ON SOME CONJECTURES ABOUT THE CHERN NUMBERS OF FILTRATIONS

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ABSTRACT. Let I be an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R,\mathfrak{m})$ of positive dimension. The coefficient $e_1(\mathcal{A})$ of the Hilbert polynomial of an I-admissible filtration $\mathcal{A}$ is called the Chern number of $\mathcal{A}$. The Positivity Conjecture of Vasconcelos for the Chern number of the integral closure filtration $\{\overline{I^n}\}$ is proved for a 2-dimensional complete local domain and more generally for any analytically unramified local ring R whose integral closure in its total ring of fractions is Cohen-Macaulay as an R-module. It is proved that if I is a parameter ideal then the Chern number of the I-adic filtration is non-negative. Several other results on the Chern number of the integral closure filtration are established, especially in the case when R is not necessarily Cohen-Macaulay.

INTRODUCTION

For a nonzero polynomial $P = P(X) \in \mathbb{Q}[X]$ of degree d such that $P(n) \in \mathbb{Z}$ for $n \gg 0$, it is customary to write P in the form

$$P = \sum_{i=0}^{d} (-1)^i e_i(P) \binom{X+d-i}{d-i}$$

with $e_i(P)$ integers, called the Hilbert coefficients of P. The top two Hilbert coefficients have special names: $e_0(P)$ is the multiplicity of P and $e_1(P)$, the subject matter of this paper, is the **Chern number** of P.

If I is an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R,\mathfrak{m})$ of positive dimension and P_I is the polynomial associated to the function $n \mapsto \lambda(R/I^{n+1})$, where λ denotes length as R-module, then the Hilbert coefficients $e_i(P_I)$ are called the Hilbert coefficients of I and are also denoted by $e_i(I)$. In particular, $e_1(I)$ is the Chern number of I.

If $\mathcal{A} = \{\mathcal{A}_n\}_{n\geq 0}$ and $\mathcal{B} = \{\mathcal{B}_n\}_{n\geq 0}$ are (decreasing) filtrations of ideals of a ring R then the **admissibility** of $\mathcal{A}$ over $\mathcal{B}$ means that there exists a nonnegative integer k such that $\mathcal{A}_{n+k} \subseteq \mathcal{B}_n \subseteq \mathcal{A}_n$ for every $n \geq 0$. We say that $\mathcal{A}$ is I-admissible, where I is an ideal of R, if $\mathcal{A}$ is admissible over the I-adic filtration.

For an ideal I of a ring R, the integral closure of I, denoted $\overline{I}$, is the ideal of R consisting of all elements of R which are integral over I, i.e. elements $a \in R$ satisfying

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an equation of the form $a^r + b_1a^{r-1} + \cdots + b_r = 0$ with r some positive integer and $b_i \in I^i$ for every i. Applying this construction to the powers I^n of an $\mathfrak{m}$-primary ideal I in a Noetherian local ring $(R, \mathfrak{m})$, we get the filtration $\{\overline{I^n}\}$ on R. If R is analytically unramified then this filtration is I-admissible by Rees [9]. It follows that the normal Hilbert function of I, namely the function $n \mapsto \lambda(R/\overline{I^{n+1}})$, is given, for $n \gg 0$, by a polynomial $\overline{P}_I$, called the **normal Hilbert polynomial** of I. The Hilbert coefficients $e_i(\overline{P}_I)$ are called the normal Hilbert coefficients of I and are also denoted by $\overline{e}_i(I)$. In particular, $\overline{e}_1(I)$ is the **normal Chern number** of I.

At a conference held in 2008 in Yokohama, Japan, Wolmer Vasconcelos [12] announced several conjectures about the the Chern number of a parameter ideal and the normal Chern number of an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m})$.

In this paper, we discuss two of these conjectures, namely the Positivity Conjecture and the Negativity Conjecture. We also provide some general estimates on the Chern number.

The Positivity Conjecture of Vasconcelos says that if I is an $\mathfrak{m}$-primary ideal of an analytically unramified Noetherian local ring $(R, \mathfrak{m})$ of positive dimension then $\overline{e}_1(I) \geq 0$.

We settle this conjecture for an analytically unramified Noetherian local ring $(R, \mathfrak{m})$ whose integral closure in its total ring of fractions is Cohen-Macaulay as an R-module. This is done in section 1. A consequence is that the Positivity Conjecture holds for a 2-dimensional complete Noetherian local domain. We also settle the conjecture in case there is a Cohen-Macaulay local ring $(S, \mathfrak{n})$ dominating $(R, \mathfrak{m})$ such that $\lambda(S/R)$ is finite.

We show in section 2 that there is a 2-dimensional analytically unramified Noetherian local ring constructed from a 1-dimensional simplicial complex for which the normal Chern number is negative. This simplicial complex is non-pure. On the other hand, we show that the normal Chern number of the maximal homogeneous ideal of the face ring of a simplicial complex Δ of dimension $d-1$ is $df_{d-1} - f_{d-2}$, where f_i is the number of i-dimensional faces of Δ. This implies that if Δ is pure then $\overline{e}_1(\mathfrak{m}) \geq 0$. These results indicate perhaps that for the Positivity Conjecture to hold, the ring needs to be quasi-unmixed, i.e. its completion $\hat{R}$ should be equidimensional.

Recall here that R is said to be unmixed if $\dim \hat{R}/\mathfrak{p} = \dim \hat{R}$ for every $\mathfrak{p} \in \operatorname{Ass} \hat{R}$.

The Negativity Conjecture of Vasconcelos says that if J is a parameter ideal of an unmixed Noetherian local ring R of positive dimension then $e_1(J) < 0$ if and only if R is not Cohen-Macaulay.

Vasconcelos [12] settled the conjecture for a domain that is essentially of finite type over a field. It was settled for a universally catenary Noetherian local domain containing a field by Ghezzi, Hong and Vasconcelos in [3]. They also proved that if S is a Cohen-Macaulay local ring and $\mathfrak{p}$ is a prime ideal of S such that $\dim S/\mathfrak{p} \geq 2$ and $S/\mathfrak{p}$ is not Cohen-Macaulay then $e_1(J) < 0$ for every parameter ideal J of $S/\mathfrak{p}$. Mandal and Verma [7] settled the Negativity Conjecture for parameter ideals in certain quotients of a regular local ring. The conjecture has been settled recently by Ghezzi, Goto, Hong, Ozeki, Phuong and Vasconcelos [2].

In section 3, we discuss the corresponding question for a finite module M (of positive dimension) over a Noetherian local ring $(R, \mathfrak{m})$ with respect to an ideal I such that $\lambda(M/IM) < \infty$. In this case, if $P_I(M, X)$ is the polynomial associated to the function $n \mapsto \lambda(M/I^{n+1}M)$, we write $e_i(I, M)$ for $e_i(P_I(M, X))$. In particular, we have the coefficient $e_1(I, M)$, which we call the Chern number of I with respect to M. While the multiplicity $e_0(I, M)$ has been studied extensively, the investigation of the Chern number $e_1(I, M)$, especially over non-Cohen-Macaulay rings, has begun only recently. We show that if J is a parameter ideal with respect to M then $e_1(J, M) \leq 0$ and, further, that $e_1(J, M) < 0$ if $\operatorname{depth} M = \dim R - 1$. We also show that if R is Cohen-Macaulay and M is an unmixed R-module with $\dim M = \dim R$ then M is Cohen-Macaulay if and only if $e_1(J, M) = 0$, for one (resp. every) parameter ideal J.

