslant d \leqslant k/2$, distinct from the d_i. Then*

$$\gamma_d \gamma_{k-d} = \sum_{i=1}^{p+1} a_i \gamma_{d_i} \gamma_{k-d_i} \tag{1.2}$$

for some rational numbers a_i.

Proof. Note that we have $d_i < k - d_i$ $(1 \leqslant i \leqslant p)$, $d_{p+1} \leqslant k - d_{p+1}$, and the $p+1$ pairs $(d_i, k-d_i)$ are distinct (as unordered pairs). First consider the case $d_{p+1} < k - d_{p+1}$ so that each pair $(d_i, k-d_i)$ consists of two distinct integers. Let $I = \{d_1, \ldots, d_{p+1}, k-d_{p+1}, \ldots, k-d_1\}$. For $p+2 \leqslant i \leqslant 2p+2$ define $d_i = k - d_{2p+3-i}$, so that $I = \{d_i\}_{1 \leqslant i \leqslant 2p+2}$. The set I contains $2p+2$ distinct integers. For $1 \leqslant i \leqslant 2p+2$ let π_i be the interpolating polynomial of degree $2p+1$, which is 1 at d_i and 0 at the remaining elements of I. Let $G(x) = \sum_{i=1}^{2p+2} \pi_i(x) F(d_i) F(k-d_i)$ and $H(x) = F(x)F(k-x)$. Then $G(c) = H(c)$ for all $c \in I$. The polynomial $G(x)$ is of degree $\leqslant 2p+1$ in x, whereas $H(x)$ is of degree $2p$ in x. These two polynomials (with coefficients in the integral domain $\mathbb{Q}[x_1, \ldots, x_p, z^{-1}]$) agree at $2p+2$ values of x, hence are equal. Setting $x = d$, $a_i = \pi_i(d) + \pi_{2p+3-i}(d)$ $(1 \leqslant i \leqslant p+1)$ and multiplying by z^k yields (1.2).

Now consider the case $d_{p+1} = k - d_{p+1}$. Let $I = \{d_1, \ldots, d_{p+1}, k-d_p, \ldots, k-d_1\}$. For $p+2 \leqslant i \leqslant 2p+1$ define $d_i \leqslant k - d_{2p+2-i}$, so that $I = \{d_i\}_{1 \leqslant i \leqslant 2p+1}$. The set I contains $2p+1$ distinct integers. For $1 \leqslant i \leqslant 2p+1$ let π_i be the interpolating polynomial of degree $2p$, which is 1 at d_i and 0 at the remaining elements of I. Let $G(x) = \sum_{i=1}^{2p+1} \pi_i(x) F(d_i) F(k-d_i)$ and $H(x) = F(x)F(k-x)$. Then $G(c) = H(c)$ for all $c \in I$. The polynomials $G(x)$ and $H(x)$ are both of degree $\leqslant 2p$ in x. These two polynomials (with coefficients in the integral domain $\mathbb{Q}[x_1, \ldots, x_p, z^{-1}]$) agree at $2p+1$ values of x, hence are equal. Setting $x = d$, $a_i = \pi_i(d) + \pi_{2p+2-i}(d)$ $(1 \leqslant i \leqslant p)$, $a_{p+1} = \pi_{p+1}(d)$, and multiplying by z^k yields (1.2). ∎

COROLLARY 1.3.

(a) *If $k \geqslant 2p$ then the monomials of degree $\leqslant 2$ and weight k in the γ_i span a vector space $V_{k,2}$ of dimension $p+1$, and any set of $p+1$ distinct monomials of degree $\leqslant 2$ is a basis for this vector space.*

(b) *If $k \leqslant 2p+1$ then any set of distinct monomials of degree $\leqslant 2$ and weight k is linearly independent.*

(c) *In any relation* (1.2) *all the a_i are uniquely determined and nonzero.*

Proof. The monomials $\gamma_k, \gamma_1\gamma_{k-1}, \ldots, \gamma_d\gamma_{k-d}$ $(d = \min(\lfloor k/2 \rfloor, p)$ are linearly independent by [4, proof of (4.1)] from which (b) follows. It also follows that if $k \geqslant 2p$ then $V_{k,2}$ is of dimension $\geqslant p+1$, and by (1.2) any $p+1$ elements span. Thus (for $k \geqslant 2p$) $\dim V_{k,2} = p+1$, and (a) and (c) follow. (Note that (c) is vacuous unless $k \geqslant 2p+2$.) ∎

Examples 1.4. Here are a few examples of the quadratic relations (obtained using a computer program that we wrote):

for $p=1$:

(1.4.1) $\gamma_4 = 4\gamma_1\gamma_3 - 3\gamma_2^2$

(1.4.2) $\gamma_5 = 3\gamma_1\gamma_4 - 2\gamma_2\gamma_3$

(1.4.3) $\gamma_1\gamma_5 = 4\gamma_2\gamma_4 - 3\gamma_3^2$

and for $p=2$:

(1.4.4) $\gamma_8 = 20\gamma_2\gamma_6 - 64\gamma_3\gamma_5 + 45\gamma_4^2$

(1.4.5) $\gamma_9\gamma_1 = 20\gamma_3\gamma_7 - 64\gamma_4\gamma_6 + 45\gamma_5^2$

(1.4.6) $\gamma_{10} = (63/5)\gamma_2\gamma_8 - (128/5)\gamma_3\gamma_7 + 14\gamma_4\gamma_6$

These examples illustrate the following.

Theorem 1.5. (1) *The quadratic relations are translation-invariant, i.e. if*

$$\gamma_d\gamma_{k-d} = \sum_{i=1}^{p+1} a_i\gamma_{d_i}\gamma_{k-d_i}$$

then also

$$\gamma_{d+j}\gamma_{k-d+j} = \sum_{i=1}^{p+1} a_i\gamma_{d_i+j}\gamma_{k-d_i+j}$$

for any integer $j \geqslant 0$ *(with the same* a_i*). (Homogenize by putting in* γ_0 *if necessary.)*

(2) *If the* d_i *are consecutive integers, then the coefficients* a_i *in* (1.2) *are integers.*

Proof. (1) In (1.1) replace d_i by $d_i' = d_i + j$ $(1 \leqslant i \leqslant p+1)$, d by $d+j$ and k by $k+2j$. Then also d_i is replaced by $d_i' = d_i + j$ $(p+2 \leqslant i \leqslant 2p+2$ or $p+2 \leqslant i \leqslant 2p+1$ respectively in the two parts of the proof of (1.1)). Formula (1.2) becomes

$$\gamma_{d+j}\gamma_{k-d+j} = \sum_{i=1}^{p+1} a_i'\gamma_{d_i+j}\gamma_{k-d_i+j}$$

where $a_i' = \pi_i'(d+j) + \pi_{2p+3-i}'(d+j)$ for $1 \leqslant i \leqslant p+1$ (respectively $a_i' = \pi_i'(d+j) + \pi_{2p+2-i}'(d+j)$ for $1 \leqslant i \leqslant p$ and $a_{p+1}' = \pi_{p+1}'(d+j)$), π_i' being the interpolating polynomial of degree $2p+1$ (respectively degree $2p$) which is 1 at d_i', and 0 at the remaining d_j'. Obviously $\pi_i'(c+j) = \pi_i(c)$ for all real numbers c, from which it follows that $a_i' = a_i$ for all i, proving (1).

(2) If the d_i $(1 \leqslant i \leqslant 2p+2$, resp. $1 \leqslant i \leqslant 2p+1$ in the two cases) are consecutive integers, then the Lagrange formula for the π_i (when evaluated at any integer) is (up to sign) the product of two binomial coefficients. Thus the $\pi_i(d)$ are integers, hence also the a_i, proving (2). ∎

Example (1.4.6) shows that in general the a_i need not be integers. We can arrange to have the d_i consecutive by taking $c = \lfloor k/2 \rfloor$ and $\{\gamma_c\gamma_{k-c}, \gamma_{c-1}\gamma_{k-c+1}, \ldots, \gamma_{c-p}\gamma_{k-c+p}\}$ as the set of quadratic monomials on the right-hand side of (1.2).

2. The Poincaré series of $R^{(N)}$

Determining the Poincaré series of $R^{(N)}$ is essentially the same as determining the dimension of the $\mathbb{Q}$-vector space $R_k^{(N)}$, the weight k part of $R^{(N)}$, for every k. In fact, we do more. Namely, using a basis interchange technique, we find in the following theorem an explicit monomial basis for $R_k^{(N)}$.

Theorem 2.1. $R_k^{(N)}$ *has* $\mathbb{Q}$-*basis*

$$\mathscr{B}_{N,k}=\left\{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d}\,|\,N\leqslant i_1\leqslant\cdots\leqslant i_{d-1}\leqslant i_d,\ i_{d-1}<N+p\quad\text{if}\quad d>1,\ \sum_{j=1}^{d} i_j=k\right\}.$$

Proof. If $p=0$ the result is trivial. For then $\gamma_i=z^i$ for all i and $R^{(N)}=\mathbb{Q}[z^i\,|\,i\geqslant N]$. If $k\geqslant N$ then $R_k^{(N)}$ has basis γ_k and $\mathscr{B}_{N,k}$ contains only γ_k, since we must have $d=1$. If $k=0$ then $R_0^{(N)}$ has basis $\gamma_0=1$ and $\mathscr{B}_{N,0}$ contains only the empty product 1 since we must have $d=0$. If $0<k<N$ then $R_k^{(N)}=0$ and $\mathscr{B}_{N,k}$ is empty. Hence assume $p\geqslant 1$. First consider the case $N=1$. In [4, (4.1)] a basis $\{z^kG_t'(k)\,|\,t\in\mathscr{T}_k\}$ for $R_k^{(1)}$ (there denoted simply as R_k) is obtained. The definition of this basis is quite technical, so we will not recall its definition completely. It suffices to note that $\mathscr{T}_k$ is a set of integers indexing all sequences of the form $a_t=(\alpha_1,\alpha_2,\ldots,\alpha_k)$ with $0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_k\leqslant p$, $\alpha_{k-1}=\alpha_k$, and $\sum\alpha_i\leqslant k$. Also, in the proof of [4, (4.1)] the above basis is put in one-to-one correspondence with another basis of $R_k^{(1)}$ that consists of monomials in the γ's. Under this bijection, $z^kG_t'(k)$, for $a_t=(\alpha_1,\alpha_2,\ldots,\alpha_k)$, corresponds to $\gamma_{\alpha_1}\cdots\gamma_{\alpha_{k-1}}\gamma_{\beta_k}$, where $\beta_k\geqslant\alpha_k$ is chosen so as to make the weight $\alpha_1+\cdots+\alpha_{k-1}+\beta_k=k$ (remember that some of the α's can be 0, and that $\gamma_0=1$). But (omitting the γ_0's, renumbering the remaining γ's and noting that $N+p-1=p$) this is just the basis $\mathscr{B}_{1,k}$ claimed for $N=1$ in the statement of the theorem.

