

ON THE THEORY OF NON-LINEAR INTEGRAL EQUATIONS.

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1. Introduction.

THE object of this paper is to study the infinite system of non-linear integral equations :

$$u_n(x) = f_n(x) + \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) u_k(y) u_l(y) dy,$$

$$(n = 1, 2, \dots \infty).$$

The sum is always taken from 1 to ∞ throughout this paper.

The method of successive approximations is employed to obtain a solution of this system. But when we proceed to establish the convergence and uniqueness of these approximations, we find that two cases arise according to the nature of the conditions which the given functions f_n , g_n , h_n satisfy.

In the first case, where

$$\sum_n |f_n(x)| = c, \quad |g_n(x, y)| \leq b, \quad \sum_n |h_n(k, l; x)| \leq a,$$

the solution exists only when x lies in a certain restricted domain. But if the last two conditions are replaced by the relation

$$\sum_n \left| \int_0^x g_n(x, y) h_n(k, l; y) dy \right| \leq a,$$

we find that the solution of the system of integral equations exists for all x however large.

These integral equations have a fundamental importance in the theory of partial differential equations of hyperbolic and parabolic type. Particular cases have been considered by the writer in several papers published recently.*

* M. R. Siddiqi, (1) "Zur Theorie der nicht-linearen partiellen Differentialgleichungen vom parabolischen Typus," *Math. Zeitschr.*, 1932, 35, 404; (2) "On a system of non-linear partial differential equations," *Journal, Osmania Uni.*, 1935, 3, 1; (3) "Cauchy's Problem in a non-linear partial differential equation of hyperbolic type," *Proc. Camb. Phil. Soc.*, 1935, 31, 195; (4) "On the theory of non-linear differential equations of parabolic type II," *Math. Zeitschr.*, 1935, 40, 484.

2. *Solution in the Restricted Domain.*

We wish to determine a solution of the system of non-linear integral equations :

$$(1) \quad u_n(x) = f_n(x) + \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) u_k(y) u_l(y) dy,$$

$$(n = 1, 2, 3, \dots \infty),$$

where x lies in the interval

$$(2) \quad 0 \leq x \leq T,$$

and where the given functions f_n, g_n, h_n satisfy the following inequalities for all n, x, y, k, l :

$$(3) \quad \sum_n |f_n(x)| = c, \quad |g_n(x, y)| \leq b, \quad \sum_n |h_n(k, l; y)| \leq a,$$

a, b, c being absolute constants.

We replace the unknown functions $u_n(x)$ by the functions $w_n(x)$ according to the relations :

$$(4) \quad w_n(x) = u_n(x) - f_n(x), \quad (n = 1, 2, \dots).$$

The equations (1) then become for all $n \geq 1$:

$$(5) \quad w_n(x) = \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{f_k(y) + w_k(y)\} \{f_l(y) + w_l(y)\} dy.$$

We shall solve this system by the method of successive approximations, and for this purpose we write :

$$(6) \quad w_n^{(r)}(x) = \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) [f_k(y) + w_k^{(r-1)}(y)] [f_l(y) + w_l^{(r-1)}(y)] dy.$$

We shall prove now that for all $r \geq 0$, the series $\sum_n |w_n^{(r)}(x)|$ lies under a fixed limit as soon as T is sufficiently small.

Let for all x in $0 \leq x \leq T$, and for all $r \geq 0$:

$$(7) \quad v_r = \text{Max}_n \sum |w_n^{(r)}(x)|.$$

Then we get from (6) on account of (3) :

$$(8) \quad v_r \leq abT(c + v_{r-1})^2.$$

Now we define an infinite sequence (A_r) as follows :

$$(9) \quad A_0 = 0, \quad A_r = 2abT(c^2 + A_{r-1}^2) \quad (r = 1, 2, \dots).$$

On account of the inequality $2ab \leq a^2 + b^2$, we see that

$$(10) \quad v_r \leq A_r \quad (r = 0, 1, 2, 3, \dots).$$

We shall prove now that the sequence A_r is bounded. Take a fixed positive integer p , and let for all $r \leq p$:

$$A_r \leq 1, (r \leq p).$$

Then from (9) we have for all $r \leq p$:

$$A_{r+1} - A_r = 2abT (A_r^2 - A_{r-1}^2) = 2abT (A_r + A_{r-1}) (A_r - A_{r-1}),$$

so that, since both A_r and A_{r-1} are supposed to be ≤ 1 ,

$$|A_{r+1} - A_r| \leq 4abT |A_r - A_{r-1}|, (r \leq p).$$

Therefore we find:

$$\begin{aligned} A_{p+1} &\leq |A_1| + |A_2 - A_1| + |A_3 - A_2| + \dots + |A_{p+1} - A_p| \\ &\leq 2abTc^2 \{1 + 4abT + (4abT)^2 + \dots + (4abT)^p\} \\ &< \frac{2abTc^2}{1 - 4abT}, \end{aligned}$$

where it is assumed that $4abT < 1$.