In section 4, we determine some bounds for the normal Chern number of an $\mathfrak{m}$-primary ideal in terms of a minimal reduction J of I. Using Serre's formula for multiplicity of a parameter ideal in terms of the Euler characteristic of the Koszul homology, we show that

$$\overline{e}_1(J) \leq \sum_{n \geq 1} \lambda(\overline{J^n}/J\overline{J^{n-1}}) + e_1(J).$$

This generalizes a formula of Huckaba and Marley [5] for the integral closure filtration in a Cohen-Macaulay local ring.

In the final section 5, we find some estimates on the Chern number of a prameter ideal J in a Noetherian local ring $(R, \mathfrak{m})$ assuming that there exists a Cohen-Macaulay local ring $(S, \mathfrak{n})$ dominating $(R, \mathfrak{m})$ with $\lambda(S/R) < \infty$. We show in this case that $\mu_R(S/R) \leq -e_1(J) \leq \lambda(S/R)$, and that if the equalities hold for every parameter ideal J then R is Buchsbaum.

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1. The Positivity Conjecture of Vasconcelos

Conjecture 1.1 (The Positivity Conjecture of Vasconcelos). *Let I be an $\mathfrak{m}$-primary ideal of an analytically unramified Noetherian local ring $(R, \mathfrak{m})$ of positive dimension. Then $\overline{e}_1(I) \geq 0$.*

In this section, we prove that the conjecture holds for a ring R which satisfies any one of the following conditions: (i) R is Cohen-Macaulay; (ii) the integral closure of R is Cohen-Macaulay as an R-module; (iii) R is a complete local domain of dimension 2; (iv) some other technical conditions. See Corollary 1.3 for details.

Let the notation and assumptions be as in the conjecture.

Put $\mathcal{A}_n = \overline{I^n}$, the integral closure of I^n in R. Then, the filtration $\mathcal{A} = \{\mathcal{A}_n\}$ is the integral closure filtration of the I-adic filtration and, as noted in the Introduction, the analytical unramifiedness of R implies by [9] that $\mathcal{A}$ is I-admissible. More generally, let $\mathcal{B} = \{\mathcal{B}_n\}$ be any filtration of ideals of R which is I-admissible. Then the function $n \mapsto \lambda(R/\mathcal{B}_{n+1})$ is given, for $n \gg 0$, by a polynomial $P_{\mathcal{B}} \in \mathbb{Q}[X]$. In this situation, we write $e_i(\mathcal{B})$ for $e_i(P_{\mathcal{B}})$. In particular, $e_i(\mathcal{A}) = \overline{e}_i(I)$.

By a **finite cover** S/R, we mean a ring extension $R \subseteq S$ such that S is a finite R-module. Then S is a Noetherian semilocal ring. We say that the finite cover S/R is **birational** if R is reduced and S is contained in the total quotient ring of R; that S/R is of **finite length** if $\lambda(S/R)$ is finite; and that S/R is **Cohen-Macaulay** if S is Cohen-Macaulay as an R-module.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. Let S/R be a finite cover such that at least one of the following two conditions holds:* (i) *S/R is of finite length; or* (ii) *S/R is birational. Let I be an $\mathfrak{m}$-primary ideal of R, and let $\mathcal{B}$ be a filtration of R such that $\mathcal{B}$ is I-admissible and $R \cap I^n S \subseteq \mathcal{B}_n$ for $n \gg 0$. Then $e_1(\mathcal{B}) \geq e_1(I, S)$.*

Proof. Let $\mathcal{C}$ denote the filtration of R given by $\mathcal{C}_n = R \cap I^n S$. For our proof, we need four length functions and their associated polynomials in $\mathbb{Q}[X]$ as listed in the following table:

Length function	Associated polynomial
$\lambda(R/I^{n+1})$	$P_I = P_I(X)$
$\lambda(S/I^{n+1}S)$	$P_{I,S} = P_{I,S}(X)$
$\lambda(R/\mathcal{B}_{n+1})$	$P_{\mathcal{B}} = P_{\mathcal{B}}(X)$
$\lambda(R/\mathcal{C}_{n+1})$	$P_{\mathcal{C}} = P_{\mathcal{C}}(X)$

By the given conditions on $\mathcal{B}$, there exists a nonnegaive integer k such that

$$\mathcal{C}_{n+k} \subseteq \mathcal{B}_{n+k} \subseteq I^n \subseteq \mathcal{C}_n \subseteq \mathcal{B}_n$$

for $n \geq 0$. Therefore

$$\lambda(R/\mathcal{C}_{n+k}) \geq \lambda(R/\mathcal{B}_{n+k}) \geq \lambda(R/I^n) \geq \lambda(R/\mathcal{C}_n) \geq \lambda(R/\mathcal{B}_n)$$

for $n \geq 0$, from which it follows that

$$d = \deg P_I = \deg P_{\mathcal{B}} = \deg P_{\mathcal{C}} \text{ and } e_0(P_I) = e_0(P_{\mathcal{B}}) = e_0(P_{\mathcal{C}}). \tag{A}$$

Now, the inequalities $\lambda(R/\mathcal{C}_n) \geq \lambda(R/\mathcal{B}_n)$ for $n \geq 0$ imply that

$$e_1(P_{\mathcal{B}}) \geq e_1(P_{\mathcal{C}}). \tag{B}$$

Assume now that (i) holds, i.e. S/R is of finite length (but may not be birational). Then, for $n \gg 0$, we have $I^n S \subseteq R$, so $\mathcal{C}_n = I^n S$. Therefore, for $n \gg 0$, we have

$$\lambda(R/\mathcal{C}_n) = \lambda(R/I^n S) = \lambda(S/I^n S) - \nu,$$

where $\nu = \lambda(S/R)$. Consequently, $P_{\mathcal{C}} = P_{I,S} - \nu$. Now, by (B), we get

$$e_1(P_{\mathcal{B}}) \geq e_1(P_{\mathcal{C}}) = e_1(P_{I,S} - \nu). \tag{C}$$

If $d = 1$ then $e_1(P_{I,S} - \nu) = e_1(P_{I,S}) + \nu \geq e_1(P_{I,S})$, while if $d \geq 2$ then $e_1(P_{I,S} - \nu) = e_1(P_{I,S})$. In either case, $e_1(P_{I,S} - \nu) \geq e_1(P_{I,S})$. Therefore, by (C), we get

$$e_1(\mathcal{B}) = e_1(P_{\mathcal{B}}) \geq e_1(P_{I,S}) = e_1(I, S),$$

which proves the assertion under condition (i).

