SINGULAR PERTURBATIONS IN ERGODIC CONTROL OF DIFFUSIONS

(RUNNING TITLE: SINGULAR ERGODIC CONTROL)

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ABSTRACT: Ergodic control of a non-degenerate diffusion with two time-scales is studied in the limiting case as the time-scale separation increases to infinity. It is shown that the limit problem is another ergodic control problem for the slow time-scale component alone with its dynamics averaged over the (controlled) invariant probability measures for the fast component. These measures in turn can be treated as the 'effective control variable'.

Key words: controlled diffusions, two time-scales, singular perturbations, ergodic control, invariant probability measures

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1 Introduction

In this paper, we consider a long run average (ergodic) problem of optimal control of non-linear singularly perturbed (**SP**) stochastic differential equations (**SDE**), in which the singular perturbations parameter $\epsilon > 0$ is introduced in such a way that the state variables are decomposed into a group of slow variables that change their values with rates of the order $O(1)$, and a group of fast ones that change their values with rates of the order $O(\frac{1}{\epsilon})$.

Singularly perturbed problems of control and optimization have been studied intensively in both deterministic and stochastic settings (see classic texts [7], [23], [25], [29] and most recent publications [1], [2], [3], [4], [5], [6], [12], [14], [15], [16], [17], [18], [20], [21], [22], [26], [28], [30], [31]). Problems of optimal control of SP SDE have been studied in [1], [7], [12], [21], [25], where earlier references can also be found.

In [12], in particular, it has been established in a very general set-up that, for the problem of optimal control of SP SDE considered on a finite time interval, the limiting problem (obtained when the singular perturbation parameter tends to zero) is an averaged problem, in which the slow dynamics is controlled by stationary marginal distributions of the fast dynamics, obtained with the slow state variables kept "frozen" (note that a deterministic counterpart of this result has been obtained in [17]).

In this article, we continue the line of research started in [12] by establishing the validity of a similar limit behavior for long run average problems of optimal control of SP SDE (referred in the sequel as SP ergodic control problems). Note that in our study we restrict ourselves to the case of non-degenerate diffusions and thus our results complement earlier results obtained in the purely deterministic setting in [18]. Our analysis is largely based on the stability and control theory for non-degenerate diffusions established in [8], [9] and [11].

The paper is organized as follows. We introduce the singularly perturbed ergodic control problem in the next section. Our objective will be to relate this problem to the ergodic control problem for the 'averaged' system obtained in the $\epsilon \to 0$ limit, i.e., to prove that the latter (lower dimensional) problem is a valid approximation to the above problem for small ϵ. The exact definition of the averaged problem is deferred till later after the appropriate terminology has been introduced. Section 3 recalls some known facts about ergodic control, notably the basic existence result (Theorem 3.1 below). Section 4 is devoted to some preliminaries, in particular the definition of the averaged control problem. Section 5 shows that the optimal cost for the averaged problem serves in general as an asymptotic lower bound for the optimal cost for the original problem in the $\epsilon \downarrow 0$ limit (Corollary 5.1). Section 6 shows that in the special case of the control entering the drift in an affine manner and the running cost strict convex in the control, it is in fact the exact limit (Theorem 6.1). This result is extended to a more general case in section 7 under some technical assumptions (Theorem 7.1). Section 8 discusses the 'stable case', where a blanket stability condition is imposed on the controlled diffusion. Section 9 concludes with some discussion, which includes some directions for future research.

2 The control problem

Let $\epsilon > 0$. We consider the coupled pair of stochastic differential equations in $\mathcal{R}^d \times \mathcal{R}^s$ given by

$$dz^\epsilon(t) = h(z^\epsilon(t), x^\epsilon(t), u(t))dt + \gamma(z^\epsilon(t))dB(t), \tag{1}$$

$$dx^\epsilon(t) = \frac{1}{\epsilon}m(z^\epsilon(t), x^\epsilon(t), u(t))dt + \frac{1}{\sqrt{\epsilon}}\sigma(z^\epsilon(t), x^\epsilon(t))dW(t). \tag{2}$$

Here:

- For a prescribed compact metric action space A, $h : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^d$, $\gamma : \mathcal{R}^d \to \mathcal{R}^{d\times d}$, $m : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^s$, $\sigma : \mathcal{R}^d \times \mathcal{R}^s \to \mathcal{R}^{s\times s}$, are Lipschitz in the first and second (if any) arguments uniformly w.r.t. the third (if any),

- The least eigenvalues of $\gamma(z)\gamma(z)^T, \sigma(z,x)\sigma(z,x)^T$ are uniformly bounded away from zero (non-degeneracy assumption).