Now, for general N, if $\gamma_i\gamma_j$ is a factor of a monomial in the γ's of weight k, with i and j both $\geqslant N+p$, then the quadratic relations (1.2) can be used to replace $\gamma_i\gamma_j$ by a linear combination of

$$\gamma_{i+j},\ \gamma_N\gamma_{i+j-N},\ \gamma_{N+1}\gamma_{i+j-N-1},\ldots,\gamma_{N+p-1}\gamma_{i+j-N-p+1}$$

(note that $i+j-N-p+1\geqslant N+p-1\geqslant N$) from which it follows that $\mathscr{B}_{N,k}$ spans $R_k^{(N)}$. Thus it suffices to prove the linear independence of $\mathscr{B}_{N,k}$. This we prove by induction on N. The idea is to produce a basis for $R_k^{(N-1)}$ that contains $\mathscr{B}_{N,k}$ as a subset.

Hence suppose that $N\geqslant 2$ and $\mathscr{B}_{N-1,k}$ is a basis for $R_k^{(N-1)}$. We have $\mathscr{B}_{N-1,k}=\{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d}\,|\,N-1\leqslant i_1\leqslant\cdots\leqslant i_{d-1}\leqslant i_d,\ i_{d-1}<N+p-1$ if $d>1, \sum_{j=1}^d i_j=k\}$. Let $\mathscr{C}=\mathscr{B}_{N,k}\cap\mathscr{B}_{N-1,k}$ (=those elements of $\mathscr{B}_{N-1,k}$ that do not contain any γ_{N-1}'s). Let $\mathscr{E}_0$ be the set of those elements of $\mathscr{B}_{N-1,k}$ which contain a certain number of γ_{N-1}'s, say $e\geqslant 1$ of them, and which have the largest subscript i_d satisfying $i_d-ep\geqslant N+p-1$. Let $\mathscr{E}$ be obtained (elementwise) from $\mathscr{E}_0$ by replacing each γ_{N-1} by γ_{N+p-1} and decreasing the highest subscript accordingly. The theorem follows from

(2.1.1) *Claim*

(2.1.2) $\mathscr{B}_{N,k}=\mathscr{C}\amalg\mathscr{E}$

(2.1.3) $(\mathscr{B}_{N-1,k}-\mathscr{E}_0)\cup\mathscr{E}$ is a basis for $R_k^{(N-1)}$.

Proof of (2.1.2). Obviously $\mathscr{C}\subseteq\mathscr{B}_{N,k}$ and $\mathscr{E}\subseteq\mathscr{B}_{N,k}$. Furthermore, any element of $\mathscr{B}_{N,k}$ that contains $e>1$ γ_{N+p-1}'s (or one γ_{N+p-1} and one γ with subscript $>N+p-1$) is obtained uniquely by the above transformation from an element of $\mathscr{E}_0$, and any element of $\mathscr{B}_N$ that contains at most one γ_{N+p-1} and has all other subscripts

$<N+p-1$ is in $\mathscr{C}$. Thus $\mathscr{B}_{N,k} \subseteq \mathscr{C} \cup \mathscr{E}$. It is obvious that $\mathscr{C} \cap \mathscr{E} = \varnothing$, which proves (2.1.2).

Proof of (2.1.3). Let

$$\mathscr{B}' = \left\{ \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d} \mid N-1 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d,\ i_{d-1} < N+p \text{ if } d>1,\ \sum_{j=1}^{d} i_j = k \right\},$$

and let $\mathscr{E}'_0$ be the set of those elements of $\mathscr{B}'$ which contain a certain number of γ_{N-1}'s, say $e \geqslant 1$ of them, and which have the largest subscript i_d satisfying $i_d - ep \geqslant N+p-1$. Then $\mathscr{B}_{N-1,k} \cup \mathscr{B}_{N,k} \subseteq \mathscr{B}'$ and $\mathscr{E}_0 \subseteq \mathscr{E}'_0$. Let $\rho: \mathscr{E}'_0 \amalg \mathscr{B}_{N,k} \to \mathscr{E}'_0 \amalg \mathscr{B}_{N,k}$ be the map which is identity on $\mathscr{B}_{N,k}$ and is defined on $\mathscr{E}'_0$ as follows: if $\gamma \in \mathscr{E}'_0$ then write $\gamma = \gamma_{N-1}\delta\gamma_c$ with $c \geqslant N+2p-1$ and δ a monomial in $\gamma_{N-1}, \gamma_N, \ldots, \gamma_{N+p-1}$, and define $\rho(\gamma) = \delta\gamma_{N+p-1}\gamma_{c-p}$. Further, for such a $\gamma = \gamma_{N-1}\delta\gamma_c \in \mathscr{E}'_0$ define

$$S(\gamma) = \{\delta\gamma_0\gamma_{c+N-1}, \delta\gamma_N\gamma_{c-1}, \delta\gamma_{N+1}\gamma_{c-2}, \ldots, \delta\gamma_{N+p-2}\gamma_{c-p+1}\}.$$

Put $\mathscr{D}_0 = \mathscr{E}_0 \amalg \mathscr{C}$, and for $i \geqslant 1$ let $\mathscr{D}_i = \{\rho(\gamma) \mid \gamma \in \mathscr{D}_{i-1}\}$. Then each $\mathscr{D}_i$ is a subset of $\mathscr{E}'_0 \amalg \mathscr{B}_{N,k}$, ρ is a bijection from $\mathscr{D}_i$ onto $\mathscr{D}_{i-1}$, and $\mathscr{E} \amalg \mathscr{C} = \mathscr{D}_i$ for $i \gg 0$. Let

$$\mathscr{D}_{ij} = \{\gamma \in \mathscr{D}_i \mid \gamma_{N-1} \text{ appears exactly to power } j \text{ in } \gamma\}.$$

Then $\mathscr{D}_i = \amalg_{j \geqslant 0} \mathscr{D}_{ij}$, and for $i, j \geqslant 1$ we have $\rho(\mathscr{D}_{i-1,j}) \subseteq \mathscr{D}_{i,j-1}$ with equality if $j \geqslant 2$. Let $\gamma \in \mathscr{D}_{ij}$ with $j \geqslant 1$. We claim that $S(\gamma) \subseteq \mathscr{D}_{i,j-1}$. This is clear for $i=0$. If $i \geqslant 1$ then $\gamma = \rho(\beta)$ with $\beta \in \mathscr{D}_{i-1,j+1}$, and clearly $S(\gamma) = \{\rho(\alpha) \mid \alpha \in S(\beta)\}$. So the claim follows by induction on i. Now, the set $\{\gamma, \rho(\gamma)\} \cup S(\gamma)$ has $p+2$ elements, and by (1.1) and (1.3) (c) any $p+1$ of these elements form a basis for the vector space spanned by this set. So, as $S(\gamma) \subseteq \mathscr{D}_{i,j-1}$, the sets $\{\gamma\} \cup \mathscr{D}_{i,j-1}$ and $\{\rho(\gamma)\} \cup \mathscr{D}_{i,j-1}$ span the same vector space. Therefore, since $\mathscr{D}_i$ can be obtained from $\mathscr{D}_{i-1}$ in stages by changing

$$(\amalg_{j \geqslant h+1} \rho(\mathscr{D}_{i-1,j})) \cup (\amalg_{j=0}^{h} \mathscr{D}_{i-1,j}) \text{ to } (\amalg_{j \geqslant h} \rho(\mathscr{D}_{i-1,j})) \cup (\amalg_{j=0}^{h-1} \mathscr{D}_{i-1,j}),$$

starting with the highest h, it follows that each $\mathscr{D}_i$ spans the same space. In particular, $\mathscr{E}_0 \amalg \mathscr{C}$ and $\mathscr{E} \amalg \mathscr{C} = \mathscr{B}_{N,k}$ span the same space. The former being a part of a basis for $\mathscr{B}_{N-1,k}$, (2.1.3) is proved. ■

COROLLARY 2.2.

The ideal of all relations among the γ's is generated by the quadratic relations (1.2).

Proof. Only the relations (1.2) were used to reduce the set of all monomials of weight k in the γ's to the basis $\mathscr{B}_{N,k}$. ■

COROLLARY 2.3.

Let $V_{k,d}$ be the subspace of $R_k^{(N)}$ spanned by monomials of weight k and degree $\leqslant d$ in the γ_i ($\deg \gamma_i = 1$ for all $i \geqslant 1$) as in [4, § 2]. *Then $V_{k,d}$ has $\mathbb{Q}$-basis of those monomials in $\mathscr{B}_{N,k}$ of degree $\leqslant d$.*

Proof. The indicated elements are linearly independent since they are part of the basis $\mathscr{B}_{N,k}$. Therefore it suffices to prove that they span $V_{k,d}$. To do this we may assume that $p \geqslant 1$. If $\gamma_i\gamma_j$ is a factor of a monomial in the γ's of weight k, and degree $\leqslant d$ with i and j both $\geqslant N+p$, then as in the proof of (2.1) the quadratic relations (1.2) can be used to replace $\gamma_i\gamma_j$ by a linear combination of $\gamma_{i+j}, \gamma_N\gamma_{i+j-N}, \gamma_{N+1}\gamma_{i+j-N-1}, \ldots, \gamma_{N+p-1}\gamma_{i+j-N-p+1}$ (note that $i+j-N-p+1 \geqslant N+p-1 \geqslant N$ and that the quadratic replacement does not increase degree), from which it follows that the claimed elements span $V_{k,d}$. ■

COROLLARY 2.4. (cf. [4, (2.1)])

We have $\dim V_{k,d} = \binom{p+d-1}{p}$ *for* $k \gg 0$. *More precisely*, $\dim V_{k,d} = \binom{p+d-1}{p}$ *if and only if* $k \geqslant m$, *where* m *is defined as follows:* (1) *if* $p \geqslant 1$ *and* $d \geqslant 2$ *then* $m = (N+p-1)d$; (2) *if* $d = 0$ *then* $m = 1$; (3) *in all other cases* $m = N$ *or* $m = 0$ *accordingly as* $N > 1$ or $N = 1$.