Now, if we take T so that

$$(11) \quad T < \text{Min} \left(\frac{1}{8ab}, \frac{1}{4abc^2} \right),$$

we get

$$A_{p+1} < 4abTc^2 < 1.$$

We have, however, $A_1 = 2abTc^2 < 1$; so that by induction we get for all $r \geq 1$:

$$(12) \quad A_r < 1 \quad (r = 1, 2, \dots, \infty).$$

From (7) and (10) we see therefore that the series $\sum_n |w_n^{(r)}(x)|$ is uniformly convergent and less than unity, provided x lies in the interval given by (11).

We go on to prove now that the doubly infinite series

$$(13) \quad \sum_n |w_n^{(1)}(x)| + \sum_r \sum_n |w_n^{(r+1)}(x) - w_n^{(r)}(x)|$$

is also uniformly convergent.

We have from (6)

$$\begin{aligned} w_n^{(r+1)}(x) - w_n^{(r)}(x) &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{ [f_k(y) + w_k^{(r)}(y)] [f_l(y) + w_l^{(r)}(y)] - \\ &\quad [f_k(y) + w_k^{(r-1)}(y)] \times [f_l(y) + w_l^{(r-1)}(y)] \} dy \\ &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{ f_k(y) [w_l^{(r)}(y) - w_l^{(r-1)}(y)] + \\ &\quad f_l(y) [w_k^{(r)}(y) - w_k^{(r-1)}(y)] + w_k^{(r)}(y) [w_l^{(r)}(y) - w_l^{(r-1)}(y)] \\ &\quad + w_l^{(r-1)}(y) [w_k^{(r)}(y) - w_k^{(r-1)}(y)] \} dy. \end{aligned}$$

From (3) and (12) we get therefore for any x :

$$(14) \quad \sum_n |w_n^{(r+1)}(x) - w_n^{(r)}(x)| < abT(2c + 2) \text{Max}_n \sum |w_n^{(r)}(x) - w_n^{(r-1)}(x)|.$$

We see, therefore, that the series $\sum_n |w_n^{(r+1)}(x) - w_n^{(r)}(x)|$ converges uniformly provided the series $\sum_n |w_n^{(r)}(x) - w_n^{(r-1)}(x)|$ does so. We take T so small that it satisfies the inequality :

$$(15) \quad T \leq \frac{\gamma}{2ab(1+c)}, \quad \gamma < 1.$$

Then applying the reduction (14) r times, we get for x in (2) :

$$(16) \quad \sum_n |w_n^{(r+1)}(x) - w_n^{(r)}(x)| < \{2abT(1+c)\}^r. \quad \text{Max}_n \sum |w_n^{(1)}(x) - w_n^{(0)}(x)| < \gamma^r. \quad \text{Max}_n \sum |w_n^{(1)}(x)|.$$

Summing both sides of (16) from $r=0$ to $r = \infty$, and remembering that $\text{Max}_n \sum |w_n^{(1)}(x)| = v_1 \leq A_1 < 1$, and that $\gamma < 1$, we see that the double series is uniformly convergent in $0 \leq x \leq T$, where T is restricted by the relation (15).

It follows therefore that all the limiting values exist :

$$(17) \quad \lim_{r \rightarrow \infty} w_n^{(r)}(x) = w_n(x) \quad (n = 1, 2, \dots \infty),$$

and that for all x in $0 \leq x \leq T$, the series $\sum_n |w_n(x)|$ is uniformly convergent.

Proceeding to the limit $r \rightarrow \infty$ in (6), we obtain that the function $w_n(x)$ satisfies the integral equation (5), and consequently $u_n(x) = w_n(x) + f_n(x)$ satisfies the integral equation (3).

We shall now establish the uniqueness of this solution.