- The initial values are fixed: $(z^\epsilon(0), x^\epsilon(0)) = (z_0, x_0)$,

- $B(\cdot), W(\cdot)$ are resp. $d-$ and $s-$dimensional independent standard Brownian motions,

- $u(\cdot)$ is an $A-$valued control process with measurable paths satisfying the *non-anticipativity* condition: for $t \geq s$, $(B(t) - B(s), W(t) - W(s))$ is independent of $\mathcal{F}_s \stackrel{def}{=}$ the completion of

$$\cap_{s'>s}\sigma(z^\epsilon(y), x^\epsilon(y), u(y), y \leq s').$$

We call such $u(\cdot)$ an *admissible control*.

We shall impose further restrictions on A, h, m later. The ergodic control problem is to minimize over all admissible $u(\cdot)$ the 'ergodic cost'

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t E[k(z^\epsilon(s), x^\epsilon(s), u(s))]ds. \tag{3}$$

Here $k : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^+$ is a continuous map satisfying

$$\lim_{||(z,x)||\to\infty} \inf_u k(z, x, u) = \infty. \tag{4}$$

We shall discuss a possible relaxation of this condition later. We also assume:

(†) There exists an $\infty > M > 0$ such that for each $\epsilon \in (0, 1)$, the cost for at least one admissible $u(\cdot)$ is $\leq M$.

We shall work with the *weak formulation* of the above control problem and assume that $u(\cdot)$ is a *relaxed* control. That is, for some compact metric space A', $A = \mathcal{P}(A') \stackrel{def}{=}$ the space of probability measures on A' with the Prohorov topology. Moreover, all functions above of the form $f(\cdots, u(t))$ (specifically, k and the components of h, m) are of the form $\int f'(\cdots, y)u(t, dy)$ for an f' satisfying the same conditions as f except that the factor A of its domain is replaced by A'. See [9], Chapter I, for more on this. As above, $\mathcal{P}(Z)$ for a Polish space Z will denote the Polish space of probability measures on Z with the Prohorov topology ([9], Chapter 2).

Furthermore, we assume that the following 'stochastic Liapunov' condition holds: Define $\mathcal{L} : C^2(\mathcal{R}^s)(\stackrel{def}{=}$ the space of twice continuously differentiable functions $\mathcal{R}^s \to \mathcal{R}$) $\to C_b(\mathcal{R}^d \times \mathcal{R}^s \times A')$ by

$$\mathcal{L}f(z, x, u) \stackrel{def}{=} \frac{1}{2}tr\left(\sigma(z, x)\sigma^T(z, x)\nabla^2 f(x)\right) + \langle\nabla f(x), m'(z, x, u)\rangle \tag{5}$$

$\forall\ f \in C^2(\mathcal{R}^s)$. Then there exists a $V \in C^2(\mathcal{R}^s), g \in C(\mathcal{R}^d \times \mathcal{R}^s)$, such that $\lim_{||x||\to\infty} V(x) = \infty$, $\lim_{||x||\to\infty} g(z, x) = \infty$ uniformly in z belonging to any compact subset of $\mathcal{R}^d$, and

$$\mathcal{L}V(z, x, u) \leq -g(z, x). \tag{6}$$

3 Ergodic control

We now recall from [9], Ch. VI, some facts about ergodic control applicable to the above framework. For this purpose, we introduce the notion of a Markov control as a $u(\cdot)$ of the form $u(t) = v(z^\epsilon(t), x^\epsilon(t))\ \forall t$ for a measurable $v : \mathcal{R}^d \times \mathcal{R}^s \to A$. By a standard abuse of terminology, we identify this $u(\cdot)$ with the map v. Note that under a Markov control, $(z^\epsilon(\cdot), x^\epsilon(\cdot))$ will be a time-homogeneous Markov process. In turn, v will be said to be a stable Markov control if the resulting Markov diffusion is positive recurrent and thus has a unique invariant probability measure $\zeta_v(dzdx)$. Furthermore, (3) will then equal $\int k'(z,x,u)v(du|z,x)\zeta_v(dzdx)$. We call $\Phi_v(dzdxdu) \stackrel{def}{=} \zeta_v(dzdx)v(du|z,x)$ the *ergodic occupation measure* associated with v and denote by $\mathcal{G}$ the set of all ergodic occupation measures Φ_v as v varies over all stable Markov controls. This has another characterization as follows: Let