Proof. Case (2) is trivial. For, if $d = 0$ then $\binom{p+d-1}{p} = 0$, and the only product of degree zero is the empty product which is 1. So assume that $d \geqslant 1$. Then if the γ_i of highest weight is removed from each element of the basis of $V_{k,d}$ described in (2.3), this basis is put in one-to-one correspondence with a subset of the monomials of degree less than or equal to $d-1$ in the p variables $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p-1}$. If k is large enough we obtain in this manner all monomials of degree less than or equal to $d-1$ in $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p-1}$. Since there are $\binom{p+d-1}{p}$ such monomials, the first part is proved. Assume now that we are in case (1), i.e. $p \geqslant 1$ and $d \geqslant 2$. Then a monomial M of degree $\leqslant d-1$ in $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p-1}$ corresponds to an element of our basis if and only if $k - wt(M)$ is bigger than or equal to any subscript occurring in M. The most critical case is γ_{N+p-1}^{d-1} which requires $k-(d-1)(N+p-1) \geqslant N+p-1$, or $k \geqslant (N+p-1)d = m$, proving case (1). The proof of case (3) is an easy and straightforward verification. ■

Example 2.5. Here is an example to illustrate the algorithm in the proof of (2.1). Let $N = p = 2$. Then $\dim_{\mathbb{Q}} R_{11}^{(1)} = 31$, $\dim_{\mathbb{Q}} R_{11}^{(2)} = 10$. Monomials in the γ's will be represented by listing the subscripts, thus $(1,1,2,7)$ represents $\gamma_1^2\gamma_2\gamma_7$. We have $\mathscr{E}_0 = \{(1,10), (1,1,9), (1,2,8), (1,1,2,7), (1,2,2,6)\}$ and $\mathscr{C} = \{(11), (2,9), (2,2,7), (2,2,2,5), (2,2,2,2,3)\}$. To understand the example it is not necessary to list the elements of $\mathscr{B}_{1,11} - \mathscr{E}_0$ explicitly. We have $\mathscr{D}_0 = \mathscr{E}_0 \amalg \mathscr{C} = \mathscr{D}_{00} \amalg \mathscr{D}_{01} \amalg \mathscr{D}_{02}$ with

$$\mathscr{D}_{00} = \{(11), (2,9), (2,2,7), (2,2,2,5), (2,2,2,2,3)\},$$
$$\mathscr{D}_{01} = \{(1,10), (1,2,8), (1,2,2,6)\} \text{ and } \mathscr{D}_{02} = \{(1,1,9), (1,1,2,7)\}.$$

The following table shows how the transformation proceeds using the linear relation among $\gamma, \rho(\gamma)$ and $S(\gamma)$:

$\gamma =$	Replaced by $\rho(\gamma) =$	Using $S(\gamma) =$
(1, 1, 9)	(1, 3, 7)	(1, 10), (1, 2, 8)
(1, 1, 2, 7)	(1, 2, 3, 5)	(1, 2, 8), (1, 2, 2, 6)
(1, 10)	(3, 8)	(11), (2, 9)
(1, 2, 8)	(2, 3, 6)	(2, 9), (2, 2, 7)

(1, 2, 2, 6)	(2, 2, 3, 4)	(2, 2, 7), (2, 2, 2, 5)
(1, 3, 7)	(3, 3, 5)	(3, 8), (2, 3, 6)
(1, 2, 3, 5)	(2, 3, 3, 3)	(2, 3, 6), (2, 2, 3, 4)

The first two rows show how $\mathscr{D}_{02}$ is transformed into $\rho(\mathscr{D}_{02})$ and the next three rows show how $\mathscr{D}_{01}$ is transformed into $\rho(\mathscr{D}_{01})$. This gives $\mathscr{D}_1 = \mathscr{D}_{10} \amalg \mathscr{D}_{11}$ with $\mathscr{D}_{11} = \rho(\mathscr{D}_{02}) = \{(1,3,7),(1,2,3,5)\}$ and $\mathscr{D}_{10} = \rho(\mathscr{D}_{01}) \amalg \mathscr{D}_{00} = \{(3,8),(2,3,6),(2,2,3,4),(11),(2,9),(2,2,7),(2,2,2,5),(2,2,2,2,3)\}$. Finally, the last two rows show how $\mathscr{D}_{11}$ is transformed into $\rho(\mathscr{D}_{11})$, giving $\mathscr{D}_2 = \mathscr{D}_{20} = \rho(\mathscr{D}_{11}) \amalg \mathscr{D}_{10} = \{(3,3,5),(2,3,3,3),(3,8),(2,3,6),(2,2,3,4),(11),(2,9),(2,2,7),(2,2,2,5),(2,2,2,2,3)\} = \mathscr{E} \amalg \mathscr{C} = \mathscr{B}_{2,11}$. Note that for fixed i, j the order in which elements of $\mathscr{D}_{ij}$ are transformed into those of $\rho(\mathscr{D}_{ij})$ is immaterial.

The basis of $V_{3,11}$ given by (2.3) is $\{(11),(2,9),(3,8),(2,2,7),(2,3,6),(3,3,5)\}$. ■

The calculation of the Poincaré series is now just a matter of counting $\mathscr{B}_{N,k}$. The number of partitions of k as sums of integers each $\geqslant N$ and $\leqslant N+p-1$ is the coefficient of t^k in

$$\frac{1}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}. \tag{2.6}$$

Allowing one integer $> N+p-1$ is the same as finding the partitions of the integers from 0 to $k-N-p$ as sums of integers each $\geqslant N$ and $\leqslant N+p-1$ (adding one more integer, which will be greater than $N+p-1$, to each partition to bring the sum up to k), and the number of such partitions is the coefficient of t^k in

$$\frac{t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}. \tag{2.7}$$

Adding (2.6) and (2.7) yields

Theorem 2.8. *Let $P(t)$ be the Poincaré series for the ring $R^{(N)}$, i.e. $P(t) = \sum_{k=0}^{\infty} H(k)t^k$, where $H(k) = \dim_{\mathbb{Q}} R_k^{(N)}$. Then*

$$P(t) = \frac{1-t+t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}.$$ ■

By a similar argument, using x to keep track of the number of terms added, we obtain that dim $V_{k,d}$ is the coefficient of $x^d t^k$ in

$$\frac{1-t+xt^{N+p}}{(1-t)(1-x)(1-xt^N)(1-xt^{N+1})\cdots(1-xt^{N+p-1})}. \tag{2.9}$$

3. Relations ideal and the structure of $R^{(N)}$

In this section we determine the structure of $R^{(N)}$ by finding a minimal presentation for it as a graded $\mathbb{Q}$-algebra. We show that $R^{(N)}$ has Krull dimension $p+1$ and embedding dimension $N+2p$, and that in a presentation of $R^{(N)}$ as a graded quotient of the polynomial algebra in $N+2p$ variables the ideal of relations is generated minimally by $\binom{N+p}{2}$ elements. As corollaries, we show that $R^{(N)}$ is always Cohen–Macaulay; that $R^{(N)}$ is Gorenstein if and only if it is a complete intersection if and only if $N+p\leqslant 2$ (which happens exactly in the three cases $p=0$, $N=1$; $p=0$, $N=2$; $p=1=N$); and that $R^{(N)}$ is regular if and only if $p=0$, $N=1$.

Let $B=\mathbb{Q}[T_N, T_{N+1},\dots, T_{2N+2p-1}]$ be the polynomial ring in $N+2p$ variables graded by weight $(T_i)=i$, and let $\varphi\colon B\to R^{(N)}$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(T_i)=\gamma_i$ for all i. Let $A=\mathbb{Q}[T_N, T_{N+1},\dots, T_{N+p}]$, let M be the A-submodule of B generated by $1, T_{N+p+1},\dots, T_{2N+2p-1}$ and let $M'=\varphi(M)$. Then M' is the A'-submodule of $R^{(N)}$ generated by $1,\gamma_{N+p+1},\dots,\gamma_{2N+2p-1}$, where $A'=\mathbb{Q}[\gamma_N,\gamma_{N+1},\dots,\gamma_{N+p}]$. (We will see later that $M'=R^{(N)}$.)

Lemma 3.1. *We have $\gamma_i\in M'$ and $\gamma_i\gamma_j\in M'$ for all $i,j\geqslant N$.*

Proof. We prove the first part by induction on i. Clearly we have $\gamma_i\in M'$ for $N\leqslant i\leqslant 2N+2p-1$. Let $i\geqslant 2N+2p$. Then $i-N-p\geqslant N+p$ so by (1.2) γ_i belongs to the $\mathbb{Q}$-span of $\gamma_N\gamma_{i-N}, \gamma_{N+1}\gamma_{i-N-1},\dots,\gamma_{N+p}\gamma_{i-N-p}$. Now $\gamma_{i-N},\gamma_{i-N-1},\dots,\gamma_{i-N-p}\in M'$ by induction, since $i>i-N\geqslant i-N-p\geqslant N$. Therefore $\gamma_i\in M'$, and the first part is proved. Now, if at least one of i and j is $\leqslant N+p$ then $\gamma_i\gamma_j\in M'$ by the first part. On the other hand, if both i and j are $>N+p$ then $i+j-N-p+1>N+p-1$ so by (1.2) $\gamma_i\gamma_j$ belongs to the $\mathbb{Q}$-span of $\gamma_{i+j},\gamma_N\gamma_{i+j-N},\dots,\gamma_{N+p-1}\gamma_{i+j-N-p+1}$ (just γ_{i+j} if $p=0$) and these $p+1$ monomials belong to M' by the first part. So $\gamma_i\gamma_j\in M'$. ∎

By the Lemma we can write, for $i,j\geqslant N+p+1$, $\gamma_i\gamma_j=\alpha'+\sum_{h=N+p+1}^{2N+2p-1}\beta'_h\gamma_h$ with $\alpha',\beta'_h\in A'$. We may assume that α',β'_h are homogeneous of appropriate weight so that the expression is homogeneous of weight $i+j$. Lift α',β'_h to homogeneous elements α,β_h of A of the same weight and let

$$P_{ij}=T_iT_j-\alpha-\sum_{h=N+p+1}^{2N+2p-1}\beta_hT_h.$$

Then P_{ij} is homogeneous of weight $i+j$.

Theorem 3.2. *The graded $\mathbb{Q}$-algebra $R^{(N)}$ has Krull dimension $p+1$ and embedding dimension $N+2p$, and has a minimal presentation with $N+2p$ generators and $\binom{N+p}{2}$ relations. More precisely, the $\mathbb{Q}$-algebra homomorphism $\varphi\colon B\to R^{(N)}$ is surjective and the ideal $\ker(\varphi)$ of B is generated minimally by the $\binom{N+p}{2}$ elements P_{ij}, $N+p+1\leqslant i\leqslant j\leqslant 2N+2p-1$.*

Proof. By [2,(1.4)], or by (3.1) above, $R^{(N)}$ is generated by $\gamma_N,\gamma_{N+1},\dots,\gamma_{2N+2p-1}$. This means that φ is surjective, and $R^{(N)}$ is a $\mathbb{Q}$-algebra of finite type. Now, since the quotient field of $R^{(N)}$ is $\mathbb{Q}(x_1,\dots,x_p,z)$ by [4,(5.2)], we get $\dim(R^{(N)})=p+1$. (That $\dim(R^{(N)})=p+1$ also follows independently from (3.3) below.)

We show next that the set $\{P_{ij}\mid N+p+1\leqslant i\leqslant j\leqslant 2N+2p-1\}$ generates $\ker(\varphi)$ minimally. To do this, let I be the ideal of B generated by this set.

Minimality. Since the P_{ij} are homogeneous, it is enough to show that no P_{ij} belongs to the ideal generated by the remaining ones. Suppose for some i,j we have $P_{ij}=\sum_{(r,s)\neq(i,j)} f_{rs}P_{rs}$ with $f_{rs}\in B$. We may assume that each f_{rs} is homogeneous with weight $(f_{rs})=i+j-r-s$ (negative weight means the element is zero). Let $Q_{rs}=P_{rs}-T_rT_s$. Then

$$T_iT_j+Q_{ij}=\sum_{(r,s)\neq(i,j)} f_{rs}(T_rT_s+Q_{rs}).$$

Since $N+p+1\leqslant i,j\leqslant 2N+2p-1$ and Q_{ij} is of degree at most one in $T_{N+p+1},\ldots,T_{2N+2p-1}$, the term T_iT_j is present on the left hand side. Let us look for this term on the right hand side. First of all, T_iT_j cannot appear in any of the terms $f_{rs}T_rT_s$ because $(r,s)\neq(i,j)$ is an unordered pair. It follows that T_iT_j must come from one of the terms $f_{rs}Q_{rs}$. Since $N+p+1\leqslant i,j\leqslant 2N+2p-1$ and Q_{rs} is of degree at most one in $T_{N+p+1},\ldots,T_{2N+2p-1}$, in order for T_iT_j to appear in the term $f_{rs}Q_{rs}$ it is necessary for f_{rs} to contain a term which is a nonzero rational times T_i or T_j or T_iT_j. Accordingly, we would get $i+j-r-s=\text{weight }(f_{rs})=i$ or j or $i+j$ whence $r+s=j$ or i or 0. This is a contradiction, since $r+s\geqslant 2N+2p+2$. This proves the minimality of the generators.