Now, drop the assumption (i) and assume (ii), so that S/R is birational (but may not be of finite length). Then S/R is annihilated by a nonzero divisor of R, so $\dim S/R \leq d - 1$. Therefore, since $\deg P_{\mathcal{C}} = d$ by (A), the exact sequence

$$0 \to R/\mathcal{C}_n \to S/I^n S \to S/(R + I^n S) \to 0$$

shows that $\deg(P_{I,S}) = \deg P_{\mathcal{C}}$ and $e_0(P_{I,S}) = e_0(P_{\mathcal{C}})$. Combining this with the inequalities $P_{I,S}(n) \geq P_{\mathcal{C}}(n)$ for $n \gg 0$, which also result from the exact sequence, we get $e_1(P_{\mathcal{C}}) \geq e_1(P_{I,S})$. Thus, using (B) again, we get

$$e_1(\mathcal{B}) = e_1(P_{\mathcal{B}}) \geq e_1(P_{\mathcal{C}}) \geq e_1(P_{I,S}) = e_1(I, S).$$

This proves the assertion under condition (ii). $\square$

Corollary 1.3. *Let $(R, \mathfrak{m})$ be an anlytically unramified Noetherian local ring of positive dimension. Then the Positivity Conjecture 1.1 holds for R if R satisfies any one of the following conditions:*

(1) *R has a finite Cohen-Macaulay cover which is of finite length or is birational.*
(2) *R is Cohen-Macaulay* (cf. [5]).
(3) $\dim R = 1$.
(4) *The integral closure of R is Cohen-Macaulay as a an R-module.*
(5) $\dim R = 2$ *and all maximal ideals of the integral closure of R have the same height.*
(6) *R is a complete local integral domain of dimension 2.*

Proof. Since a minimal reduction of an $\mathfrak{m}$-primary ideal I gives rise to the same integral closure filtration as I does, it is enough to prove the conjecture (under any of the above conditions on R) for a parameter ideal of R. So, let I be a parameter ideal of R, and let $\mathcal{A}$ be the integral closure filtration of the I-adic filtration of R. Then, as noted earlier, $\mathcal{A}$ is I-admissible, and we have $\overline{e}_1(I) = e_1(\mathcal{A})$. Thus we have to show that $e_1(\mathcal{A}) \geq 0$ under each of the six condtions.

(1) Let S/R be a finite Cohen-Macaulay cover which is of finite length or is birational. Since S is integral over R, we have $R \cap I^n S \subseteq \mathcal{A}_n$ for every $n \geq 0$ by Proposition 1.6.1 of [11]. So, by the above theorem applied with $\mathcal{A}$ in place of $\mathcal{B}$, we get $e_1(\mathcal{A}) \geq e_1(I, S)$. Since S is Cohen-Macaulay as an R-module and I is a parameter ideal of R, we have $e_1(I, S) = 0$. Thus $e_1(\mathcal{A}) \geq 0$.

(2) Apply (1) to the trivial cover R/R.

(3) Since R is reduced and one dimensional, it is Cohen-Macaulay, so we can use (2).

For the remaining part of the proof, let R' be the integral closure of R in its total quotient ring. Then R'/R is a finite birational cover by [9], and $\dim R' = \dim R$.

(4) Since R'/R is a finite birational cover which is Cohen-Macaulay, we are done by (1).

(5) $\dim R' = 2$ implies that R' is Cohen-Macaulay as a ring. Now, it is easy to see that the assumption that all maximal ideals of R' have the same height implies that R' is Cohen-Macaulay as an R-module. So the assertion follows from (4).

(6) In this case, it is well known that R' is local, so (5) applies. $\square$

2. The Positivity Conjecture for the Maximal Homogeneous Ideal of a Face Ring

In this section, we show that the Positivity Conjecture holds for the filtration $\overline{\mathfrak{m}^n}$ where $\mathfrak{m}$ is the maximal homogeneous ideal of the face ring of a pure simplicial complex Δ. Let Δ be a $(d-1)-$dimensional simplicial complex. Let f_i denote the number of i-dimensional faces of Δ for $i = -1, 0, \ldots, d-1$. Here $f_{-1} = 1$. Let Δ have n vertices $\{v_1, v_2, \ldots, v_n\}$. Let $x_1, x_2, \ldots, x_n$ be indeterminates over a field k. The ideal I_Δ of Δ is the ideal generated by the square free monomials $x_{a_1}x_{a_2}\ldots x_{a_m}$ where $1 \leq a_1 < a_2 < \cdots < a_m \leq n$ and $\{v_{a_1}, v_{a_2}, \ldots, v_{a_m}\} \notin \Delta$. The face ring of Δ over a field k is defined as $k[\Delta] = k[x_1, x_2, \ldots, x_n]/I_\Delta$.

Lemma 2.1. *Let R be a Noetherian ring and I be an ideal of R such that the associated graded ring $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ is reduced. Then $\overline{I^n} = I^n$ for all n.*

Proof. Let $\mathcal{R}(I) = \oplus_{n\in\mathbb{Z}} I^n t^n$ denote the extended Rees ring of I. Since $G(I) = \mathcal{R}(I)/(u)$ where $u = t^{-1}$, and $G(I)$ is reduced, $(u) = P_1 \cap P_2 \cap \ldots \cap P_r$ for some height one prime ideals $P_1, \ldots, P_r$ of $\mathcal{R}(I)$. Therefore (u) is integrally closed in $\mathcal{R}(I)$. As $P_i\mathcal{R}(I)_{P_i} = (u)\mathcal{R}(I)_{P_i}$ for all i, $\mathcal{R}(I)_{P_i}$ is a DVR for all i. Since u is regular, $\operatorname{Ass}(\mathcal{R}(I)/(u^n)) = \{P_1, P_2, \ldots, P_r\}$ for all $n \geq 1$. Thus $(u^n) = \cap_{i=1}^r P_i^{(n)}$ is integrally closed. Hence $I^n = (u^n) \cap R$ is integrally closed for all n. $\square$

Lemma 2.2. *Let Δ be a $(d-1)-$dimensional simplicial complex. Let $\mathfrak{m}$ denote the maximal homogeneous ideal of the face ring $k[\Delta]$ over a field k. Then $\mathfrak{m}^n = \overline{\mathfrak{m}}^n$ for all n. and*

$$e_1(\mathfrak{m}) = \overline{e}_1(\mathfrak{m}) = df_{d-1} - f_{d-2}.$$

Proof. Since $k[\Delta]$ is standard graded $k-$algebra, $G(\mathfrak{m}) = k[\Delta]$. Hence $G(\mathfrak{m})$ is reduced and consequently $\mathfrak{m}$ is a normal ideal. Moreover, $\lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \dim_k k[\Delta]_n$. The Hilbert Series of the face ring is written as

$$H(k[\Delta], t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}.$$

Put $h(t) = h_0 + h_1 t + \cdots + h_s t^s$ where the face vector $(f_1, f_0, \ldots, f_{d-1})$ and the h-vector are related by the equation

$$\sum_{i=0}^{s} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{(d-i)}$$

by [1, Lemma 5.1.8]. Then by [1, Proposition 4.1.9] we have

$$e_1(\mathfrak{m}) = h'(1) = df_{d-1} - f_{d-2}.$$

$\square$

Theorem 2.3. *Let Δ be a pure simplicial complex. Then*

$$\overline{e}_1(\mathfrak{m}) = e_1(\mathfrak{m}) \geq 0.$$

Proof. Let $\dim \Delta = d-1$. We prove that if Δ is a pure simplicial complex then $df_{d-1} \geq f_{d-2}$. Let σ be a facet. For any $v_i \in \sigma = \{v_1, \ldots, v_d\}$, $\sigma \setminus \{v_i\}$ is a $(d-2)-$dimensional face and $\sigma \setminus \{v_i\}$ are distinct for all $i = 1, \ldots, d$. Therefore each facet gives rise to d, $(d-2)$-dimensional faces. But different facets may produce same faces of dimension $d-2$. Since Δ is pure each $(d-2)-$dimensional face is contained in a facet. Hence $df_{d-1} \geq f_{d-2}$. Therefore $\overline{e}_1(\mathfrak{m}) \geq 0$ by Lemma 2.2 $\square$

Example 2.4. The above theorem indicates that the the maximal homogeneous ideal of the face ring of a non-pure simplicial complexe may have negative Chern number. Indeed, consider the simplicial complex Δ_n on the vertices $\{v_1, v_2, \ldots v_{n+2}\}$ where $n \geq 2$ and

$$\Delta_n = \{\{v_1, v_2\}, v_3, \ldots, v_{n+2}\}.$$

Then $e_1(\mathfrak{m}) = df_{d-1} - f_{d-2} = -n$. Hence we need to add the assumption of quasi-unmixedness on the ring in Vasaconccelos' Positivity conjecture.