$$\begin{aligned}
&\hat{\mathcal{L}}f(z,x,u) \stackrel{def}{=} \\
&\quad \frac{1}{2}tr\left(\gamma(z)\gamma^T(z)\nabla_z^2 f(z,x)\right) + \langle \nabla_z f(z,x), h'(z,x,u)\rangle \\
&+ \quad \frac{1}{2\epsilon}tr\left(\sigma(z,x)\sigma^T(z,x)\nabla_x^2 f(z,x)\right) + \frac{1}{\epsilon}\langle \nabla_x f(z,x), m'(z,x,u)\rangle,
\end{aligned}$$

where ∇_y, ∇_y^2 denote resp. the gradient and the Hessian in the variable y. Also let $C_0^2(\mathcal{R}^{d+s}) \stackrel{def}{=}$ the space of twice continuously differentiable functions $\mathcal{R}^{d+s} \to \mathcal{R}$ that vanish at infinity, along with their first and second order partial derivatives.

Lemma 3.1 $\mathcal{G} = \{\Phi \in \mathcal{P}(\mathcal{R}^{d+s} \times A') : \int \hat{\mathcal{L}} f d\Phi = 0\ \forall f \in C_0^2(\mathcal{R}^{d+s})\}$.

This follows by Lemma 1.1, p. 144, [9] (see [8] for a more general result). Define the empirical measures $\nu_t, t > 0$, and the average empirical measures $\bar{\nu}_t, t > 0$, by

$$\begin{aligned}
\int f d\nu_t &\stackrel{def}{=} \frac{1}{t}\int_0^t f(z^\epsilon(s), x^\epsilon(s), u(s))ds, \\
\int f d\bar{\nu}_t &\stackrel{def}{=} \frac{1}{t}\int_0^t E[f(z^\epsilon(s), x^\epsilon(s), u(s))]ds,
\end{aligned}$$

for $f \in C_b(\mathcal{R}^{d+s} \times A')$. Let $\mathcal{R}^*$ denote the one point compactification of $\mathcal{R}^d \times \mathcal{R}^s$ with '∞' the point at infinity. Finally, let:

$\mathcal{G}^* \stackrel{def}{=} \{\Phi \in \mathcal{P}(R^* \times A')$: there exist some $0 \leq a \leq 1$, $\phi \in \mathcal{G}$ and $\phi' \in \mathcal{P}(\{\infty\} \times A')$ such that $\Phi(B \times B') = a\phi((B \times B') \cap (\mathcal{R}^{d+s} \times A')) + (1 - a)\phi'((B \times B') \cap (\{\infty\} \times A'))$ $\forall$ B Borel in $\mathcal{R}^*$, B' Borel in $A'\}$.

Lemma 3.2 As $t \uparrow \infty$, $\bar{\nu}_t \to \mathcal{G}^*$ and $\nu_t \to \mathcal{G}^*$ a.s. in $\mathcal{P}(R^* \times A')$.

Lemma 3.2 is proved as in Ch. VI, [9]. The following consequence thereof also follows as in *ibid.*

Theorem 3.1 There exists a stable optimal Markov control v^*_ϵ such that if Φ^*_ϵ is the corresponding ergodic occupation measure, then under any admissible $u(\cdot)$,

$$\begin{aligned}
\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t k(z^\epsilon(s), x^\epsilon(s), u(s))ds &\geq \int k' d\Phi^*_\epsilon \quad \text{a.s.,} \\
\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t E[k(z^\epsilon(s), x^\epsilon(s), u(s))]ds &\geq \int k' d\Phi^*_\epsilon.
\end{aligned}$$

Remark One can in fact show that the v^*_ϵ can be taken to be *precise*, i.e., $v^*_\epsilon(z, x)$ is a Dirac measure for all z, x. This is because the extreme points of $\mathcal{G}$ correspond to precise controls, as proved in [13].