Generation. By construction, we have $I\subseteq\ker(\varphi)$. So we have the surjective map $\psi\colon B/I\to R^{(N)}$ induced by φ. We have to show that ψ is an isomorphism. Note that M is a free A-module of rank $N+p$, with basis $T_0:=1, T_{N+p+1},\ldots,T_{2N+2p-1}$. The module M is graded by weight $(T_i)=i$. Let $\zeta\colon M\to B/I$ be the restriction of the natural map $B\to B/I$ to M. Given any polynomial in B, we can reduce it modulo I to an element of M. This means that ζ is surjective. Now, let $\sigma=\psi\circ\zeta$. Then $\sigma\colon M\to R^{(N)}$ is an A-linear map which is homogeneous of degree zero and is surjective. Now, denoting by $P_L(t)$ the Poincaré series of a graded A-module L and writing $R=R^{(N)}$, it is enough to prove that $P_R(t)=P_M(t)$. For, since σ is surjective, this would show that σ is an isomorphism whence also ψ is an isomorphism. Now, by (2.8) we have

$$P_R(t)=\frac{1-t+t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}.$$

On the other hand, since A is the polynomial ring $\mathbb{Q}[T_N,T_{N+1},\ldots,T_{N+p}]$ with weight $(T_i)=i$, we have

$$P_A(t)=\frac{1}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p})}.$$

Therefore, since M is A-free with basis $1, T_{N+p+1},\ldots,T_{2N+2p-1}$ and weight $(T_i)=i$, we get

$$P_M(t)=P_A(t)+\sum_{i=N+p+1}^{2N+2p-1} t^iP_A(t)=\frac{1+t^{N+p+1}+t^{N+p+2}+\cdots+t^{2N+2p-1}}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p})}.$$

Now, it is checked readily that $P_R(t)=P_M(t)$. This completes the proof of the equality $I=\ker(\varphi)$.

Finally, we show that the embedding dimension of $R^{(N)}$ is $N+2p$. Recall that for a finitely generated graded ring $C=\oplus_{k\geqslant 0}C_k$ with C_0 a field its embedding dimension emdim(C) is the minimal number of homogeneous C_0-algebra generators of C, or equivalently the minimal number of homogeneous generators of the ideal $C_+=\oplus_{k\geqslant 1}C_k$. In our situation we have $R^{(N)}=B/I$ with I generated by the P_{ij},

$N+p+1 \leqslant i \leqslant j \leqslant 2N+2p-1$. For such i, j we have $i+j \geqslant 2N+2p+2$. Therefore in the expression $P_{ij} = T_i T_j - \alpha - \sum_{h=N+p+1}^{2N+2p-1} \beta_h T_h$ we have $\alpha \in A_+^2$ and each $\beta_h \in A_+$. This shows that $I \subseteq B_+^2$. Therefore by (graded) Nakayama the minimal number of homogeneous generators of the ideal $R_+^{(N)}$ of $R^{(N)}$ is the same as that of the ideal B_+ of B, which is $N+2p$, since B is the polynomial ring in $N+2p$ variables. This proves that $\operatorname{emdim}(R^{(N)}) = N+2p$. ■

COROLLARY 3.3.

The ring $A' = \mathbb{Q}[\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}]$ *is the polynomial ring in* $p+1$ *variables over* $\mathbb{Q}$, *and* $R^{(N)}$ *is a finite free* A'*-module with basis* $1, \gamma_{N+p+1}, \ldots, \gamma_{2N+2p-1}$.

Proof. The restriction of the isomorphism $\sigma: M \to R^{(N)}$ to A is a $\mathbb{Q}$-algebra isomorphism of A onto A', sending T_i to γ_i $(N \leqslant i \leqslant N+p)$. This implies the first part. The second part follows since $\sigma(T_i) = \gamma_i$ $(i = 0$ or $N+p+1 \leqslant i \leqslant 2N+2p-1)$. ■

COROLLARY 3.4.

A $\mathbb{Q}$*-basis for* $R^{(N)}$ *in terms of monomials in* $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{2N+2p-1}$ *is*

$$\{\gamma_N^{q_N} \gamma_{N+1}^{q_{N+1}} \cdots \gamma_{2N+2p-1}^{q_{2N+2p-1}} \mid q_{N+p+1} + q_{N+p+2} + \cdots + q_{2N+2p-1} \leqslant 1\}.$$

Consequently, a $\mathbb{Q}$*-basis for* $R_k^{(N)}$ *is*

$$\left\{ \gamma_N^{q_N} \gamma_{N+1}^{q_{N+1}} \cdots \gamma_{2N+2p-1}^{q_{2N+2p-1}} \mid q_{N+p+1} + q_{N+p+2} + \cdots + q_{2N+2p-1} \leqslant 1, \quad \sum_{j=N}^{2N+2p-1} N_j q_j = k \right\}$$

which can also be written, for comparison with (2.1), *as*

$$\left\{ \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_d} \mid N \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d \leqslant 2N+2p-1, \quad i_{d-1} \leqslant N+p \text{ if } d > 1, \ \sum_{j=1}^{d} i_j = k \right\}.$$

Proof. Immediate from (3.3). ■

COROLLARY 3.5.

The sequence $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}$ *is* $R^{(N)}$*-regular, and the ring* $R^{(N)}$ *is Cohen–Macaulay.*

Proof. The regularity of the sequence is immediate from (3.3). Therefore the localization of $R^{(N)}$ at the irrelevant maximal ideal $R_+^{(N)}$ of $R^{(N)}$ is Cohen–Macaulay. It is well known that this implies that $R^{(N)}$ is Cohen-Macaulay (e.g. [1, (33.27)]). ■

COROLLARY 3.6.

The following three conditions are equivalent:
(1) $R^{(N)}$ *is Gorenstein;* (2) $R^{(N)}$ *is a complete intersection;* (3) $N+p \leqslant 2$.

Note that, since $N \geqslant 1$, (3) occurs in exactly the following three cases: $p=0$, $N=1$; $p=0$, $N=2$; $p=1=N$.

Proof. (1)⇔(3): Since $R^{(N)}$ is graded, it is well known that $R^{(N)}$ is Gorenstein if and only if its localization at the irrelevant maximal ideal $R_+^{(N)}$ is Gorenstein (e.g. [1, (33.27)]). Let C denote this localization and put $D=C/(\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p})$. Then, since C is Cohen–Macaulay and $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}$ is a regular C-sequence by (3.5), C is Gorenstein if and only if D is Gorenstein. Let $\mathfrak{m}$ be the maximal ideal of D. Then, since $\dim(D)=0$, D is Gorenstein if and only if $\mathrm{ann}(\mathfrak{m})$, the annihilator of $\mathfrak{m}$, is a 1-dimensional space over $D/\mathfrak{m}$. Now, it follows from (3.2) that $\mathfrak{m}$ is generated minimally by $\delta_{N+p+1}, \ldots, \delta_{2N+2p-1}$, where δ_i denotes the natural image of γ_i in D. Consider two cases:

Case 1: $\mathfrak{m}=0$. In this case D is Gorenstein, and this case occurs $\Leftrightarrow \{\delta_{N+p+1}, \ldots, \delta_{2N+2p-1}\}=\varnothing \Leftrightarrow 2N+2p \leqslant N+p+1 \Leftrightarrow N+p \leqslant 1$.

Case 2: $\mathfrak{m} \neq 0$. Then $\mathrm{ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. If $N+p+1 \leqslant i, j \leqslant 2N+2p-1$ then, as noted in the proof of (3.2), we have $P_{ij}=T_iT_j-\alpha-\sum_{h=N+p+1}^{2N+2p-1} \beta_h T_h$ with $\alpha \in A_+^2$ and each $\beta_h \in A_+$. It follows that $\mathfrak{m}^2=0$. Thus $\mathfrak{m} \subseteq \mathrm{ann}(\mathfrak{m})$ whence $\mathrm{ann}(\mathfrak{m})=\mathfrak{m}$. So D is Gorenstein $\Leftrightarrow \mathfrak{m}$ is generated by one element $\Leftrightarrow 2N+2p-1=N+p+1 \Leftrightarrow N+p=2$.

(2)⇔(3): Since $\dim(R^{(N)})=p+1$ and $R^{(N)}=B/I$ with I minimally generated by $\binom{N+p}{2}$ homogeneous elements, $R^{(N)}$ is a complete intersection if and only if $N+2p=p+1+\binom{N+p}{2}$. The solutions of this equation with integers $p \geqslant 0$, $N \geqslant 1$ are exactly those given by $N+p \leqslant 2$. ■

COROLLARY 3.7.

The ring $R^{(N)}$ is regular if and only if $p=0$, $N=1$.

Proof. $R^{(N)}$ is regular $\Leftrightarrow \mathrm{emdim}(R^{(N)})=\dim(R^{(N)}) \Leftrightarrow N+2p=p+1 \Leftrightarrow N+p=1 \Leftrightarrow p=0$, $N=1$. ■

Example 3.8. We illustrate the structure theorem (3.2) by computing P_{ij} explicitly in the cases $p=1=N$ and $p=1$, $N=2$.

First, let $p=1=N$. In this case $B=\mathbb{Q}[T_1, T_2, T_3]$, $A=\mathbb{Q}[T_1, T_2]$, $A'=\mathbb{Q}[\gamma_1, \gamma_2]$, M' is the A'-module generated by $1, \gamma_3$, and there is only one relation P_{33}. To find it we have to express γ_3^2 as an A'-linear combination of $1, \gamma_3$. We do this by eliminating γ_4, γ_5 among the relations (1.4.1), (1.4.2), (1.4.3) obtaining

$$\gamma_3^2=3\gamma_1^2\gamma_2^2-4\gamma_2^3-4\gamma_1^3\gamma_3+6\gamma_1\gamma_2\gamma_3$$

as the desired linear combination. So $P_{33}=T_3^2-3T_1^2T_2^2+4T_2^3+4T_1^3T_3-6T_1T_2T_3$ and $R^{(1)} \cong \mathbb{Q}[T_1, T_2, T_3]/(T_3^2-3T_1^2T_2^2+4T_2^3+4T_1^3T_3-6T_1T_2T_3)$.

A similar computation for the case $p=1$, $N=2$ gives

$$R^{(2)} \cong \mathbb{Q}[T_2, T_3, T_4, T_5]/(P_{44}, P_{45}, P_{55})$$

with $P_{44}=T_4^2-(8/3)T_2T_3^2+3T_2^2T_4-(4/3)T_3T_5$, $P_{45}=T_4T_5+12T_3^3-16T_2T_3T_4+3T_2^2T_5$ and $P_{55}=T_5^2-32T_2^2T_3^2+36T_2^3T_4+9T_3^2T_4-14T_2T_3T_5$.