Suppose that the system of integral equations (5) has another solution $\bar{w}_n(x)$ ($n = 1, 2, \dots$), such that the series $\sum_n |\bar{w}_n(x)|$ converges uniformly:

$$(18) \quad \sum_n |\bar{w}_n(x)| = \bar{c}.$$

From the definitions of $\bar{w}_n(x)$ and $w_n^{(r+1)}(x)$, we get

$$\begin{aligned} \bar{w}_n(x) - w_n^{(r+1)}(x) &= \int_0^x g_n(x, y) \sum_{k,l} h_n(k, l; y) \{ [f_k(y) + \bar{w}_k(y)] [f_l(y) + \bar{w}_l(y)] \\ &\quad - [f_k(y) + w_k^{(r)}(y)] [f_l(y) + w_l^{(r)}(y)] \} dy \\ &= \int_0^x g_n(x, y) \sum_{k,l} h_n(k, l; y) \{ f_k(y) [\bar{w}_l(y) - w_l^{(r)}(y)] \\ &\quad + f_l(y) [\bar{w}_k(y) - w_k^{(r)}(y)] + w_k^{(r)}(y) [\bar{w}_l(y) - w_l^{(r)}(y)] \\ &\quad + \bar{w}_l(y) [\bar{w}_k(y) - w_k^{(r)}(y)] \} dy. \end{aligned}$$

From (3), (12) and (18) we obtain therefore :

$$(19) \quad \sum_n |\bar{w}_n(x) - w_n^{(r+1)}(x)| < abT(2c + 1 + \bar{c}). \quad \text{Max}_n \sum |\bar{w}_n(x) - w_n^{(r)}(x)|.$$

Now, we take T so small that besides (15) it satisfies also :

$$(20) \quad T < \frac{q}{ab(2c + 1 + \bar{c})}, \quad q < 1.$$

Then applying the reduction (19) $r + 1$ times, we get

$$\begin{aligned} \sum_n |\bar{w}_n(x) - w_n^{(r+1)}(x)| &< \{abT(2c + 1 + \bar{c})\}^r. \quad \text{Max}_n \sum |\bar{w}_n(x) - w_n^{(0)}(x)| \\ &< q^r \text{Max}_n \sum |\bar{w}_n(x)| = \bar{c} q^r. \end{aligned}$$

Since $q < 1$, we get

$$\lim_{r \rightarrow \infty} \sum_n |\bar{w}_n(x) - w_n^{(r+1)}(x)| = 0,$$

which gives

$$\begin{aligned} \bar{w}_n(x) &= \lim_{r \rightarrow \infty} w_n^{(r+1)}(x) = w_n(x), \\ (n &= 1, 2, \dots \infty). \end{aligned}$$

This shows that the two solutions are really identical.

3. Solution in the Unrestricted Domain.

If the system

$$(1) \quad u_n(x) = f_n(x) + \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) u_k(y) u_l(y) dy, \quad (n = 1, 2, \dots)$$

has to be solved for the unrestricted domain :

$$(2) \quad 0 \leq x < \infty,$$

we must assume that the given functions satisfy :

$$(3) \quad \sum_n |f_n(x)| = c, \quad \sum_n \left| \int_0^x g_n(x, y) h_n(k, l; y) dy \right| \leq a,$$

where c and a are constants independent of x, k, l .

We write for the successive approximations :

$$(4) \quad u_n^{(0)}(x) = f_n(x), \quad \text{and for } r \geq 1$$

$$(5) \quad u_n^{(r)}(x) = f_n(x) + \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) u_k^{(r-1)}(y) u_l^{(r-1)}(y) dy.$$

We have now to show that these approximations converge to a unique limit. For this purpose we prove first that the series $\sum_n |u_n^{(r)}(x)|$ converges uniformly for all x and all r . Evidently, $\sum_n |u_n^{(r)}(x)|$ converges uniformly for all x provided $\sum_n |u_n^{(r-1)}(x)|$ does so, for we have :

$$(6) \quad \sum_n |u_n^{(r)}(x)| \leq c + a [\text{Max}_n \sum_n |u_n^{(r-1)}(x)|]^2.$$

From (3) and (4) we find $\sum_n |u_n^{(0)}(x)| = c$, so that

$$\sum |u_n^{(1)}(x)| \leq c + ac^2.$$

We assume now that

$$(7) \quad c < \frac{1}{4a},$$

so that we find $\sum_n |u_n^{(1)}(x)| < 2c$ for all x . We substitute this in (6) and put $r = 2$, then we get

$$\sum_n |u_n^{(2)}(x)| < c + 4ac^2 < c + c = 2c,$$

and in general, for all $r \geq 1$, and all $x \geq 0$:

$$(8) \quad \sum_n |u_n^{(r)}(x)| < c + 4ac^2 < 2c.$$

We shall employ this inequality in proving further the uniform convergence of the double series

$$(9) \quad \sum_n |u_n^{(1)}(x) - u_n^{(0)}(x)| + \sum_r \sum_n |u_n^{(r+1)}(x) - u_n^{(r)}(x)|.$$