4 The averaged system

Setting $\tau = \frac{t}{\epsilon}, x'(\tau) = x^\epsilon(\epsilon\tau), z'(\tau) = z^\epsilon(\epsilon\tau), u'(\tau) = u(\epsilon\tau), W'(\tau) = \frac{1}{\sqrt{\epsilon}}W(\epsilon\tau)$, (2) becomes

$$dx'(\tau) = m(z'(\tau), x'(\tau), u'(\tau))d\tau + \sigma(z'(\tau), x'(\tau))dW'(\tau),$$

which does not depend on ϵ explicitly. To this we associate the '*associated system*'

$$dx'(\tau) = m(z', x'(\tau), u'(\tau))d\tau + \sigma(z', x'(\tau))dW'(\tau), \tag{7}$$

where z' is fixed, $W'(\cdot)$ a standard Brownian motion independent of $x'(0)$, and admissibility of $u'(\cdot)$ is defined by: for $t > s$, $W'(t) - W'(s)$ is independent of $\Xi_s \stackrel{def}{=}$ the completion of $\cap_{s'>s}\sigma(x'(\tau), u'(\tau), W'(\tau), \tau \leq s')$.

Let $D_z \stackrel{def}{=} \{\mu \in \mathcal{P}(\mathcal{R}^s \times A') : \int \mathcal{L}f(z,x,u)\mu(dxdu) = 0 \ \forall \ f \in C_0^2(\mathcal{R}^s)\}$ where $\mathcal{L}$ is as in (5). The next lemma in particular characterizes this as the set of ergodic occupation measures for the associated system.

Lemma 4.1 $D_z =$ the set of $\mu(dxdu)$ of the form $\mu(dxdu) = \eta(dx)v(du|x)$, where η is the unique stationary distribution for the time-homogeneous Markov diffusion $X(\cdot)$ given by (7) when $u(\cdot) = v(X(\cdot)) \stackrel{def}{=} v(du|X(\cdot))$. The set valued map $z \to D_z$ is convex compact valued and continuous. Furthermore, for compact $B \subset \mathcal{R}^d$, $\cup_{z \in B} D_z$ is compact.

Proof The first claim follows from Lemma 1.1, p. 144, of [9]. That D_z is convex closed for each z is easily verified from the definition. Thus we need to verify its relative compactness in $\mathcal{P}(\mathcal{R}^s \times A')$. Since A' is compact, it suffices to verify the compactness of the corresponding marginals $\eta(dx)$ in $\mathcal{P}(\mathcal{R}^s)$. Under our assumption (6), this is proved in [11]. Next, let $z_n \to z_\infty$ and $\mu_n \in D_{z_n} \ \forall \ n, 1 \leq n < \infty$. Then: (i) $\{\mu_n\}$ are tight by arguments similar to those used in [11], and, (ii) any limit point μ thereof is in D_{z_∞} - this is easily verified from the definition of D_z. Thus $z \to D_z$ is upper semi-continuous. Now fix a $\mu(dxdu) = \eta_{z_\infty}(dx)v(du|x) \in D_{z_\infty}$. Under $z = z_n$, the stationary Markov control $v(du|x)$ leads to a unique stationary distribution $\eta_{z_n}(dx), 1 \leq n \leq \infty$. By our non-degeneracy assumption, the transition probabilities for $t > 0$ of the corresponding time-homogeneous Markov processes have densities w.r.t. the Lebesgue measure. Therefore so do the corresponding invariant probability measures $\{\eta_{z_n}\}$. Let $\{\chi_{z_n}(\cdot)\}$ denote these densities. We claim that they are pointwise bounded and equicontinuous. If pointwise boundedness does not hold, $\chi_{z_n}(x^*) \uparrow \infty$ for some x^*. But $\{\chi_{z_n}\}$ satisfy $(\mathcal{L}_v^{z_n})^* \chi_{z_n} \equiv 0$, where

$$\mathcal{L}_v^z \stackrel{def}{=} \frac{1}{2} tr\left(\sigma(z,x)\sigma^T(z,x)\nabla_x^2\right) + \langle m(z,x,v(x)), \nabla_x \rangle,$$

and '$(\mathcal{L}_v^z)^*$' denotes its formal adjoint given by

$$(\mathcal{L}_v^z)^* f \stackrel{def}{=} \frac{1}{2} \sum_{i,j,k} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sigma_{ik}(z,x)\sigma_{jk}(z,x)f(x)\right) - \sum_i \frac{\partial}{\partial x_i} \left(m_i(z,x,v(x))f\right).$$