4. The independence of $\mathscr{B}_{N,k}$

In this section we give a new proof of the linear independence of $\mathscr{B}_{N,k}$, which does not depend upon the proof of (2.1). The matrix approach used here gives additional insight into the nature of $R^{(N)}$. In particular, we obtain a sharpening of the independence part of (2.1), in that we prove that a specific minor of a certain matrix is nonzero. Our matrix theoretic techniques are perhaps of interest in their own right.

Before stating our result precisely (Theorem (4.1)) we would like to describe more carefully the relationship between the two bases $\mathscr{B}_{1,k}$ and $\mathscr{G}_k := \{z^k G_t'(k) \mid t \in \mathscr{T}_k\}$ of $R_k^{(1)}$. In §2 we noted that $\mathscr{T}_k$ is a set of integers indexing (as t ranges over $\mathscr{T}_k$) all sequences of the form $a_t = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant p$, $\alpha_{k-1} = \alpha_k$, $\sum \alpha_i \leqslant k$. In [4, §3] we also introduced monomials $b_t = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}$ (with $x_0 = 1$). If we wish to write an element M of $\mathscr{B}_{1,k}$ (or more generally, any monomial M of weight k in the γ_i) as a linear combination of $\mathscr{G}_k$ we just expand M in terms of monomials b_t. Then the coefficient of $z^k G_t'(k)$ in M is the rational coefficient of b_t (ignoring the power of z). See [4, (3.7) (5)], and for some explicit examples [4, (4.3)]. We shall think of the basis element $z^k G_t'(k)$ as also being indexed by the monomial b_t.

Put $\mathscr{B}'_{N,k} = \{x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} \mid 0 \leqslant q \leqslant p,\ a_q \geqslant 2,\ \sum_{i=1}^{q} (i+N-1)a_i \leqslant k\}$. Then $\mathscr{B}'_{N,k}$ has the same cardinality as $\mathscr{B}_{N,k}$. An explicit bijection between $\mathscr{B}'_{N,k}$ and $\mathscr{B}_{N,k}$ is given by $x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} \leftrightarrow \gamma_N^{a_1} \gamma_{N+1}^{a_2} \cdots \gamma_{N+q-2}^{a_{q-1}} \gamma_{N+q-1}^{a_q-1} \gamma_\varepsilon$, where $\varepsilon \geqslant N+q-1$ is chosen to yield weight k. Give the set $\{x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} \mid 0 \leqslant q \leqslant p, a_q \geqslant 2\}$ the reverse lexicographic order and let $\mathscr{B}'_{N,k}$ have the induced order. Let $\mathscr{B}_{N,k}$ be given the order corresponding to that of $\mathscr{B}'_{N,k}$ under the above-mentioned bijection between $\mathscr{B}_{N,k}$ and $\mathscr{B}'_{N,k}$. This done, let ζ be the matrix over $\mathbb{Q}$ whose ij entry is the coefficient of the jth element of $\mathscr{G}_k$ in the expression of the ith element of $\mathscr{B}_{N,k}$ written as a $\mathbb{Q}$-linear combination of $\mathscr{G}_k$. The linear independence of $\mathscr{B}_{N,k}$ follows immediately from the following theorem.

Theorem 4.1. *Let $p \geqslant 0$ and let ζ be the matrix (with entries in $\mathbb{Q}$) defined above. Let η be the submatrix of ζ consisting of the columns corresponding to $\mathscr{B}'_{N,k}$. Then* $\det(\eta) \neq 0$.

Our first attempt to prove the linear independence of the $\mathscr{B}_{N,k}$ was by proving (4.1), but this turned out to be somewhat elusive. So we ended up proving (2.1) using the basis interchange technique given in §2. However, we were still intrigued by the equality $\operatorname{card}(\mathscr{B}_{N,k}) = \operatorname{card}(\mathscr{B}'_{N,k})$, and we were finally able to prove (4.1), showing that this equality is not a coincidence. This gives an independent, but more difficult, proof of (the linear independence part of) (2.1). In the proof of [4, (4.1)] (the case $N = 1$) the matrix ζ ($=\eta$ in this case) was triangular with nonzero entries down the diagonal so non-singularity was easy to establish. We have not been able to find such a simple argument in the case $N > 1$.

The following example will help explain the meaning of (4.1), as well as illustrate (2.3).

Example 4.2. Let $N = p = 2$, $k = 10$. Then $\mathscr{B}'_{2,10} = \{1, x_1^2, x_1^3, x_1^4, x_1^5, x_2^2, x_1 x_2^2, x_1^2 x_2^2, x_2^3\}$ and in the corresponding order $\mathscr{B}_{2,10} = \{\gamma_{10}, \gamma_2\gamma_8, \gamma_2^2\gamma_6, \gamma_2^3\gamma_4, \gamma_2^5, \gamma_3\gamma_7, \gamma_2\gamma_3\gamma_5, \gamma_2^2\gamma_3^2, \gamma_3^2\gamma_4\}$. Then $V_{10,1}$ has basis $\{\gamma_{10}\}$, $V_{10,2}/V_{10,1}$ has basis $\{\gamma_2\gamma_8, \gamma_3\gamma_7\}$, $V_{10,3}/V_{10,2}$ has basis $\{\gamma_2^2\gamma_6, \gamma_2\gamma_3\gamma_5, \gamma_3^2\gamma_4\}$, $V_{10,4}/V_{10,3}$ has basis $\{\gamma_2^3\gamma_4, \gamma_2^2\gamma_3^2\}$ and $V_{10,5}/V_{10,4}$ has basis $\{\gamma_2^5\}$. The complete list of monomials corresponding to $\mathscr{G}_k$ is $\{1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6, x_1^7, x_1^8, x_1^9, x_1^{10}, x_2^2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2, x_1^4x_2^2, x_1^5x_2^2, x_1^6x_2^2, x_2^3, x_1x_2^3, x_1^2x_2^3, x_1^3x_2^3, x_1^4x_2^3, x_2^4, x_1x_2^4, x_1^2x_2^4, x_2^5\}$ so the matrix ζ is 9 by 26. Monomials of degree greater than 5 can be omitted since all entries in their columns will be 0. This leaves $\{1, x_1^2, x_1^3, x_1^4, x_1^5, x_2^2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2,$

$x_2^3, x_1x_2^3, x_1^2x_2^3, x_2^4, x_1x_2^4, x_2^5\}$ so the non-trivial part of ζ is 9 by 15. We shall not write this matrix down, but the possibly nonzero entries by degree considerations (a row of degree d can have nonzero entries only in a column of degree $\leqslant d$) are indicated by *'s, and only the subscript digits are indicated for the row indices (x being 10). The column indices of $\mathscr{B}'_{2,10}$ (i.e. the columns of η) are underlined.

	$\underline{1}$	$\underline{x_1^2}$	$\underline{x_1^3}$	$\underline{x_1^4}$	$\underline{x_1^5}$	$\underline{x_2^2}$	$\underline{x_1x_2^2}$	$\underline{x_1^2x_2^2}$	$x_1^3x_2^2$	$\underline{x_2^3}$	$x_1x_2^3$	$x_1^2x_2^3$	x_2^4	$x_1x_2^4$	x_2^5
x	*														
28	*	*				*									
226	*	*	*			*	*			*					
2224	*	*	*	*		*	*	*		*	*		*		
22222	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
37	*	*				*									
235	*	*	*			*	*			*					
2233	*	*	*	*		*	*	*		*	*		*		
334	*	*	*			*	*			*					

Theorem (4.1) in this case is sharper than (2.1) in that there are several other maximal minors that could be nonzero.

The proof of (4.1) will now occupy the rest of this section. The various constructions involved are illustrated by Example (4.12) below, to which the reader might refer while working through the proof. Suppose $p=0$. Then $\mathscr{G}_k=\{z^k\}$ and $\mathscr{B}'_{1,k}=\{1\}$. Further, $\mathscr{B}_{N,k}=\varnothing$ if $0<k<N$ and $\mathscr{B}_{N,k}=\{\gamma_k\}$ otherwise. So ζ is either the 0×1 empty matrix or the 1×1 identity matrix, and (4.1) holds trivially in either case. Similarly, (4.1) is trivial in case $k=0$. Assume therefore that $p\geqslant 1$ and $k\geqslant 1$. The integers p, k and $N\geqslant 1$ are fixed in what follows. Let $d=\lfloor (k/N)\rfloor$. In the notation of [4, (3.5)] let $\mathscr{A}'=\{(\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d \mid 0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_{d-1}=\alpha_d\leqslant p\}$. For $i\geqslant 1$ let a_i be the number of times i occurs in $(\alpha_1,\ldots,\alpha_d)$. Then the correspondence $(\alpha_1,\ldots,\alpha_d)\leftrightarrow(a_1,\ldots,a_p)$ identifies $\mathscr{A}'$ with the following subset U of $(\mathbb{Z}^+)^p$:

$$U=\{(0,\ldots,0)\}\cup\left\{(a_1,\ldots,a_p)\,\middle|\,\sum_{i=1}^{p} a_i\leqslant d, \exists\, j \text{ with } a_j\geqslant 2 \text{ and } a_i=0\,\forall\, i>j\right\}.$$

For $a=(a_1,\ldots,a_p)\in(\mathbb{Z}^+)^p$ define the *weight* of a to be $wt(a)=\sum_{i=1}^p (N+i-1)a_i$. Let $V=\{a\in U \mid wt(a)\leqslant k\}$. Define $V_0=W_0=\{(0,\ldots,0)\}$ and for $1\leqslant j\leqslant p$ define $V_j=\{(a_1,\ldots,a_p)\in V \mid a_j\geqslant 2 \text{ and } a_i=0\,\forall\, i>j\}$ and $W_j=\{(a_1,\ldots,a_{j-1},a_j-1,0,\ldots,0)\mid (a_1,\ldots,a_{j-1},a_j,0,\ldots,0)\in V_j\}$. Put $W=\coprod_{j=0}^p W_j$.

We use the reverse lexicographic order on U. Namely, $(a_1,\ldots,a_p)<(b_1,\ldots,b_p)$ (or $(a_1,\ldots,a_p)$ "precedes" $(b_1,\ldots,b_p)$) if the last nonzero entry of $(a_1,\ldots,a_p)-(b_1,\ldots,b_p)$ is negative. Let V and W have the induced order. This order is such that the elements of V_{j-1} (resp. W_{j-1}) precede those of V_j (resp. W_j).