3. The Negativity Conjecture of Vasconcelos

In this section we show that the Chern number of any parameter ideal with respect to a finite module over a Noetherian local ring is non-negative. For this purpose, we need to generalize a result of Goto-Nishida [4, Lemma 2.4] to modules.

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finite R-module with* $\dim M = 1$*. If a is a parameter for M then*

$$e_1((a), M) = -\lambda(H^0_{\mathfrak{m}}(M)).$$

Proof. Let $N = H^0_{\mathfrak{m}}(M)$ and $\overline{M} = M/N$. Notice that $H^0_{\mathfrak{m}}(\overline{M}) = 0$ and $\dim \overline{M} = \dim M = 1$, which implies $\operatorname{depth} \overline{M} = 1$. Thus $\overline{M}$ is Cohen-Macaulay R-module. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0.$$

By taking tensor product with $R/(a)^n$ we get the exact sequence for all $n \geq 1$

$$0 \longrightarrow \ker \phi_n \longrightarrow N/a^nN \xrightarrow{\phi_n} M/a^nM \longrightarrow \overline{M}/a^n\overline{M} \longrightarrow 0. \tag{1}$$

By Artin-Rees Lemma, there is a k such that

$$a^nM \cap N = a^{n-k}(a^kM \cap N) \subseteq a^{n-k}N \subseteq \mathfrak{m}^{n-k}N = 0$$

for large n. Hence $\ker \phi_n = 0$ for all large n. Thus, for all large n, we get the exact sequence:

$$0 \longrightarrow N \longrightarrow M/a^nM \longrightarrow \overline{M}/a^n\overline{M} \longrightarrow 0.$$

Hence we have $\lambda(N) + \lambda(\overline{M}/a^n\overline{M}) = \lambda(M/a^nM)$. Since $\overline{M}$ is Cohen-Macaulay,

$$\lambda(\overline{M}/a^n\overline{M}) = e_0((a^n), \overline{M}) = e_0((a), \overline{M})n = e_0((a), M)n.$$

For large n, $\lambda(M/a^nM) = ne_0((a), M) - e_1((a), M)$. Therefore

$$e_1((a), M) = -\lambda(H^0_{\mathfrak{m}}(M)).$$

□

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finite R-module with* $\dim M = 1$*. Let a be a parameter for M. Then $e_1((a), M) = 0$ if and only if M is a Cohen-Macaulay module.*

In order to investigate the Chern number for finite modules of dimension $d \geq 2$ we use induction on dimension. The principal tool for this purpose is the concept of superficial element of an ideal with respect to a module. The next theorem is found in Nagata [8, 22.6] for Noetherian local rings. It is proved for modules over Noetherian local rings in [6].

Nagata's Theorem: Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be a finite A-module with $\dim M = d \geq 2$. Let I be an ideal of definition of M and let a be a superficial element for I with respect to M. Set $\overline{M} = M/aM$. Then

$$P_{\overline{I}}(\overline{M}, n) = \triangle P_I(M, n) + \lambda(0 :_M a).$$

In particular,

$$e_i(\overline{I}, \overline{M}) = \begin{cases} e_i(I, M) & \text{if } 0 \leq i < d-1. \\ e_{d-1}(I, M) + (-1)^{d-1}\lambda(0 :_M a) & \text{if } i = d-1. \end{cases}$$

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finite R-module with* $\dim M = d$. *Let I be an ideal of definition for M generated by $\mathbf{x} = x_1, \ldots, x_d$ which is a superficial sequence for I with respect to M. If M is not Cohen-Macaulay then M/x_1M is not Cohen-Macaulay.*

Proof. Suppose $\overline{M} = M/x_1M$ is Cohen-Macaulay. Then $\overline{x_2}, \ldots, \overline{x_d}$ is an $\overline{M}$-regular sequence. Thus $\lambda(\overline{M}/(\overline{x_2}, \ldots \overline{x_d})\overline{M}) = e_0(\overline{x_2}, \ldots \overline{x_d}, \overline{M})$. Since x_1 is superficial for M, $e_0(\mathbf{x}, M) = e_0(\overline{x_2}, \ldots, \overline{x_d}, M)$. Hence $\lambda(M/(\mathbf{x})M) = e_0(\mathbf{x}, M)$. Therefore M is Cohen-Macaulay which is a contradiction. □

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finite R-module with* $\dim M = d$ *and* $\operatorname{depth} M = d - 1$. *Let J be generated by a system of parameters for M. Then $e_1(J, M) < 0$.*

Proof. Apply induction on d. The $d = 1$ case is already done. Suppose $d = 2$. Let $J = (a, b)$. We may assume that (a, b) is a superficial sequence for J with respect to M and since $\operatorname{depth} M = 1$, a is M-regular. Let $\overline{M} = M/aM$. Then $\dim \overline{M} = 1$. By Nagata's Theorem, we have $e_1(\overline{J}, \overline{M}) = e_1(J, M)$. By Lemma 3.3, $\overline{M}$ is not Cohen-Macaulay. Thus $e_1(\overline{J}, \overline{M}) < 0$. Therefore $e_1(J, M) < 0$.

Next assume that $d \geq 3$ and $J = (x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a superficial sequence with respect to M. Let $\overline{M} = M/x_1M$ then $\dim \overline{M} = d-1$. By Nagata's Theorem we get $e_1(\overline{J}, \overline{M}) = e_1(J, M)$. If M is not Cohen-Macaulay then $\overline{M}$ is also not Cohen-Macaulay and hence by induction hypothesis $e_1(\overline{J}, \overline{M}) < 0$, which implies $e_1(J, M) < 0$. □

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finite R-module with* $\dim M = d$. *Let J be an ideal generated by a system of parameters for M. Then* $e_1(J, M) \leq 0$.