By Harnack's inequality (Theorem 8.20, p. 199, [19]), the ratio of the maximum to the minimum of $\chi_{z_n}(\cdot)$ on any compact set must remain bounded uniformly in n. Thus $\chi_{z_n}(\cdot) \uparrow \infty$ uniformly on compacts, which contradicts the fact that they are probability densities. Hence they are pointwise bounded. By Theorem 8.24, p. 202, [19], they satisfy a uniform Hölder continuity condition, which gives equicontinuity. In particular, $\chi_{z_n}(\cdot|x)$ are uniformly continuous on compacts. The equation

$$\int \mathcal{L}_v^{z_n} f(x)\eta_{z_n}(dx) = 0 \ \forall \ f \in C_0^2(\mathcal{R}^s) \tag{8}$$

characterizes $\eta_{z_n}(dx)$ and therefore $\chi_{z_n}(\cdot)$. Let η^* denote a limit point of $\eta_{z_n}(dx)$ in $\mathcal{P}(\mathcal{R}^s)$ as $n \uparrow \infty$. By Arzela-Ascoli theorem, we may drop to a subsequence if necessary and suppose that $\chi_{z_n}(\cdot) \rightarrow \chi^*(\cdot)$ in $C(\mathcal{R}^s)$. Then for compactly supported $f \in C(\mathcal{R}^s)$,

$$\int f(x)\chi_{z_n}(x)dx \rightarrow \int f(x)\chi^*(x)dx,$$

implying that $\eta^*(dx) = \chi^*(x)dx$. By Scheffe's theorem ([10], p. 26), we have $\eta_{z_n}(dx) \rightarrow \eta^*$ in total variation. Hence we can let $n \rightarrow \infty$ in (8) to obtain

$$\int \mathcal{L}_v^{z_\infty} f d\eta^* = 0 \ \forall \ f \in C_0^2(\mathcal{R}^s),$$

implying $\eta^*(dx) = \eta_{z_\infty}(dx)$. Thus the lower semi-continuity of $z \rightarrow D_z$ follows. Together, upper and lower semi-continuity imply continuity of this set-valued map. Compactness of $\cup_{z \in B} D_z$ is proved by an argument similar to that used for proving upper semi-continuity. □

In particular, it follows that $\{(z, \mu) : z \in \mathcal{R}^s, \mu \in D_z\}$ is closed and $\{(z, \mu) : z \in B, \mu \in D_z\}$ is compact for compact $B \subset \mathcal{R}^s$. Define

$$\begin{aligned} \bar{h}(z, \mu) &\stackrel{def}{=} \int h'(z, x, u)\mu(dxdu), \\ \bar{k}(z, \mu) &\stackrel{def}{=} \int k'(z, x, u)\mu(dxdu). \end{aligned}$$

The *averaged system* is defined by

$$dz(t) = \bar{h}(z(t), \mu(t))dt + \gamma(z(t))dB'(t), \tag{9}$$

$$\mu(t) \in D_{z(t)} \ \forall \ t. \tag{10}$$

Here $z(0) = z_0$ (the same as in (1)), $B'(\cdot)$ is a standard Brownian motion in $\mathcal{R}^d$, and $\mu(\cdot)$ satisfies (10) and the non-anticipativity condition: for $t \geq s \geq 0$,

$B'(t) - B'(s)$ is independent of $z(y), B'(y), \mu(y), y \leq s$. We may view $\mu(\cdot)$ as the 'effective control process' for the averaged system. The objective for the averaged control problem is to minimize

$$\limsup_{t\uparrow\infty} \frac{1}{t} E[\int_0^t \bar{k}(z(s), \mu(s))]ds$$

over all admissible $\mu(\cdot)$. By analogy with section 2, we call $\mu(\cdot)$ a Markov control if $\mu(t) = q(z(t)) \stackrel{def}{=} q(dxdu|z(t))\ \forall t$, identified with the measurable map q. Call it a stable Markov control if in addition the resulting time homogeneous Markov process $z(\cdot)$ is positive recurrent. In the latter case, $z(\cdot)$ will have a unique invariant probability distribution $\varphi_q(dz)$ and the corresponding ergodic occupation measure $\Gamma(dzdxdu) \stackrel{def}{=} \varphi_q(dz)q(dxdu|z)$. Let $\mathcal{Q}$ denote the set of such Γ. Then as before, one has the following characterization. Define $\tilde{\mathcal{L}} : C_0^2(\mathcal{R}^d) \to C_b(\mathcal{R}^d \times \mathcal{P}(\mathcal{R}^s \times A')))$ by

$$\tilde{\mathcal{L}} f(z, \mu) = \frac{1}{2} tr\left(\gamma(z)\nabla^2 f(z)\right) + \langle \nabla f(z), \bar{h}(z, \mu)\rangle.$$