Let $S=\mathbb{Q}[x_1,\ldots,x_p,T]$ and let $F(T)=\sum_{i=0}^p \binom{T}{i} x_i$ (where $x_0=1$). If $(a_1,\ldots,a_p)$ is the ith element of W define

$$F_i(T)=F(N)^{a_1}F(N+1)^{a_2}\cdots F(N+p-1)^{a_p} \times F(T-a_1N-a_2(N+1)-\cdots-a_p(N+p-1)).$$

Note that $F(n)=(\gamma_n)_{z=1}$, and that $F_i(k)$ is the ith element of $\mathscr{B}_{N,k}$ (with z set equal to 1). The reason for decreasing the last index in defining the elements of W is to take into account the adjustment of the last index to obtain weight k when defining the elements of $\mathscr{B}_{N,k}$. If $(b_1,\ldots,b_p)$ is the jth element of V then $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ is the jth element of $\mathscr{B}'_{N,k}$ (where the latter has the same order as before). Let $r=\mathrm{card}(V)=\mathrm{card}(W)$. Let $M(T)$ be the $r\times r$ matrix $(M_{ij}(T))_{1\leqslant i,j\leqslant r}$ with $M_{ij}(T)\in\mathbb{Q}[T]$ the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $F_i(T)$, where $(b_1,\ldots,b_p)$ is the jth element of V. (Note that the rows of M are indexed by W and that the columns are indexed by V.) By the discussion preceding Theorem (4.1), $M_{ij}(k)$ is the coefficient of $z^kG'_t(k)$ (t corresponding to the jth element of $\mathscr{B}'_{N,k}$) in the expansion of the ith element of $\mathscr{B}_{N,k}$. Therefore $\eta=M(k)$, so (4.1) is equivalent to $M(k)$ being invertible. If $p=1$ then $M(k)$ is lower triangular with nonzero entries down the diagonal, hence trivially invertible. The argument that follows is needed only for $p\geqslant 2$.

Note that if j corresponds to an element of V_h then

$$\deg_T M_{ij}(T)\leqslant h. \tag{4.3}$$

Therefore

$$\deg_T\det(M(T))\leqslant\delta:=\sum_{h=1}^{p}h\cdot\mathrm{card}(V_h)=\sum_{h=1}^{p}h\cdot\mathrm{card}(W_h). \tag{4.4}$$

Our intention is to show that $M(k)$ is invertible by finding δ roots for $\det(M(T))$, each less than k, and then showing that the coefficient of T^δ in $\det(M(T))$ is not identically zero. The δ roots will be found by obtaining coincidences of the rows of the matrix $M(s)$, as s ranges between N and $k-1$.

We begin by proving a few lemmas.

Lemma 4.5. *Let* $\tilde{M}(T)$ *be an* $r\times r$ *matrix with entries in* $\mathbb{Q}[T]$. *Let* $\mu\in\mathbb{Q}$. *Let* $\mathscr{R}$ *be the set of rows of* $\tilde{M}(T)$ *and let* $\mathscr{S}$ *be the set of all nonempty subsets of* $\mathscr{R}$. *Suppose there exists a subset* $\mathscr{E}$ of $\mathscr{S}$ *such that*

(1) *The sets in* $\mathscr{E}$ *are disjoint.*

(2) *For each* $E\in\mathscr{E}$, *all the rows in E coincide when T is specialized to* μ.

Let $c=c(\mathscr{E})=\sum_{E\in\mathscr{E}}(\mathrm{card}(E)-1)$. *Then* $(T-\mu)^c$ *divides* $\det(\tilde{M}(T))$.

Proof. It is clear that $\mathrm{rank}(\tilde{M}(\mu))\leqslant r-c$. By elementary row and column operations over $\mathbb{Q}[T]$ the matrix $\tilde{M}(T)$ can be reduced to a diagonal matrix $\tilde{D}(T)$ with diagonal entries $\{f_1(T),\ldots,f_r(T)\}$. (This is well known, and easily proved using that $\mathbb{Q}[T]$ is an Euclidean domain.) Then (since the same operations can be carried out with T set equal to μ) we have $\mathrm{rank}(\tilde{D}(\mu))=\mathrm{rank}(\tilde{M}(\mu))\leqslant r-c$. Thus $(T-\mu)$ divides at least c of the f_i. Since (up to a nonzero scalar) $\det(\tilde{M}(T))=\det(\tilde{D}(T))=\Pi_{i=1}^r f_i$, the lemma follows. ■

Before stating the next lemma we introduce some notation. For $a=(a_1,\ldots,a_p)\in(\mathbb{Z}^+)^p$ put $\gamma^a=\gamma_N^{a_1}\gamma_{N+1}^{a_2}\cdots\gamma_{N+p-1}^{a_p}$. Then $\mathscr{B}_{N,k}=\{\gamma^a\gamma_{k-wt(a)}\,|\,a\in W\}$. Since the rows of $M(T)$ correspond to $\mathscr{B}_{N,k}$, those of $M(s)$ correspond to $\mathscr{B}_{N,k}(s):=\{\gamma^a\gamma_{s-wt(a)}\,|\,a\in W\}$. Here the elements $\gamma^a\gamma_{s-wt(a)}$ are treated as *symbolic monomials* with $s-wt(a)$ allowed to be negative. Given symbolic monomials $\gamma^a\gamma_t$, $\gamma^b\gamma_u$ with $a,b\in(\mathbb{Z}^+)^p$ and $t,u\in\mathbb{Z}$, we say they are *formally equal* if at least one of the following two conditions holds: (1) $(a,t)=(b,u)$; (2) both t and u belong to the set $\{0\}\cup[N,N+p-1]$ and $\gamma^a\gamma_t$ and $\gamma^b\gamma_u$ coincide as formal monomials in $\gamma_N,\ldots,\gamma_{N+p-1}$ on replacing γ_0 by 1. We say that a row R of $M(s)$ is *labeled* by a symbolic monomial $\gamma^a\gamma_t$ if the symbolic monomial in

$\mathscr{B}_{N,k}(s)$ corresponding to R formally equals $\gamma^a\gamma_t$. Clearly two rows of $M(s)$ labeled by the same symbolic monomial are equal.

Let $Q_j = \{(b_1,\ldots,b_j,0,\ldots,0)\in(\mathbb{Z}^+)^p \mid b_j\neq 0\}$. For $b\in Q_j$ put $E(b)=W\cap\{b-e_i \mid 0\leqslant i\leqslant j\}$, where $e_0=(0,\ldots,0)$ and for $1\leqslant i\leqslant p$, $e_i=(0,\ldots,1,\ldots,0)$ is the standard basis vector with 1 in the ith place.

Lemma 4.6. *Let* $b\in Q_j$. *Then the rows of* $M(wt(b))$ *which are labeled by* $\gamma^b(=\gamma^b\gamma_0)$ *are precisely those indexed by* $E(b)$. *Moreover, if* b, $c\in Q_j$ *with* $b\neq c$ *and* $wt(b)=wt(c)$ *then* $E(b)\cap E(c)=\varnothing$.

Proof. It is clear that the rows of $M(wt(b))$ indexed by $E(b)$ are labeled by γ^b. Let R be a row of $M(wt(b))$ which is labeled by γ^b. Let a be the element of W corresponding to R. Then the symbolic monomial of $\mathscr{B}_{N,k}(wt(b))$ corresponding to R is $\gamma^a\gamma_{wt(b)-wt(a)}$. Comparing the subscripts and exponents of this symbolic monomial with those of γ^b we conclude that $a\in E(b)$. This proves the first part. Now, let b, $c\in Q_j$ with $wt(b)=wt(c)=s$, say. Suppose $E(b)$ and $E(c)$ have a common element, say a. Let R be the row of $M(s)$ indexed by a. Then R is labeled by γ^b as well as by γ^c whence we get $b=c$. ■

Lemma 4.7. *For an element* b *of* Q_j *the following three conditions are equivalent:* (1) $card(E(b))\geqslant 2$; (2) $b-e_i\in W$ *for some* i, $0\leqslant i<j$; (3) $b-e_j\in W$ *and* $b-e_i\in W$ *for some* i, $0\leqslant i<j$.
Moreover, if any of these conditions holds then $wt(b)<k$.

Proof. Assume (2). Then $b-e_j\in Q_h$ for some h, $i\leqslant h\leqslant j$. Since $b-e_i\in W_j$ we have $wt(b-e_j)<wt(b-e_i)\leqslant k-(N+j-1)\leqslant k-(N+h-1)$ whence $b-e_j\in W_h$. This proves (2)$\Rightarrow$(3). Also, the inequality $wt(b-e_j)<k-(N+j-1)$ gives $wt(b)<k$. The implications (1)$\Rightarrow$(2) and (3)$\Rightarrow$(1) are trivial. ■

Put $Q=\{b\in\amalg_{j=1}^p Q_j \mid \mathrm{card}(E(b))\geqslant 2\}$.

Lemma 4.8. *The product* $\amalg_{b\in Q}\,(T-wt(b))^{\mathrm{card}(E(b))-1}$ *divides* $\det(M(T))$.

Proof. Writing $Q(s)=\{b\in Q \mid wt(b)=s\}$, it is enough to prove that $\amalg_{b\in Q(s)}(T-wt(b))^{\mathrm{card}(E(b))-1}$ divides $\det(M(T))$ for every s. But this is immediate from (4.6) and (4.5), since rows labeled by the same symbolic monomial are equal. ■

Lemma 4.9. $\sum_{b\in Q}(\mathrm{card}(E(b))-1)=\delta$.

Proof. For $b\in Q\cap Q_j$ put $E'(b)=\{(b,b-e_i) \mid 0\leqslant i<j,\ b-e_i\in W\}$. It follows from (4.7) that $\mathrm{card}(E'(b))=\mathrm{card}(E(b))-1$. Let $\mathscr{E}=\amalg_{b\in Q}E'(b)$. The second projection induces a map $\eta\colon\mathscr{E}\to W$. Let $a\in W_j$ and let i be an integer with $0\leqslant i<j$. Then $a+e_i\in Q$ by (4.7). It follows that $\eta^{-1}(a)=\{(a+e_i,a) \mid 0\leqslant i<j\}$. Thus there are exactly j elements in the fibre of η over each element of W_j. Therefore we get $\sum_{b\in Q}(\mathrm{card}(E(b))-1)=\sum_{b\in Q}\mathrm{card}(E'(b))=\mathrm{card}(\mathscr{E})=\sum_{j=1}^p j\cdot\mathrm{card}(W_j)=\delta$. ■

Now, since $\deg_T\det(M(T))\leqslant\delta$ by (4.4), and since (4.7)–(4.9) taken together exhibit δ roots of $\det(M(T))$ each less than k, it remains only to show that $\det(M(T))$ is not identically zero. We do this by showing that the coefficient of T^δ is not zero. Let σ_{ij} be the coefficient in $M_{ij}(T)$ of T^h if j corresponds to an index in V_h (by (4.3) h is highest power of T with a potentially nonzero coefficient in $M_{ij}(T)$). It then suffices to show that $\det((\sigma_{ij}))\neq 0$. For $1\leqslant i,\ j\leqslant r$ (where as before M is $r\times r$) let $H_i(T)=F(T)^{a_1}F(T+1)^{a_2}\cdots F(T+p-1)^{a_p}$, where $(a_1,\ldots,a_p)$ is the ith element of W, and let τ_{ij} be the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $H_i(N)$, where $(b_1,\ldots,b_p)$ is the jth element of W. Let h be the index for which $(b_1,\ldots,b_p)\in W_h$. Then, since $(b_1,\ldots,1+b_h,0,\ldots,0)$ is the corresponding element of V_h and since $F_i(T)=H_i(N)F(T-c)$ for some integer c, we get $\sigma_{ij}=(1/h!)\tau_{ij}$. So it suffices to prove that $\det(\tau)\neq 0$ (where $\tau=(\tau_{ij})$). Rearrange the rows and columns of τ by reordering W by degree (where degree $(a_1,\ldots,a_p)=\sum a_i$). Then τ is lower-block triangular with degree blocks down the diagonal. It suffices to show that each of these blocks has a nonzero determinant. Therefore for u $(0\leqslant u\leqslant d-1)$ let S_u be the submatrix of τ with rows and columns indexed by elements of W and V of degree u. It suffices to show that $\det(S_u)\neq 0$.