Proof. Apply induction on d. The $d = 1$ case is already proved. Suppose $d = 2$. Let $J = (x, y)$ where x, y is a superficial sequence for J with respect to M. Consider the exact sequence

$$0 \longrightarrow M/(0 :_M x) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0.$$

Applying $H^0_{\mathfrak{m}}(.)$ we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(M/(0 :_M x)) \xrightarrow{x} H^0_{\mathfrak{m}}(M) \xrightarrow{g} H^0_{\mathfrak{m}}(M/xM) \longrightarrow C \longrightarrow 0 \tag{2}$$

where $C = \operatorname{coker} g$. Consider the exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0.$$

Applying $H^0_{\mathfrak{m}}(.)$ on the exact sequence we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(0 :_M x) \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(M/(0 :_M x)) \longrightarrow 0.$$

Since $H^0_{\mathfrak{m}}(0 :_M x) = 0 :_M x$, we have

$$\lambda(0 :_M x) = \lambda(H^0_{\mathfrak{m}}(M)) - \lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))).$$

Subtracting $\lambda(H^0_{\mathfrak{m}}(M/xM))$ from both sides of the above equation we get

$$\lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM)) = \lambda(H^0_{\mathfrak{m}}(M)) - \lambda(H^0_{\mathfrak{m}}(M/xM)) - \lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))).$$

From the exact sequence (2) we get

$$\lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))) - \lambda(H^0_{\mathfrak{m}}(M)) + \lambda(H^0_{\mathfrak{m}}(M/xM) = \lambda(C).$$

Therefore we have $\lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM) = -\lambda(C)$. By Theorem **??**, we get

$$e_1(\overline{J}, \overline{M}) = e_1(J, M) - \lambda(0 :_M x).$$

By Proposition 3.1, $e_1(\overline{J}, \overline{M}) = -\lambda(H^0_{\mathfrak{m}}(M/xM))$. Therefore

$$e_1(J, M) = \lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM)) = -\lambda(C) \leq 0.$$

Let $d \geq 3$ and $a \in J$ be a superficial for J with respect to M. Since $e_1(J, M) = e_1(J/(a), M/aM)$, we are done by induction. $\square$

Proposition 3.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finite R-module with $\dim M = d \geq 2$. Let J be a parameter for M. If $M/H^0_{\mathfrak{m}}(M)$ is Cohen-Macaulay then $e_1(J, M) = 0$.*

Proof. Let $W = H^0_{\mathfrak{m}}(M)$ and $\overline{M} = M/W$. Since $\lambda(W) < \infty$, for $n >> 0$, $J^nM \cap W = 0$. We have for large n,

$$\begin{aligned} H_{\overline{J}}(\overline{M}, n) &= \lambda(\overline{M}/J^n\overline{M}) \\ &= \lambda(M/J^nM + W) \\ &= \lambda(M/J^nM) - \lambda(J^nM + W/J^nM) \\ &= \lambda(M/J^nM) - \lambda(W/J^nM \cap W) \\ &= H_J(M, n) - \lambda(W). \end{aligned}$$

Therefore

$$P_{\overline{J}}(\overline{M}, n) = P_J(M, n) - \lambda(W).$$

Hence $e_1(J, M) = e_1(\overline{J}, \overline{M})$. Since $\overline{M}$ is Cohen-Macaulay, $e_1(\overline{J}, \overline{M}) = 0$. Thus $e_1(J, M) = 0$. $\square$

Example 3.7. Let $S = k[|X, Y, Z|]$ be a power series ring over a field k and $J = (XZ, YZ, Z^2)$. Put $R = S/J = k[[x, y, z]]$. Then $\dim R = 2$ and depth $R = 0$. Consider the parameter ideal $I = (x, y)$. We calculate the Hilbert coefficients of I. Let '$-$' denote the image in $\overline{R} = R/H^0_{\mathfrak{m}}(R)$ where $\mathfrak{m}$ is the maximal ideal of R. Notice that for large n,

$$H^0_{\mathfrak{m}}(R) = \frac{J : (X, Y, Z)^n}{J} = \frac{(J : X^n) \cap (J : X^{n-1}Y) \cap \ldots \cap (J : Y^n)}{J} = \frac{(Z)}{J}.$$

Therefore $R/H^0_{\mathfrak{m}}(R) = \frac{k[|X,Y,Z|]}{J} / \frac{(Z)}{J} = k[|X, Y|]$ which is Cohen-Macaulay. Thus $e_1(x, y) = e_1(\overline{x}, \overline{y}) = 0$. Notice that $e_2(\overline{I}) = e_2(I) - \lambda(H^0_{\mathfrak{m}}(R))$. Since $e_2(\overline{I}) = 0$, $e_2(I) = \lambda(H^0_{\mathfrak{m}}(R)) = 1$.

Example 3.8. Let $S = \mathbb{Q}[[x, y, z, u]]$, be the power series ring over $\mathbb{Q}$. Let $\phi : \mathbb{Q}[[x, y, z, u]] \longrightarrow \mathbb{Q}[|x, t|]$ defined by

$$\phi(x) = x, \phi(y) = t^2, \phi(z) = t^5 \text{ and } \phi(u) = t^7.$$

Then

$$I_1 := \ker \phi = (y^6 - uz, z^3 - y^4u, u - yz)$$

is a height 2 prime ideal. Let $\chi : \mathbb{Q}[[x, y, z, u]] \longrightarrow \mathbb{Q}[|u, t|]$ be defined by

$$\chi(x) = t^2, \chi(y) = t^3, \chi(z) = t^4 \text{ and } \chi(u) = u.$$

Then

$$I_2 := \ker \chi = (y^2 - xz, x^2 - z)$$

is also a height 2 prime ideal. Put $I = I_1 \cap I_2$ and $R = S/I$. Then $\dim R = 2$ and R is not Cohen-Macaulay. The ideal $J = (x, u)R$ is a parameter ideal in R and $e_1(J) = -3$. This example has been calculated using Cocoa. We thank M. Rossi for sending this

CoCoA procedure to find Hilbert polynomial. The code is given below.

```
Alias P:=$contrib/primary;
Use S ::= Q[x, y, z, u];
I1 := Ideal(y^6 − uz, z^3 − y^4u, u − yz);
I2 :=Ideal(y^2 − xz, x^2 − z);
I := Intersection(I1, I2);
I;
Ideal(y^3z − xyz^2 − y^2u + xzu, x^2yz − yz^2 − x^2u + zu, x^2y^3u^2 − x^2z^4 − xyz^2u^2 + z^5 −
y^2u^3 + xzu^3, y^5u^2 − y^2z^4 + xz^5 − yz^3u^2 − xy^2u^3 + z^2u^3, y^6u − y^2z^3u − xy^3u^2 − y^2z^3 + xz^4 +
yz^2u^2, x^2y^4u − xy^2z^2u − x^2z^3 − y^3u^2 + xyzu^2 + z^4, x^2z^5 − x^2y^2u^3 − z^6 + y^2zu^3, y^7 − xyz^4 −
xy^4u + xz^3u − y^2z^2 + xz^3, x^2y^6 − y^2z^4 − y^5u + yz^3u − x^2zu + z^2u, y^2z^5 − xz^6 − y^4u^3 + xy^2zu^3)
Q := Ideal(x, u) + I;
Dim(S/Q);
0
J := I1 + I2;
Dim(S/J);
0
PS := P.PrimaryPoincare(I, Q); PS;
(12 − 3x)/(1 − x)^2
Hilbert(S/J);
H(0) = 1, H(1) = 4, H(2) = 7, H(3) = 5, H(t) = 0 for t ≥ 4.
```

Recently the Negativity Conjecture has been settled in [2] for unmixed local rings. We generalize this to finite unmixed modules over Cohen-Macaulay local rings.

Definition 3.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. A finite $R-$module M is called unmixed if for each associated prime P of its $\mathfrak{m}$-adic completion $\hat{M}$, $\dim R/P = d$.

