On infinitesimal σ-fields generated by random processes

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Abstract. It is proved that the infinitesimal look-ahead and look-back σ-fields of a random process disagree at almost countably many time instants.

Keywords. Random processes; look-ahead σ-field; look-back σ-field; Markov processes.

Let $X(t), t \geq 0$, be a Polish space-valued random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the $P$-completion of a countably generated σ-field $\mathcal{F}_0$.

Let $\xi(t+) \ (\text{the 'look-ahead' } \sigma \text{-field at } t)$ and $\xi(t-) \ (\text{the 'look-back' } \sigma \text{-field at } t)$ denote the $P$-completions of $\bigcap_{s>t} \sigma(X(y), t \leq y \leq s)$ and $\bigcap_{s<t} \sigma(X(y), s \leq y \leq t)$ respectively.

The aim of this note is to prove the following fact:

**Theorem 1.** $\xi(t+) = \xi(t-) \ (\text{for all but at most countably many } t \geq 0)$.

We shall prove this through a sequence of lemmas. Let $\{A_n, n \geq 1\}$ be a dense collection of sets in $\mathcal{F}_0$ which generates $\mathcal{F}_0$.

**Lemma 1.** The space of all $P$-complete sub-σ-fields of $\mathcal{F}$ forms a metric space under the metric

$$d(\mathcal{G}_1, \mathcal{G}_2) = \sum_1^n 2^{-n} E[|P(A_n/\mathcal{G}_1) - P(A_n/\mathcal{G}_2)|].$$

The proof is easy.

**Remark.** This topology was first introduced in Cotter [3].

For $t \geq 0$, let

$$\mathcal{F}(t) = \sigma(X(s), 0 \leq s \leq t), \quad \mathcal{F}(t+) = \bigcap_{s \geq t} \mathcal{F}(s), \quad \mathcal{F}(t-) = \bigvee_{s < t} \mathcal{F}(s),$$

all completed with respect to $P$. For $n \geq 1, t \geq 0$, define

$$h_n(t) = E[(E[I_{A_n}/\mathcal{F}(t)])^2].$$

**Lemma 2.** (i) $t \rightarrow h_n(t)$ is bounded nondecreasing for all $n$.

(ii) $\lim_{t \uparrow t} h_n(t) = E[(E[I_{A_n}/\mathcal{F}(t+))]^2].$

(iii) $\lim_{t \downarrow t} h_n(t) = E[(E[I_{A_n}/\mathcal{F}(t-))]^2].$

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Proof. (i) follows from the conditional Jensen’s inequality and (ii), (iii) follow from the convergence theorems for regular martingales and reversed martingales [2]. □

From (i) above, it follows that each $h_n(t)$ has at most a countable set of points of discontinuity. Let $D \subset [0, \infty)$ be the at most countable set of points where one or more of the $h_n(t)$’s is discontinuous.

**Lemma 3.** For $t \notin D$, $\mathcal{F}(t+) = \mathcal{F}(t-)$.  

**Proof.** For $t \notin D$, Lemma 2 (ii), (iii) imply that  

$$E[(E[I_{A_n}/\mathcal{F}(t+)])^2] = E[(E[I_{A_n}/\mathcal{F}(t-)])^2], \quad n \geq 1.$$ 

Thus  

$$E[(E[I_{A_n}/\mathcal{F}(t+)]) - E[(E[I_{A_n}/\mathcal{F}(t-)])]^2] = 0, \quad n \geq 1,$$

implying $E[I_{A_n}/\mathcal{F}(t+)] = E[I_{A_n}/\mathcal{F}(t-)]$ a.s., $n \geq 1$. The claim follows from Lemma 1. □

**COROLLARY 1**

$\xi(t+) \subset \xi(t-)$ for all but at most countably many $t$.

**Proof.** Let $\{r_m\}$ be an enumeration of rationals in $[0, \infty)$. Define $\mathcal{F}^m(t)$, $\mathcal{F}^m(t+)$, $\mathcal{F}^m(t-)$, and $h^m(t)$ as in (1)–(4) resp. with $X(r_m+)$ replacing $X(t)$, $m \geq 1$. The foregoing results hold for each $X(r_m+)$ as well. For every rational $r \geq 0$, let  

$$D_r = \{t > r | P\text{-completion of } \bigvee_{\varepsilon > 0} \sigma(X_s, r \leq s \leq t - \varepsilon) \neq$$

$$P\text{-completion of } \bigcap_{\varepsilon > 0} \sigma(X_s, r \leq s \leq t + \varepsilon)\},$$