The matrix S_u is obtained as follows: Let $W(u)$ be the elements of W of degree u, and let $r(u)=\mathrm{card}(W(u))$. For $1\leqslant i,\ j\leqslant r(u)$ let $(a_1,\ldots,a_p)$ be the ith element of $W(u)$ and let $(b_1,\ldots,b_p)$ be the jth element of $W(u)$. Define an $r_u\times r_u$ matrix $L_u(T)$ by setting the (i,j) entry to be the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $H_i(T)$. Then $S_u=L_u(N)$. Thus it suffices to show that N is not a root of $\det(L_u(T))$. Since we are now dealing with the homogeneous case we can replace $F(T)$ by $\tilde F(T)=\sum_{i=1}^p\binom{T}{i}x_i$ and $H_i(T)$ by $\tilde H_i(T)=\tilde F(T)^{a_1}\tilde F(T+1)^{a_2}\cdots\tilde F(T+p-1)^{a_p}$ without changing $L_u(T)$. We now note that $\tilde H_i(T)$ is divisible by $T^{a_1}(T+1)^{a_2}\cdots(T+p-1)^{a_p}$, or equivalently, the ith row of L_u is divisible by $T^{a_1}(T+1)^{a_2}\cdots(T+p-1)^{a_p}$. Factoring out these entries from the rows of $L_u(T)$ we obtain a matrix $K_u(T)$ which can be defined directly as follows: let $G(T)=\sum_{i=1}^p(1/i)\binom{T-1}{i-1}x_i$ (so that $TG(T)=\tilde F(T)$) and define $\tilde L_i(T)=G(T)^{a_1}G(T+1)^{a_2}\cdots G(T+p-1)^{a_p}$. Then the (i,j) entry of $K_u(T)$ is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $\tilde L_i(T)$. Noting that the roots of the factors $(T+i)^{a_i}$ are all $\leqslant 0$, it suffices to prove that N is not a root of $\det(K_u(T))$. In fact, $\det(K_u(T))$ is a nonzero constant, as we show next.

For $a=(a_1,\ldots,a_j,0,\ldots,0)\in Q_j$ define $aw(a)$, the *augmented weight* of a, to be $N+j-1+\sum_{i=1}^j(N+i-1)a_i$. Also define $aw(0)=0$. If $a\in W$ then $aw(a)$ is the weight of the corresponding element of V, and $aw(a)\leqslant k$ for all $a\in W$. Now, order the elements of $W(u)$ by augmented weight with small weights coming first, and order elements of the same weight by reverse lexicographic order as was done previously. This ordering is such that

(4.10) if for $j<i$ we decrease a_i by one and increase a_j by one then we get an earlier element in the ordering.

Furthermore $W(u)$ is a leading segment in the set $\tilde W(u)$ of all elements of degree u in $(\mathbb{Z}^+)^p$ (where $\tilde W(u)$ is ordered in the same manner). The matrix $K_u(T)$ can be constructed with $W(u)$ ordered in this way without changing the value of $\det(K_u(T))$.

Now we shall work with $\tilde W(u)$. Let $\tilde r(u)=\mathrm{card}(\tilde W(u))$ and let $\tilde K_u(T)$ be the $\tilde r_u\times\tilde r_u$ matrix whose (i,j) entry is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $\tilde L_i(T):=G(T)^{a_1}G(T+1)^{a_2}\cdots G(T+p-1)^{a_p}$ where $(a_1,\ldots,a_p)$ and $(b_1,\ldots,b_p)$ are respectively the ith and the jth elements of $\tilde{W}(u)$ (for convenience of notation we are changing the meaning of $\tilde{L}_i$ rather than introducing a new symbol). Let $u=1$. If we take out the factors $1/i$ from the columns then $\tilde{K}_1(T)$ is reduced to the matrix $J=\left[\binom{T+i-2}{j-1}\right]_{1\leqslant i,j\leqslant p}$. If we subtract each row of J from the next (performing the operations in the order replace pth row by pth $-(p-1)$st, replace $(p-1)$st by $(p-1)$st $-(p-2)$nd etc.) and use the binomial identities $\binom{T+i-1}{j-1}-\binom{T+i-2}{j-1}=\binom{T+i-2}{j-2}$ then J row reduces to $\begin{pmatrix}1 & * \\ 0 & J'\end{pmatrix}$ where $J'=\left[\binom{T+i-2}{j-1}\right]_{1\leqslant i,j\leqslant p-1}$. (Performing row operations in this manner was suggested to us by Sue Geller.) Continued row reduction of this type (subtracting from a row $\mathbb{Q}$-linear combinations of previous rows) will reduce $\tilde{K}_1(T)$ to an upper triangular matrix with ones down the diagonal. We conclude that $\det(J)=1$ whence $\det(\tilde{K}_1(T))=1/p!$, a nonzero constant. Now, let $E=(E_{ij})$ by any $p\times p$ matrix with entries in $\mathbb{Q}[T]$. If R_i is the ith row of E let us identify R_i with the element $E_{i1}x_1+E_{i2}x_2+\cdots+E_{ip}x_p$ of $\mathbb{Q}[T,x_1,\ldots,x_p]$. Let $f_u(E)$ be the $\tilde{r}_u\times\tilde{r}_u$ matrix whose (i,j) entry is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $R_1^{a_1}R_2^{a_2}\cdots R_p^{a_p}$, where as before $(a_1,\ldots,a_p)$ and $(b_1,\ldots,b_p)$ are respectively the ith and the jth elements of $\tilde{W}(u)$. This construction is such that $f_u(\tilde{K}_1(T))=\tilde{K}_u(T)$. Furthermore if we change E into a matrix E' by row operations of the above type (i.e. subtracting from a row $\mathbb{Q}$-linear combinations of previous rows) then because of (4.10) $f_u(E)$ is changed into $f_u(E')$ by row operations of the same type. We have that f_u of an upper triangular matrix is upper triangular, so $\tilde{K}_u(T)$ can be converted to an upper triangular matrix with nonzero constant entries down the diagonal by a succession of row operations in which from a given row we subtract a $\mathbb{Q}$-linear combination of previous rows. These row operations leave invariant the subspaces spanned by the first i rows $(1\leqslant i\leqslant\tilde{r}_u)$. Since $W(u)$ is an initial segment of $\tilde{W}(u)$ we conclude that $\det K_u(T)$ is a nonzero constant, completing the proof of (4.1). ■

Example 4.11. If $N=3$, $p=4$, then in reverse lexicographic order we have $(1,0,2,0)<(0,1,2,0)<(0,0,3,0)<(2,0,0,1)$ with augmented weights respectively 18, 19, 20, 18. Therefore if we order reverse lexicographically instead of by augmented weights the argument above will fail for $k=18$ since then $W(3)$ will not be an initial segment of $\tilde{W}(3)$.

Example 4.12. Let us return to (4.2), where $N=p=2$, $k=10$. Here we have $V_0=\{(0,0)\}$, $V_1=\{(2,0),(3,0),(4,0),(5,0)\}$, $V_2=\{(0,2),(1,2),(2,2),(0,3)\}$, $W_0=\{(0,0)\}$, $W_1=\{(1,0),(2,0),(3,0),(4,0)\}$ and $W_2=\{(0,1),(1,1),(2,1),(0,2)\}$. The rows of $M(10)$ are indexed by the monomials $\mathscr{B}_{2,10}=\{\gamma_{10},\gamma_2\gamma_8,\gamma_2^2\gamma_6,\gamma_2^3\gamma_4,\gamma_2^5,\gamma_3\gamma_7,\gamma_2\gamma_3\gamma_5,\gamma_2^2\gamma_3^2,\gamma_3^2\gamma_4\}$ as noted in (4.2). Thus the rows of $M(s)$ are indexed by $\mathscr{B}_{2,10}(s)=\{\gamma_s,\gamma_2\gamma_{s-2},\gamma_2^2\gamma_{s-4},\gamma_2^3\gamma_{s-6},\gamma_2^4\gamma_{s-8},\gamma_3\gamma_{s-3},\gamma_2\gamma_3\gamma_{s-5},\gamma_2^2\gamma_3\gamma_{s-7},\gamma_3^2\gamma_{s-6}\}$. The polynomial $\det(M(T))$ is of degree $\mathrm{card}(V_1)+2\mathrm{card}(V_2)=4+2\cdot 4=12$, and we have $Q=\{(1,0),(2,0),(3,0),(4,0),(0,1),(1,1),(2,1),(3,1),(0,2),(1,2)\}$. Taking $b=(1,0)$ we get $E(b)=\{(0,0),(1,0)\}$. This corresponds to the pair $\gamma_s,\gamma_2\gamma_{s-2}$, indexing the first two rows, which become equal when we set $s=2$. The complete set of row coincidences is obtained similarly and is given by the following table:

b	$E(b)$	elts of $\mathscr{B}_{2,10}(s)$	rows	roots of $\det(M(T))$
(1,0)	(0,0), (1,0)	$\gamma_s, \gamma_2\gamma_{s-2}$	1,2	2
(2,0)	(1,0), (2,0)	$\gamma_2\gamma_{s-2}, \gamma_2^2\gamma_{s-4}$	2,3	4
(3,0)	(2,0), (3,0)	$\gamma_2^2\gamma_{s-4}, \gamma_2^3\gamma_{s-6}$	3,4	6
(4,0)	(3,0), (4,0)	$\gamma_2^3\gamma_{s-6}, \gamma_2^4\gamma_{s-8}$	4,5	8
(0,1)	(0,0), (0,1)	$\gamma_s, \gamma_3\gamma_{s-3}$	1,6	3
(1,1)	(1,0), (0,1), (1,1)	$\gamma_2\gamma_{s-2}, \gamma_3\gamma_{s-3}, \gamma_2\gamma_3\gamma_{s-5}$	2,6,7	5,5
(2,1)	(2,0), (1,1), (2,1)	$\gamma_2^2\gamma_{s-4}, \gamma_2\gamma_3\gamma_{s-5}, \gamma_2^2\gamma_3\gamma_{s-7}$	3,7,8	7,7
(3,1)	(3,0), (2,1)	$\gamma_2^3\gamma_{s-6}, \gamma_2^2\gamma_3\gamma_{s-7}$	4,8	9
(0,2)	(0,1), (0,2)	$\gamma_3\gamma_{s-3}, \gamma_3^2\gamma_{s-6}$	6,9	6
(1,2)	(1,1), (0,2)	$\gamma_2\gamma_3\gamma_{s-5}, \gamma_3^2\gamma_{s-6}$	7,9	8

By direct computation $\det(M(T))$ turns out to be

$$2^4 3^5 (T-9)(T-8)^2(T-7)^2(T-6)^2(T-5)^2(T-4)(T-3)(T-2),$$

which is in agreement with the roots (together with multiplicities) obtained from the above table. We have that $\det(M(T))$ does not vanish at $T = 10$, as claimed.