We use Nagata's technique of idealization [8]. Let M be an $R-$module. Let $R^* = R \oplus M$ be the direct sum of the $R-$modules R and M. Define multiplication in R^* by

$$(r, m)((s, n) = (rs, rn + ms) \text{ for all } r, s \in R; m, n \in M.$$

In the next lemma we prove that the associated primes of the idealization R^* come from those of R and M.

Lemma 3.10. *Let $(R, \mathfrak{m})$ be a local ring and M be a finite R-module. Let $A = R * M$ be the idealization of M over R. Then*

$$\operatorname{Ass} A \subseteq \{P * M \mid P \in \operatorname{Ass} R \cup \operatorname{Ass}_R M\}.$$

Moreover if $P \in \operatorname{Ass}_R M$ *then* $P * M \in \operatorname{Ass} A$.

Proof. Let $\mathcal{P} \in \operatorname{Spec} A$ then $\mathcal{P} \supseteq 0 * M$ as $(0*M)^2 = 0$. Hence $\mathcal{P}/(0*M) \in \operatorname{Spec}(A/0*M) = \operatorname{Spec} R$. Therefore there exists a prime $P \in R$ such that $\mathcal{P}/0*M = P*M/0*M$ which implies $\mathcal{P} = P*M$. Thus every prime ideal of A is of the form $P*M$ where P is a prime ideal of R.

Let $P*M \in \operatorname{Ass} A$ then $P = (0 : (r, m))$, where $r \in R$ and $m \in M$. Let $a \in P$ then $(a, 0) \in P*M$ which implies that $(ar, am) = (0, 0)$. Thus $a \in (0 : r) \cap (0 : m)$. Hence $P \subseteq (0 : r) \cap (0 : m)$. Let $b \in (0 : r) \cap (0 : m)$ then $(b, 0)(r, m) = (0, 0)$ which implies $(b, 0) \in P*M$. Thus $b \in P$. Hence $P = (0 : r) \cap (0 : m)$. Therefore either $P = (0 : r)$ or $P = (0 : m)$. Hence $P \in \operatorname{Ass} R \cup \operatorname{Ass}_R M$. Therefore $\operatorname{Ass} A \subseteq \{P*M \mid P \in \operatorname{Ass} R \cup \operatorname{Ass}_R M\}$.
Let $P \in \operatorname{Ass}_R M$ then $P = (0 : m)$ where $m \in M$. Want to show that $P*M = (0 : (0, m))$. Let $(a, n) \in P*M$. Since $(a, n)(0, m) = (0, am) = (0, 0)$ therefore $(a, n) \in (0 : (0, m))$. Conversely if $(b, m') \in (0 : (0, m))$ then $b \in (0 : m)$. Thus $P*M = (0 : (0, m))$ and hence $P*M \in \operatorname{Ass} A$. ☐

Theorem 3.11. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring and let* M *be an unmixed module with* $\dim R = \dim M = d$. *If* $e_1(J, M) = 0$ *for some parameter ideal* J *for* M. *Then* M *is is a Cohen-Macaulay* $R-$*module.*

Proof. Let $A = R*M$ be the idealization of M over R. Then $\dim A = \dim R$. Note that $\widehat{A} = \widehat{R*M} = \widehat{R} * \widehat{M}$. If $P*\widehat{M} \in \operatorname{Ass}\widehat{A}$, then $P \in \operatorname{Ass}\widehat{R} \cup \operatorname{Ass}\widehat{M}$ by Lemma 3.10. Since R is Cohen-Macaulay and M is unmixed $\dim \widehat{R}/Q = d$ for all $Q \in \operatorname{Ass}\widehat{R} \cup \operatorname{Ass}\widehat{M}$. Therefore $\dim \widehat{A}/(P*\widehat{M}) = \dim \widehat{R}/P = d$. Hence A is unmixed. Consider the exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow A \longrightarrow R \longrightarrow 0. \tag{3}$$

Tensoring the above sequence with R/J^n we get the following exact sequence

$$0 \longrightarrow M/J^nM \longrightarrow A/J^nA \longrightarrow R/J^n \longrightarrow 0.$$

Since the length function is additive, we get

$$\lambda(A/J^nA) = \lambda(M/J^nM) + \lambda(R/J^n).$$

Hence $P_J(A, n) = P_J(M, n) + P_J(R, n)$. Equating the coefficients of the Hilbert polynomials we get

$$e_1(J, A) = e_1(J, M) + e_1(J).$$

Since R is Cohen-Macaulay $e_1(J) = 0$. Thus $e_1(J, A) = 0$. Hence by [2, Theorem 2.1] A in Cohen-Macaulay ring. Applying depth lemma on the exact sequence (3) we get that

$$\operatorname{depth} M \geq \min\{\operatorname{depth} A, \operatorname{depth} R + 1\}$$

which implies depth $M = d$. Thus M is Cohen-Macaulay. $\square$

4. Some Bounds for the Chern Number

In this section we find an upper bound for the Chern number of an admissible filtration $\mathcal{F}$. This bound yields the Huckaba-Marley bound in Cohen-Macaulay case. We use Rees algebra of $\mathcal{F}$ and Serre's multiplicity formula in terms of lengths of Koszul homology modules.

Let $A = \oplus_{n\geq 0} A_n$ be a standard graded algebra with $A_0 = (R, \mathfrak{m})$ be a local ring. Let $M = \oplus_{n\geq 0} M_n$ be a finitely generated graded $A-$module of dimension d such that $\lambda(M_n) < \infty$ for all $n \geq 0$. Let $P_M(x)$ be the polynomial corresponding to the function $H_M(n) = \lambda(M_n)$. Write

$$P_M(x) = \sum_{i=0}^{d-1} (-1)^i e_i(M) \binom{x+d-i}{d-i}.$$

Lemma 4.1. *Let $A = \oplus_{n\geq 0} A_n$ be a standard graded algebra with $A_0 = (R, \mathfrak{m})$ be a local ring and let $M = \oplus_{n\geq 0} M_n$ be a finitely generated graded $A-$module of dimension d such that $\lambda(M_n) < \infty$ for all $n \geq 0$. Then*

$$e_0(A_1, M) = e_0(M).$$

Proof. Let n_0 be the largest degree of a homogeneous set of generators of M as an A-module. Then

$$M_{n_0+1} = A_{n_0+1} M_0 + A_{n_0} M_1 + \cdots + A_1 M_{n_0}.$$

Since A is standard graded $A_r = (A_1)^r$ for all $r \geq 1$. Therefore we have

$$M_{n_0+1} = (A_1)^{n_0+1} M_0 + (A_1)^{n_0} M_1 + \ldots + (A_1) M_{n_0} = A_1 M_{n_0}.$$

Hence for all $k \geq 1$, $M_{n_0+k} = (A_1)^k M_{n_0}$. Let $H(n) = \lambda(M_n)$. Since

$$\frac{M}{(A_1)^n M} = \frac{\oplus_{r\geq 0} M_r}{\oplus_{r\geq 0} A_1^n M_r} = M_0 \oplus \cdots \oplus M_{n-1} \oplus \frac{M_n}{A_1^n M_0} \oplus \cdots \oplus \frac{M_{n+n_0}}{A_1^n M_{n_0}},$$

we get

$$\lambda(M/(A_1)^n M) = \sum_{i=0}^{n-1} H(i) + \sum_{j=0}^{n_0} \lambda\left(\frac{M_{n+j}}{A_1^n M_j}\right).$$