$$D = \bigcup D_r.$$  

Then by Lemma 3, $D_r$ and therefore $D$ is at most countable. Fix $t \notin D$ and let $\{r_{m(i)}\}$ be a collection of rationals increasing to $t$. Then for $i \geq 1$,  

$$\bigvee_{\varepsilon > 0} \sigma(X_s, r_{m(i)} \leq s \leq t - \varepsilon) = \bigcap_{\varepsilon > 0} \sigma(X_s, r_{m(i)} \leq s \leq t + \varepsilon)$$

on $P$-completion. Thus  

$$\xi(t+) \subset P\text{-completion of } \bigvee_{\varepsilon > 0} \sigma(X_s, r_{m(i)} \leq s \leq t - \varepsilon), \quad i \geq 1,$$

and hence  

$$\xi(t+) \subset P\text{-completion of } \bigcap_i \bigvee_{\varepsilon > 0} \sigma(X_s, r_{m(i)} \leq s \leq t - \varepsilon) \subset \xi(t-).$$
The claim follows.

Proof of Theorem 1. It suffices to consider $t \in [0, T]$ for some finite $T > 0$. Applying the above corollary to the process $X(T-t), t \in [0, T]$, we conclude that $\xi(t-) \subset \xi(t+)$ except at most countably many $t$. Combine this with the corollary to conclude.

COROLLARY 2

If $X(\cdot)$ is a Markov process, then $\xi(t+) = \xi(t-) = \xi(t)$ ($=\text{ the P-completion of } \sigma(X(t))$), at all but at most countably many $t$.

Proof. Note that $\xi(t) \subset \xi(t+) \cap \xi(t-)$ for all $t \geq 0$. Let $t \geq 0$ be such that $\xi(t+) = \xi(t-)$. Since $X(\cdot)$ is Markov, $\xi(t+), \xi(t-)$ are conditionally independent given $X(t)$. Thus $\xi(t+)$ is conditionally independent of itself given $X(t)$, implying $\xi(t+) \subset \xi(t)$. Similarly $\xi(t-) \subset \xi(t)$.

It is conjectured that the conclusions of Corollary 2 hold even in absence of the Markov property. If true, this result will have important implications in stochastic control theory [1]. We conclude with an example to show that one cannot improve on Theorem 1 in general.

Example Let $\Omega = [0, 1]^\infty$, $\mathcal{F}_0$ the product Borel $\sigma$-field, $P$ the product Lebesgue measure and $\mathcal{F}$ the product $\sigma$-field completed with respect to $P$. Let $w = (w_1, w_2, \ldots)$ denote a typical element of $\Omega$. Let $\{r_n\}$ be an enumeration of rationals in $(0, 1)$. Define an $R^\infty$-valued process $X(t) = [X_1(t), X_2(t), \ldots], t \in [0, 1]$ as follows: $[X_1(t), X_2(t), \ldots]$ evaluated at the sample point $[w_1, w_2, \ldots]$ is given by

$$X_{2i}(t) = w_{2i}[(t - r_i)^+], i \geq 1,$$

$$X_{2i-1}(t) = w_{2i-1}[(t - r_i)^-], i \geq 1,$$

for $t \in [0, 1]$. Then it is easy to see that $\xi(t+), \xi(t-), \xi(t)$ are the $P$-completions of $\Pi_n G^+(t), \Pi_n G^-(t), \Pi_n G^0(t)$ respectively, where $G^+(t), G^-(t), G^0(t)$ are as described below. Let $\mathcal{B}$ be the Borel $\sigma$-field of $[0, 1]$, $\beta$ the trivial $\sigma$-field $\{\phi, [0, 1]\}$ on $[0, 1]$.

Then for $n \geq 1$,

$$G^+(t) = \mathcal{B} \text{ if } t \geq r_n, = \beta \text{ otherwise},$$

$$G^+(t) = \mathcal{B} \text{ if } t < r_n, = \beta \text{ otherwise},$$

$$G^-(t) = \mathcal{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G^-(t) = \mathcal{B} \text{ if } t \leq r_n, = \beta \text{ otherwise},$$

$$G^0(t) = \mathcal{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G^0(t) = \mathcal{B} \text{ if } t < r_n, = \beta \text{ otherwise}.$$
Remark: One may replace $X()$ above by a real valued process without altering the conclusions by virtue of the isomorphism theorem for Polish spaces (Theorem 2.12, p. 14, of [4]).

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References