We have from (5)

$$\begin{aligned} u_n^{(r+1)}(x) - u_n^{(r)}(x) &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{u_k^{(r)}(y)u_l^{(r)}(y) - u_k^{(r-1)}(y)u_l^{(r-1)}(y)\} dy \\ &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{u_k^{(r)}(y)[u_l^{(r)}(y) - u_l^{(r-1)}(y)] \\ &\quad + u_l^{(r-1)}(y)[u_k^{(r)}(y) - u_k^{(r-1)}(y)]\} dy. \end{aligned}$$

From (3) and (8) we get therefore for all x :

$$\sum_n |u_n^{(r+1)}(x) - u_n^{(r)}(x)| < 4ac. \quad \text{Max}_n \sum_n |u_n^{(r)}(x) - u_n^{(r-1)}(x)|.$$

Applying this reduction formula r times, we obtain:

$$(10) \quad \sum_n |u_n^{(r+1)}(x) - u_n^{(r)}(x)| < (4ac)^r. \quad \text{Max}_n \sum_n |u_n^{(1)}(x) - u_n^{(0)}(x)|.$$

But

$$u_n^{(1)}(x) - u_n^{(0)}(x) = \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) u_k^{(0)}(y) u_l^{(0)}(y) dy,$$

so that for all $x \geq 0$, we have

$$\sum_n |u_n^{(1)}(x) - u_n^{(0)}(x)| \leq ac^2.$$

Moreover, since $4ac < 1$, we get $\sum_{r=0}^{\infty} (4ac)^r = \frac{1}{1-4ac}$. Thus, summing

both sides of (10) from $r = 0$ to $r = \infty$, we see that the double series (9) converges uniformly for all x .

We deduce therefore that the limiting functions

$$(11) \quad \lim_{r \rightarrow \infty} u_n^{(r)}(x) = u_n(x) \quad (n = 1, 2, \dots, \infty)$$

exist and are continuous for all $x \geq 0$, and that the series $\sum_n |u_n(x)| = \lim_{r \rightarrow \infty} \sum_n |u_n^{(r)}(x)|$ is uniformly convergent. Proceeding now to the limit $r \rightarrow \infty$ in (5), we see that the function $u_n(x)$ found in (11) satisfies the integral equation (1), and is therefore the required solution.

To establish the uniqueness of this solution, we suppose that the integral equation (1) has another solution $\bar{u}_n(x)$ ($n = 1, 2, \dots$), which is such that the series $\sum_n |\bar{u}_n(x)|$ converges uniformly:

$$(12) \quad \sum_n |\bar{u}_n(x)| \leq 2c.$$

From the definitions of $\bar{u}_n(x)$ and $u_n^{(r+1)}(x)$ we get then

$$\begin{aligned} \bar{u}_n(x) - u_n^{(r+1)}(x) &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{ \bar{u}_k(y) \bar{u}_l(y) - u_k^{(r)}(y) u_l^{(r)}(y) \} dy \\ &= \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \{ \bar{u}_k(y) [\bar{u}_l(y) - u_l^{(r)}(y)] \\ &\quad + u_l^{(r)}(y) [\bar{u}_k(y) - u_k^{(r)}(y)] \} dy. \end{aligned}$$

On account of (3), (8) and (12) we get therefore for all x :

$$\sum_n |\bar{u}_n(x) - u_n^{(r+1)}(x)| < 4ac. \quad \text{Max}_n \sum_n |\bar{u}_n(x) - u_n^{(r)}(x)|;$$

and repeating this reduction $(r+1)$ times:

$$(13) \quad \sum_n |\bar{u}_n(x) - u_n^{(r+1)}(x)| < (4ac)^{r+1}. \quad \text{Max}_n \sum_n |\bar{u}_n(x) - u_n^{(0)}(x)|.$$

Now, since

$$\bar{u}_n(x) - u_n^{(0)}(x) = \int_0^x g_n(x, y) \sum_{k, l} h_n(k, l; y) \bar{u}_k(y) \bar{u}_l(y) dy,$$

we have from (3) and (12) for all $x \geq 0$:

$$\sum_n |\bar{u}_n(x) - u_n^{(0)}(x)| \leq 4ac^2.$$

Thus, since $4ac < 1$, we obtain

$$\begin{aligned} \lim_{r \rightarrow \infty} \sum_n |\bar{u}_n(x) - u_n^{(r+1)}(x)| &< 4ac^2 \lim_{r \rightarrow \infty} (4ac)^r \\ &= 0, \end{aligned}$$

showing that for all $x \geq 0$:

$$(14) \quad \bar{u}_n(x) = \lim_{r \rightarrow \infty} u_n^{(r+1)}(x) = u_n(x) \quad (n = 1, 2, \dots, \infty).$$

The two solutions are therefore identical.