Since the 2nd sum is a finite sum for large n it is a polynomial function of degree at most $d-1$. Hence $\lambda(M/A_1^n M)$ is a polynomial function of degree d since $\sum_{i=0}^{n-1} H(i)$ is a polynomial function of degree d. Thus $e_0(A_1, M) = e_0(M)$. $\square$

Theorem 4.2. *Let $(R,\mathfrak{m})$ be a d-dimensional local ring and let J be a parameter ideal of R. Let $\mathcal{F}=\{J_n\}$ be a $J-$admissible filtration. Let $A=R[Jt]=\oplus_{n\geq 0}J^nt^n$ and $B=\mathcal{R}(\mathcal{F})=\oplus_{n\geq 0}J_nt^n$ and $M=B/A=\oplus_{n\geq 1}J_n/J^n$. If $\operatorname{ht}(A:_A B)=1$ then*

$$e_1(\mathcal{F})\leq e_1(J)+\sum_{n\geq 1}\lambda(J_n/JJ_{n-1}).$$

Proof. We may assume that R is complete. Since $\mathcal{F}$ is an admissible filtration B is a finitely generated A-module and hence M is also a finitely generated A-module. Since $\operatorname{ht}(A:_A B)=1$, $\dim M=d$. Note that

$$\begin{aligned}\lambda(M_n) &= \lambda(J_n/J^n)\\ &= \lambda(R/J^n)-\lambda(R/J_n)\\ &= [e_1(\mathcal{F})-e_1(J)]\binom{n+d-2}{d-1}+\text{ lower degree terms.}\end{aligned}$$

Therefore $\lambda(M_n)$ is a polynomial for large n of degree $d-1$ with leading coefficient $e_1(\mathcal{F})-e_1(J)$. Note that $M/JtM=\oplus_{n\geq 1}J_n/JJ_{n-1}$ and for large n, $J_n=JJ_{n-1}$. Thus $\lambda(M/JtM)<\infty$. By Lemma 4.1, $\lambda(M/J^nt^nM)$ is a polynomial for large n of degree d and $e_0(Jt,M)=e_1(\mathcal{F})-e_1(J)$. By Serre's Theorem we have

$$e_0(Jt,M)=\sum_{i=0}^{d}(-1)^i\lambda(H_i(Jt,M))$$

where $H_i(Jt,M)$ is the i^{th} Koszul homology of M with respect to Jt. Note that

$$H_0(Jt,M)=M/JtM=\bigoplus_{n\geq 1}J_n/JJ_{n-1}.$$

Let $\chi_1=\sum_{i=1}^{d}(-1)^{i+1}\lambda(H_i(Jt,M))$. By [1, Theorem 4.7.10] $\chi_1\geq 0$. Hence

$$e_1(\mathcal{F})-e_1(J)\leq\sum_{n\geq 1}\lambda(J_n/JJ_{n-1}).$$

Thus we have

$$e_1(\mathcal{F})\leq e_1(J)+\sum_{n\geq 1}\lambda(J_n/JJ_{n-1}).$$

□

Corollary 4.3. *Let $(R,\mathfrak{m})$ be a d-dimensional analytically unramified local ring and let J be a parameter ideal of R. Let $\mathcal{F}=\{\overline{J^n}\}$ denote the integral closure filtration. Let $A=R[Jt]=\oplus_{n\geq 0}J^nt^n$ and $B=\mathcal{R}(\mathcal{F})=\oplus_{n\geq 0}\overline{J^n}t^n$ and $M=B/A=\oplus_{n\geq 1}\overline{J^n}/J^n$. If $\operatorname{ht}(A:_A B)=1$ then*

$$\overline{e}_1(J)\leq\sum_{n\geq 1}\lambda(\overline{J^n}/J\overline{J^{n-1}})+e_1(J).$$

Proof. Since R is analytically unramified $\mathcal{F} = \{\overline{J^n}\}$ is a J-admissible filtration. Hence by Theorem 4.2, we have

$$\overline{e}_1(J) \leq \sum_{n\geq 1} \lambda(\overline{J^n}/J\overline{J^{n-1}}) + e_1(J).$$

□

Corollary 4.4 (Huckaba-Marley)**.** [5, Theorem 4.7] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, let J be a parameter ideal of R and let $\mathcal{F} = \{J_n\}$ be a J-admissible filtration. Then*

$$e_1(\mathcal{F}) \leq \sum_{n\geq 1} \lambda(J_n/J\overline{J_{n-1}}).$$

Proof. Since R is Cohen-Macaulay $e_1(J) = 0$. Hence by Theorem 4.2, we have

$$e_1(\mathcal{F}) \leq \sum_{n\geq 1} \lambda(J_n/J\overline{J_{n-1}}).$$

□

5. Some Further Estimates for the Chern Number

In this section, we provide some estimates for the Chern number in terms of a cover S/R of finite length such that S is local.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$, and let S/R be a cover of finite length such that S is local. Let $\mathfrak{n}$ be the maximal ideal of S, let $\rho = [S/\mathfrak{n} : R/\mathfrak{m}]$, and let $\nu = \lambda(S/R)$. Let J be a parameter ideal of R. Then JS is a parameter ideal of S.

Let $P_J(X)$ and $P_{JS}(X)$ be the polynomials associated to the functions $n \mapsto \lambda(R/J^{n+1})$ and $n \mapsto \lambda_S(S/J^{n+1}S)$, respectively.

For a finitely generated R-module M, let $\mu_R(M)$ denote the minimum number of generators of M.

Proposition 5.1. (1) *For every $n \geq 1$ we have*

$$\begin{aligned}
\mu_R(S/R)\binom{n+d-1}{d-1} &\leq \lambda(S/(R+JS))\binom{n+d-1}{d-1} \\
&\leq \lambda(J^nS/J^n) \\
&\leq \lambda(S/R)\binom{n+d-1}{d-1}.
\end{aligned}$$

(2) *The function* $n \mapsto \lambda(J^{n+1}S/J^{n+1})$ *is of polynomial type with associated polynomial* $P_J(X) + \nu - \rho P_{JS}(X)$, *and further,*

$$P_J(X) + \nu - \rho P_{JS}(X) = -e_1(J)\binom{X+d}{d-1} + f(X) \text{ with } \deg f(X) \leq d-2.$$

(3) $e_0(J) = \rho e_o(JS)$.

(4) $\mu_R(S/R) \leq \lambda(S/(R+JS)) \leq -e_1(J) \leq \lambda(S/R)$.

(5) *If* $\mu_R(S/R) = \lambda(S/R)$ *(equivalently, if* $\mathfrak{m}S \subseteq R$*) then*

$$e_1(J) = -\mu_R(S/R) = -\lambda(S/R)$$

and

$$\lambda(J^nS/J^n) = -e_1(J)\binom{n+d-1}{d-1} \quad \text{for every } n \geq 1.$$

Proof. (1) The first inequality holds trivially because

$$\mu_R(S/R) = \mu_R(S/(R+JS)) \leq \lambda(S/(R+JS)).$$

To prove the second inequality, let $m = \lambda(S/(R+JS))$, and choose $y_1, \dots, y_m \in S$ such that if $M_i = R + JS + (y_1, \dots, y_i)R$ then $S = M_m$ and $\lambda(M_i/M_{i-1}) = 1$ for every i.

