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LIMIT DISTRIBUTIONS OF POLYNOMIAL TRAJECTORIES ON HOMOGENEOUS SPACES

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1. Introduction. Let G be a Lie group and Γ a lattice in G; that is, Γ is a discrete subgroup of G such that G/Γ admits a finite G-invariant measure. Let $u: \mathbf{R} \to G$ be a unipotent one-parameter subgroup of G; that is, $\operatorname{Ad} u(t)$ is a unipotent linear automorphism of $\operatorname{Lie}(G)$ for all $t \in \mathbf{R}$. The action of $\{u(t): t \in \mathbf{R}\}$ on G/Γ is called a unipotent flow.

Through the series of four fundamental papers [R1], [R2], [R3], [R4] proving the Raghunathan conjectures on "nice algebraic" behaviour of unipotent flows, Marina Ratner proved also the following result: For any $x \in G/\Gamma$, there exists a closed subgroup F of G such that the orbit Fx is closed and admits a unique F-invariant probability measure, say μ_F, and the trajectory $\{u(t)x: t > 0\}$ is uniformly distributed with respect to μ_F. That is, for any bounded continuous function f on G/Γ,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u(t)x)\, dt = \int_{Fx} f\, d\mu_F.$$

Essentially the basic property of a unipotent one-parameter subgroup used in the work of Ratner is that the map $t \mapsto \operatorname{Ad} u(t)$ is a polynomial function in each coordinate of $\operatorname{End}(\operatorname{Lie}(G))$. Therefore it is natural to ask the following question. Let G be a closed subgroup of $\mathrm{SL}_n(\mathbf{R})$, and Γ a lattice in G. Let $\theta: \mathbf{R} \to G$ be a map which is a polynomial function, namely, each matrix coordinate is a polynomial. Then is it true that the trajectory $\{\theta(t)\Gamma: t > 0\}$ is uniformly distributed with respect to a measure of the form μ_F as above? In the case when $G = \mathbf{R}^n$ and $\Gamma = \mathbf{Z}^n$, this indeed holds, as can be deduced from a classical result due to Weyl. In this paper we answer the question affirmatively in a more general setup.

A group G is called *real algebraic* if it is an open subgroup of $\mathbf{R}$-points of an algebraic group $\mathbf{G}$ defined over $\mathbf{R}$. A map $\Phi: \mathbf{R}^k \to G$ is called *regular algebraic* if it is the restriction of a morphism $\Phi: \mathbf{C}^k \to \mathbf{G}$ of algebraic varieties defined over $\mathbf{R}$. We caution the reader that a map such as $\phi: \mathbf{R} \to \mathbf{R}^*$ given by $\phi(t) = 1 + t^2$ for all $t \in \mathbf{R}$ is *not* regular algebraic according to our definition, as ϕ does not extend to an algebraic map from $\mathbf{C}$ to $\mathbf{C}^*$.

The following is the main result.

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