Lemma 4.2

$$\begin{aligned} \mathcal{Q} \;\; = \;\; & \{\xi = q(dxdu|z)\phi(dz) \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A') : q(\cdot|z) \in D_z\ \forall z, \\ & \int \tilde{\mathcal{L}} f(z, q(dxdu|z))\phi(dz) = 0\ \forall\ f \in C_0^2(\mathcal{R}^d)\}. \end{aligned}$$

This is immediate from by Lemma 1.1, p. 144, [9]. We have then the following counterpart of Theorem 3.1, proved analogously.

Theorem 4.1 There exists a stable optimal Markov control q^* for the averaged system such that if $\Gamma^*(dzdxdu) = q^*(dxdu|z)\varphi^*(dz)$ is the corresponding ergodic occupation measure, then for any admissible $\mu(\cdot)$ as above,

$$\begin{aligned} \liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t \bar{k}(z(s), \mu(s))ds \;\; &\geq \;\; \int k' d\Gamma^* \text{ a.s.}, \\ \liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t E[\bar{k}(z(s), \mu(s))]ds \;\; &\geq \;\; \int k' d\Gamma^*. \end{aligned}$$

Let $\mathcal{Q}_{opt}$ denote the set of optimal ergodic occupation measures, i.e., Argmin$\{\int k' d\xi : \xi \in \mathcal{Q}\}$. Also, write $q^*(dxdu|z)$ above as $q^*(dxdu|z) = v^*(du|z, x)\eta^*(dx|z)$.

5 A lower bound

We now consider the $\epsilon \downarrow 0$ limit. Let Φ^*_ϵ be as in Theorem 3.1 above. Then by (†) and (4), it follows that $\{\Phi^*_\epsilon, \epsilon \in (0,1)\}$ is tight. Let Φ^*_0 be a limit point thereof in $\mathcal{P}(\mathcal{R}^{d+s} \times A')$.

Theorem 5.1 $\Phi^*_0 \in \mathcal{Q}$.

Proof Disintegrate Φ^*_0 as

$$\begin{aligned}\Phi^*_0(dzdxdu) &= \varphi(dz)\mu(dxdu|z) \\ &= \varphi(dz)\eta(dx|z)v(du|z,x).\end{aligned}$$

(In particular, $\mu(dxdu|z) = \eta(dx|z)v(du|z,x)$.) Let $f \in C^2_0(\mathcal{R}^d), g \in C^2_0(\mathcal{R}^s)$. Let $\epsilon \downarrow 0$ in the equation $\epsilon \int \hat{\mathcal{L}}(fg)d\Phi^*_\epsilon = 0$ to obtain

$$\int f(z) \int \mathcal{L}g(z,x,u)\mu(dxdu|z)\varphi(dz) = 0. \tag{11}$$

Then as (11) holds for all $f \in C^2_0(\mathcal{R}^d)$, we conclude that for $\varphi-$a.s. z,

$$\int \mathcal{L}g(z,x,u)d\mu(dxdu|z) = 0,$$

implying that $\mu(dxdu|z) \in D_z$. The qualification '$\varphi-$a.s.' may be dropped by choosing a suitable version. Now for $h \in C^2_0(\mathcal{R}^d)$ (i.e., h is a function of $z \in \mathcal{R}^d$ alone), let $\epsilon \downarrow 0$ in $\int \hat{\mathcal{L}}hd\Phi^*_\epsilon = 0$ to obtain

$$\int \hat{\mathcal{L}}hd\Phi^*_0 = \int \tilde{\mathcal{L}}h(z,\mu(\cdot|z))\varphi(dz) = 0. \tag{12}$$

By Lemma 1.1, p. 144, [9], (12) implies that φ is the unique stationary distribution under μ for the averaged system. It follows that $\Phi^*_0 \in \mathcal{Q}$. □

Corollary 5.1 $\liminf_{\epsilon\downarrow 0} \int k' d\Phi^*_\epsilon \geq \int \bar{k} d\Gamma^*$.