Let $J = (x_1, \dots, x_d)R$. For a fixed n, let $s = \binom{n+d-1}{d-1}$, and let $\alpha_1, \dots, \alpha_s$ be all the monomials of degree n in $x_1, \dots, x_d$. Then $J^n = (\alpha_1, \dots, \alpha_s)R$. We have to show that $ms \leq \lambda(J^nS/J^n)$.

Since $S = R + JS + (y_1, \dots, y_m)R$, we have $J^nS = J^n + J^{n+1}S + J^n(y_1, \dots, y_m)R$. Let $N_i = J^n + J^{n+1}S + J^n(y_1, \dots, y_i)R$. Then $N_0 = J^n + J^{n+1}S$ and $N_m = J^nS$, and we have the sequence $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m$. So it is enough to prove that $\lambda(N_i/N_{i-1}) \geq s$ for every $i \geq 1$.

For a fixed $i \geq 1$ and for $0 \leq j \leq s$, let $P_j = N_{i-1} + (\alpha_1, \dots, \alpha_j)y_i$. Then $P_0 = N_{i-1}$ and $P_s = N_i$ and we have the sequence $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_s$. So it is enough to prove that all the inclusions in this sequence are proper.

Suppose, to the contrary, that $P_j = P_{j+1}$ for some $j \leq s-1$. Then

$$\alpha_{j+1}y_i \in P_j = J^n + J^{n+1}S + J^n(y_1, \dots, y_{i-1}) + (\alpha_1, \dots, \alpha_j)y_i.$$

So we can write

$$\alpha_{j+1}y_i = \beta + \sum_{k=1}^{s} a_k\alpha_k + \sum_{k=1}^{s} b_k\alpha_k + \sum_{k=1}^{j} c_k y_i \alpha_k$$

with $\beta \in J^{n+1}S,\ a_k, c_k \in R$ and $b_k \in (y_1, \dots y_{i-1})R$. Since JS is a parameter ideal in the Cohen-Macaulay local ring S, $\mathrm{gr}_{JS}(S)$ is a polynomial ring in the images of $x_1, \dots, x_d$. Therefore, since $\alpha_1, \dots, \alpha_s$ are distinct monomials in $x_1, \dots, x_d$, the coefficient of each α_k on the two sides of the above equality are congruent modulo JS. In particular, looking at the coefficient of α_{j+1}, we get $y_i \in R + JS + (y_1, \ldots, y_{i-1})R$. This contradicts the condition $\lambda(M_i/M_{i-1}) = 1$, so the second inequality of (1) is proved.

To prove the third inequality, choose a sequence

$$R = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\nu = S$$

of R-submodules such that $M_i/M_{i-1} \cong R/\mathfrak{m}$ for every $i \geq 1$. Then each $M_i = Rz_i + M_{i-1}$ for some $z_i \in S$ such that $\mathfrak{m}z_i \subseteq M_{i-1}$. For a fixed n, we have $J^n = (\alpha_1, \ldots, \alpha_s)R$ as above. Therefore

$$J^n M_i = J^n z_i + J^n M_{i-1} = (\alpha_1 z_i, \ldots, \alpha_s z_i) + J^n M_{i-1}.$$

Further, $\mathfrak{m}\alpha_j z_i \subseteq M_{i-1}\alpha_j \subseteq J^n M_{i-1}$. Therefore $\lambda(J^n M_i/J^n M_{i-1}) \leq s$ for every $i \geq 1$. Now, the sequence

$$J^n = J^n M_0 \subseteq J^n M_1 \subseteq \cdots \subseteq J^n M_\nu = J^n S$$

shows that $\lambda(J^n S/J^n) \leq \nu s = \lambda(S/R)\binom{n+d-1}{d-1}$.

This completes the proof of (1).

(2) From the commutative diagram

$$\begin{tikzcd} J^{n+1}S \arrow[r] & S \\ J^{n+1} \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

of inclusions, we get

$$\begin{aligned} \lambda(J^{n+1}S/J^{n+1}) &= \lambda(R/J^{n+1}) + \nu - \lambda(S/J^{n+1}S) \\ &= \lambda(R/J^{n+1}) + \nu - \rho\lambda_S(S/J^{n+1}S). \end{aligned}$$

Therefore the function $n \mapsto \lambda(J^{n+1}S/J^{n+1})$ is of polynomial type with associated polynomial

$$Q(X) := P_J(X) + \nu - \rho P_{JS}(X).$$

Since this function is squeezed between two polynomial functions of the same degree $d-1$ appearing in (1), we get

$$Q(X) = e\binom{X+d-1}{d-1} + f(X) \tag{$*$}$$

with $\lambda(S/(R+JS)) \leq e \leq \lambda(S/R)$ and deg $f(X) \leq d-2$. Since JS is a parameter ideal in the Cohen-Macaulay local ring S, we have

$$P_{JS}(X) = e_0(JS)\binom{X+d}{d}.$$

Substituting the above expressions for $P_{JS}(X)$ and $Q(X)$ in the formula

$$Q(X) = P_J(X) + \nu - \rho P_{JS}(X),$$

we get

$$Q(X) = -e_1(J)\binom{X+d-1}{d-1} + f(X)$$

with $\deg f(X) \leq d-2$, as required.

(3) We have $\deg P_J(X) = d = \deg P_{JS}(X)$. Therefore, since $\deg(P_J(X) - \rho P_{JS}(X)) \leq d-1$ by (2), we get $e_0(J) = \rho e_o(JS)$.

(4) This is immediate from (1) and (2).

(5) This is immediate from (1) and (4). □

Corollary 5.2. *In the above set up, assume further that S is Cohen-Macaulay. If $\mu_R(S/R) = \lambda(S/R)$ (equivalently, if $\mathfrak{m}S \subseteq R$) then*

$$e_1(J) = -\mu_R(S/R) = -\lambda(S/R)$$

for every parameter ideal J of R. Further, in this case R is Buchsbaum.

Proof. The first part is immediate from the above proposition. Taking $n = 0$ in the commutative square appearing in the above proof, we get

$$\lambda(S/R) + \lambda(R/J) = \lambda(S/JS) + \lambda(JS/J).$$

Since JS is a parameter ideal in the Cohen-Macaulay local ring S, we have $\lambda_S(S/JS) = e_0(JS)$. Therefore $\lambda(S/JS) = \rho e_0(JS) = e_0(J)$ by the above proposition. Further, taking $n = 1$ in part (5) of the above proposition, we get

$$\lambda(JS/J) = -e_1(J)d = \lambda(S/R)d = \nu d.$$

Substituting these values in the formula displayed above, we get $\lambda(R/J) - e_0(J) = (d-1)\nu$. Thus $\lambda(R/J) - e_0(J)$ is independent of the parameter ideal J, so R is Buchsbaum. □

Example 5.3. These are examples to show that for $d = 2$ the Chern number $e_1(J)$ can attain every value in the range given by Proposition (6.1). More precisely, given any integers r, p with $1 \leq r \leq p$, there exists a Noetherian local ring R of dimension 2, a Cohen-Macaulay cover S/R of finite length and a parameter ideal J of R such that $\mu_R(S/R) = 1$, $\lambda(S/R) = p$ and $e_1(J) = -r$. In fact, it can be verified by a direct computation that these equalities hold in the following situation: $R = k[[t^2, t^3, x, tx^p]] \subseteq S = k[[t, x]]$ and $J = (t^2, x^r)R$, where k is a field and t and x are indeterminates.

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