Now we shall illustrate some features of the last part of the proof. Here $G(T) = x_1 + ((T-1)/2)x_2$, and $G(T+1) = x_1 + (T/2)x_2$, so $K_1(T) = \tilde{K}_1(T) = \begin{pmatrix} 1 & (T-1)/2 \\ 1 & (T/2) \end{pmatrix}$. We have $W(3) = \{(3,0), (2,1)\}$ and $\tilde{W}(3) = \{(3,0), (2,1), (1,2), (0,3)\}$. The respective augmented weights of the elements of $\tilde{W}(3)$ are 8 $(=4\cdot 2)$, 10 $(=2\cdot 2 + 2\cdot 3)$, 11 $(= 1\cdot 2 + 3\cdot 3)$ and 12 $(=4\cdot 3)$. The last two have weights greater than 10 and so are not included in $W(3)$. If $p = 2$ the reverse lexicographic ordering is also an ordering by weight, but this need not be the case for larger p, as we saw in (4.11). Set $R_1 = G(T)$ and $R_2 = G(T+1)$. Then the matrix $\tilde{K}_3$ has rows $\{R_1^3, R_1^2R_2, R_1R_2^2, R_2^3\}$, (or more precisely the 4×4 matrix obtained by taking the coefficients of $\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$ in these polynomials). The rows of $\tilde{K}_3$ will be denoted as $\{r_1, r_2, r_3, r_4\}$. The row operation that reduces $\tilde{K}_1(T)$ to upper triangular form is to replace $\{R_1, R_2\}$ by $\{R_1, R_2 - R_1\}$. Then $f_3(\{R_1, R_2 - R_1\})$ has row corresponding to $\{R_1^3, R_1^2(R_2 - R_1), R_1(R_2 - R_1)^2, (R_2 - R_1)^3\} = \{R_1^3, R_1^2R_2 - R_1^3, R_1R_2^2 - 2R_1^2R_2 + R_1^3, R_2^3 - 3R_1R_2^2 + 3R_1^2R_2 - R_1^3\}$ so the row operation to reduce $\tilde{K}_3$ to upper triangular form (with nonzero diagonal entries) replaces $\{r_1, r_2, r_3, r_4\}$ by $(r_1, r_2 - r_1, r_3 - 2r_2 + r_1, r_4 - 3r_3 + 3r_2 + r_1\}$. The matrix K_3 is the upper left 2×2 submatrix of $\tilde{K}_3$, to which these row operations restrict, so det $\tilde{K}_3$ is also a nonzero constant. If we had used weight 11 rather than 10, then $\tilde{K}_3$ would have been the upper left 3×3 block of K_3, which also has determinant a nonzero constant, for the same reason.

5. Hilberty polynomials

The graded ring $R^{(N)}$ has Hilbert function H given by $H(n) = \dim_{\mathbb{Q}} R_n^{(N)}$. We consider the problem of expressing $H(n)$ as one or more polynomials in n. The Hilbert function of a graded ring which is standard (i.e. finitely generated over a field by elements of weight 1) is given for $n \gg 0$ by its Hilbert polynomial. Our ring $R^{(N)}$ is finitely generated but is not standard except in the trivial case $p=0$, $N=1$. For such a ring there exist, by [5, Corollary 2], a positive integer d and polynomials $H_0, H_1, \ldots, H_{d-1}$ such that

$$H(n) = H_i(n) \quad \text{if} \quad n \gg 0 \text{ and } n \equiv i \pmod{d}. \tag{$*$}$$

In general, it is of interest to quantify precisely the condition "$n \gg 0$". In particular, in the standard case, if the Hilbert function coincides with the Hilbert polynomial for all $n \geqslant 0$ then the ring is called a *Hilbertian* ring. So we may call a general finitely generated graded ring Hilbertian if ($*$) holds for all $n \geqslant 0$. In our first result (5.1) we show that $R^{(N)}$ is Hilbertian if $p \geqslant 2$, and determine the minimal d satisfying ($*$).

If $p=0$ then $H(n)=1$ if $n=0$ or $n \geqslant N$, so in this case ($*$) holds with $d=1$, $H_0=1$, and $R^{(N)}$ is Hilbertian if and only if $N=1$.

Now, in general, to say that $R^{(N)}$ is Hilbertian is the same as saying that its Hilbert function H is a *quasi-polynomial* in the language of [6, (4.4)]. The integer d appearing in ($*$) is then a *quasi-period* of H.

Theorem 5.1. *Let* $d = \operatorname{lcm}(N, N+1, \ldots, N+p-1)$. *If* $p \geqslant 2$ *then* H *is a quasi-polynomial with minimum quasi-period* d, *and in particular* $R^{(N)}$ *is Hilbertian. If* $p=1$ *then the function* $\tilde{H}$ *given by* $\tilde{H}(n) = H(n)$ *for* $n \geqslant 1$ *and* $\tilde{H}(0) = H(0) - 1 = 0$ *is a quasi-polynomial with minimum quasi-period* d.

Proof. Let $P(t) = \sum_{n=0}^{\infty} H(n)t^n$ and $\tilde{P}(t) = \sum_{n=0}^{\infty} \tilde{H}(n)t^n$, where we put $\tilde{H} = H$ if $p \geqslant 2$. Then by (2.8) we have

$$\tilde{P}(t) = P(t) = \frac{1 - t + t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})} \quad \text{if } p \geqslant 2,$$

and $\tilde{P}(t) = P(t) - 1 = t^N/((1-t)(1-t^N))$ if $p=1$. In either case write $\tilde{P}(t) = f(t)/g(t)$ with $f(t)$, $g(t)$ polynomials without a common factor. Then $\deg f(t) < \deg g(t)$ and the zeros of $g(t)$ are the dth roots of 1. So by [6, (4.4.1)] $\tilde{H}$ is a quasi-polynomial with quasi-period d.

To prove the minimality of d, we claim first that d is the lcm of the orders of the roots of unity which occur as zeros of $g(t)$. This is clear if $p=1$. Hence assume that $p \geqslant 2$. If λ is a root of unity as well as a zero of $1-t+t^{N+p}$ then 1, $-\lambda$ and λ^{N+p} are three roots of unity whose sum is zero. This is the case if and only if $\{1, -\lambda, \lambda^{N+p}\}$ are the three cube roots of unity. Thus $-\lambda$ is a primitive cube root of unity, so λ is a primitive sixth root of unity and $\lambda^{N+p} = (-\lambda)^2$ is the other primitive cube root of unity, whence $N+p \equiv 2 \pmod{6}$. Obviously $1-t+t^{N+p}$ has no repeated factors, so if $N+p \equiv 2 \pmod{6}$ then we can cancel the cyclotomic polynomial $1-t+t^2$ of primitive sixth roots of unity once, otherwise there is no cancellation. The cancellation still leaves us with roots of unity of order 2 and 3 as zeros of $g(t)$, proving our claim.

Now let D be the minimum quasi-period. Then we can write $\tilde{P}(t)$ in the form $\tilde{P}(t) = \sum_{j=0}^{D-1} \sum_{i \geqslant 0} H_j(j+Di)t^{j+Di}$ for some polynomials H_j. Multiplying by $1-t^D$ amounts to differencing the coefficients (except in low degrees) so $(1-t^D)^e \tilde{P}(t)$ is a polynomial in t for some positive integer e. Therefore the roots of unity that occur as zeros of $g(t)$ must have orders which divide D. Thus d divides D, proving the minimality of d. ■

Theorem 5.2. *The polynomials H_i in* (5.1) *are all of degree p.*

Proof. This is seen by examining the partial fraction expansion of $\tilde{P}(t)$. We have that 1 is a root of the denominator of $\tilde{P}(t)$ of multiplicity $p+1$, and that all other roots are of smaller multiplicity. Setting $X=\lambda T$ in the well-known expansion $\frac{1}{(1-X)^r}=\sum_{n\geqslant 0}\binom{n+r-1}{r-1}X^n$ (in which the coefficient of X^n is a polynomial in n of degree $r-1$), we see that a root λ of multiplicity m of the denominator contributes a polynomial of degree $m-1$ to each of the H_j. Thus 1 contributes degree p to each H_j and the other roots contribute a lower degree, so the highest degree terms cannot cancel leaving all the H_j of degree p. ∎

Now, we give an example to show that the various H_j need not be distinct. Consider the case $N=2$, $p=3$, where our Poincaré series

$$\frac{1-t+t^5}{(1-t)(1-t^2)(1-t^3)(1-t^4)}$$

has partial fraction expansion

$$\frac{a}{(1-t)^4}+\frac{b}{(1+t)^2}+\frac{1/8}{1+t^2}+\frac{2/9}{1+t+t^2}$$

with a of degree 3 and b of degree 1 which need not be stated explicitly. The power series expansions of $1/(1+t^2)$ and $1/(1+t+t^2)$ are

$$1-t^2+t^4-t^6+t^8-\cdots$$
$$1-t+t^3-t^4+t^6-\cdots$$

of periods 4 and 3 respectively, with coefficients in each period being $1,0,-1,0$ and $1,-1,0$ respectively. The "non-polynomial" contribution to the various $H(i)$ are given by the following table (with rows corresponding to t^i for $i=0,1,2,\ldots$ and columns corresponding respectively to the roots of order $1,2,4,3$):

i				
0	1	1	1	1
1	1	−1	0	−1
2	1	1	−1	0
3	1	−1	0	1
4	1	1	1	−1
5	1	−1	0	0
6	1	1	−1	1
7	1	−1	0	−1
8	1	1	1	0
9	1	−1	0	1
10	1	1	−1	−1
11	1	−1	0	0
12	1	1	1	1

The polynomials coincide if and only if the rows are the same. By inspection of the table we see that the period is indeed 12, as given by (5.1), and that $H_1=H_7$, $H_3=H_9$, and $H_5 = H_{11}$, with the polynomials H_j $(0 \leqslant j \leqslant 11)$ being otherwise distinct. The equality of the H_j's here comes from the 0's in the power series expansion of the cyclotomic polynomial of primitive fourth roots of unity. Note that the possibilities are determined only by the columns corresponding to roots of order 4 and 3. Obviously the first column plays no role in deciding on the cases, and the second does not either since whenever entries in columns three and four are equal, so are the entries in column two.

By explicit computation we obtain $H_0(t) = 1 + \frac{t}{6} + \frac{t^2}{48} + \frac{t^3}{144}$, $H_1 = H_7 = -\frac{19}{144} + \frac{5t}{48} + \frac{t^2}{48} + \frac{t^3}{144}$, etc. with the polynomials all of degree 3 as claimed by our theorem, and with polynomials equal and distinct as claimed above. The coefficients of t^2 and t^3 are the same in all polynomials, which can be explained by the fact that only the root 1 has multiplicity greater than two, and the coefficient of t is periodic with period 2 since only the root -1 has multiplicity 2.

In another example that we have worked out, equality of the various H_j arose in a seemingly accidental way from primitive roots of unity of order other than powers of two. The general situation seems to be quite complicated.

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