This shows that the optimal ergodic cost for the averaged problem provides an asymptotic lower bound (as $\epsilon \downarrow 0$) for the optimal ergodic cost of the original problem. To show that it is in fact a valid approximation, we must replace the 'lim inf' by 'lim' in the above and the inequality by an equality. We shall do so under additional assumptions in the following sections.

6 Main results - the affine case

Assume the following:

- **(*)** A' is a compact subset of $\mathcal{R}^m$ for some $m \geq 1$ and for each z, x, $h'(z, x, \cdot)$, $m'(z, x, \cdot)$ are componentwise affine and $k'(z, x, \cdot)$ is strictly convex.
- **(**)** $||h'(z, x, u)|| = o(k'(z, x, u))$ as $||(z, x)|| \uparrow \infty$ and

$$\sup_u |k'(z, x, u)|^{1+a} \leq Kg(z, x)$$

  for some $K, a > 0$ and g as in (6).

The next lemma, which uses only **(*)** and **(**)**, shows in particular that v^* above is unique. Thus we can state our third assumption:

- **(***)** $v^*(z, x) \stackrel{def}{=} v^*(du|z, x)$ is a stable Markov control for (1), (2) for sufficiently small $\epsilon > 0$ (say, $\epsilon < \epsilon_0$) and the corresponding stationary distributions, denoted $\zeta^\epsilon(dzdx), 0 < \epsilon < \epsilon_0$, are tight.

A stochastic Liapunov condition along the lines of (6) can be given to ensure this.

Lemma 6.1 $v^*(du|z, x)$ above is unique and continuous in z, x.

Proof By Theorem 3.3, p. 163, [9], a necessary and sufficient condition for the optimality of q^* is that $q^*(z)$ minimize the function

$$\mu \to \bar{k}(z, \mu) + \langle \nabla \Psi(z), \bar{h}(z, \mu) \rangle, \tag{13}$$

over D_z for a.e. z, where $\Psi \in C^2(\mathcal{R}^d)$ is the value function for the ergodic control problem for the averaged system[3]. We may drop the qualification

[3] [9] proves the existence of a C^2 value function and the associated 'verification theorem' for nondegenerate diffusions with bounded coefficients and the so called 'near-monotone' cost, for the case when the control space is state-independent. The latter would correspond to D_z being independent of z in the present set-up. Condition (4) is a special case of near-monotonicity. The modifications required to handle the more general Lipschitz coefficients and state-dependent control space needed here are minor in view of the continuity of the set-valued map $z \to D_z$ already established.

'for a.e. z' by taking an appropriate version. Now for fixed z, consider the ergodic control problem for the associated system (7) with cost

$$\limsup_{t\uparrow\infty} \frac{1}{t}\int_0^t E[\ell_z(x'(s), a(t))]ds,$$

where $\ell_z \in C(\mathcal{R}^s \times A')$ is defined by

$$\ell_z(x,a) \stackrel{def}{=} k'(z,x,a) + \langle \nabla\Psi(z), h'(z,x,a)\rangle.$$

Since D_z is precisely the set of ergodic occupation measures for the associated system, q^* is the optimal ergodic occupation measure for the above problem. By **(**)**, Theorem 3.3, p. 163, can be applied again to this new control problem, in order to conclude as above that $v^*(du|z,x)$ minimizes

$$\kappa \to \int (\ell_z(x,\cdot) + \langle \nabla\tilde{\Psi}_z(x), m'(z,x,\cdot)\rangle) d\kappa,$$

where $\tilde{\Psi}_z \in C^2(\mathcal{R}^s)$ is the value function for this new ergodic control problem[4]. By Theorem 2.1, p. 183, [9], it follows that the map $(z,x) \to \nabla\tilde{\Psi}_z(x)$ is continuous. By **(*)**, the above minimum is attended at a unique point. It is easy to see then that this point will depend continuously on z, x. That is, $(z,x) \to v^*(z,x)$ is continuous. □

Remark: Note that $v^*(z,x)$ will in fact be Dirac for all z, x.

Recall the measures q^*, Γ^* from Theorem 4.1.

Corollary 6.1 q^*, Γ^* are unique.

Proof Since $q^*(dxdu|z) = \eta^*(dx|z)v^*(du|z,x)$ where $\eta^*(dx|z)$ is the unique stationary distribution for the associated system under $v^*(du|z,x)$, uniqueness of q^* follows. In turn, $\Gamma^*(dzdxdu) = q^*(dxdu|z)\varphi^*(dz)$ where φ^* is the unique stationary distribution of the averaged system under $q^*(dxdu|z)$. Thus Γ^* is unique. □

Let

$$\tilde{\Phi}_\epsilon(dzdxdu) \stackrel{def}{=} \zeta^\epsilon(dzdx)v^*(du|z,x), \epsilon \in (0,\epsilon_0),$$

[4]Note that for each z, this cost function satisfies a condition akin to (4) and thus the above remarks apply.

and v^* as above.

Theorem 6.1 $\tilde{\Phi}_\epsilon \to \mathcal{Q}_{opt}$ in $\mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A')$.

Proof In view of Theorem 5.1, it suffices to prove that

$$\lim_{\epsilon \downarrow 0} \int k' d\tilde{\Phi}_\epsilon = \int k' d\Gamma^*. \tag{14}$$

Let $\zeta^\epsilon(dzdx) \to \hat{\zeta}(dzdx) = \hat{\varphi}(dz)\hat{\eta}(dx|z)$ along a subsequence as $\epsilon \downarrow 0$. In view of the continuity of $v^*(du|\cdot,\cdot)$, we may pass to the limit along this subsequence in

$$\epsilon \int \hat{\mathcal{L}} f(z,x,u) v^*(du|z,x)\zeta^\epsilon(dzdx) = 0, \ f \in C_0^2(\mathcal{R}^{d+s}),$$

to obtain

$$\int \mathcal{L} f(z,x,u) v^*(du|z,x)\hat{\zeta}(dzdx) = 0, \ f \in C_0^2(\mathcal{R}^{d+s}).$$

Argue as in Theorem 5.1 to conclude that $\hat{\eta}(dx|z)$ is in fact the unique stationary distribution for the associated system controlled by $v^*(du|z,x)$ (i.e., $\hat{\eta}(dx|z) = \eta^*(dx|z)$) for $\hat{\varphi}-$a.s. z. The latter qualification may be dropped by choosing an appropriate version. Recall that $q^*(dxdu|z) = \eta^*(dx|z)v^*(du|z,x)$ for all z. Let $\epsilon \downarrow 0$ in

$$\int \hat{\mathcal{L}} f(z,x,u) v^*(du|z,x)\zeta^\epsilon(dzdx) = 0,$$

for $f \in C_0^2(\mathcal{R}^d)$ (i.e., f is a C^2 function of the z variable alone). An argument similar to the above then yields

$$\int \tilde{\mathcal{L}} f(z, q^*(\cdot|z))\hat{\varphi}(dz) = 0, \ f \in C_0^2(\mathcal{R}^d).$$

Thus $\hat{\varphi}(dz)$ is the unique stationary distribution for the averaged system controlled by the stable Markov control q^*, i.e., $\hat{\varphi} = \varphi^*$. Then

$$v^*(du|z,x)\hat{\zeta}(dzdx) = \Gamma^*(dzdxdu).$$

That is, $\tilde{\Phi}_\epsilon \to \Gamma^*$. By (6) and Theorem 8.3 of [11], $\int g d\Phi$ is uniformly bounded as Φ varies over $\mathcal{Q}$. By the second half of (**), it then follows that k' is uniformly integrable over $\mathcal{Q}$. Hence (14) holds. □

7 Main results - the general case

Now we drop **(*)**. Define $v_\delta^*(du|z,x), \delta > 0$ small (say, $\delta \in (0, \delta_0]$), by

$$\int f v_\delta^*(du|z,x) \stackrel{def}{=} \int\int f v^*(du|z',x')\pi_\delta(z-z',x-x')dz'dx', \ f \in C(A'),$$

where $\{\pi_\delta : \mathcal{R}^{d+s} \to \mathcal{R}, \delta \in (0, \delta_0]\}$ are smooth approximations to the Dirac measure, i.e., compactly supported C^∞ probability density functions such that $\pi_\delta(z,x)dzdx \to \delta_{(0,0)}$ in $\mathcal{P}(\mathcal{R}^{d+s})$ as $\delta \downarrow 0$. In the following, $v_0^*(du|z,x) \stackrel{def}{=} v^*(du|z,x)$ and all quantities with subscript $\delta = 0$ correspond to it. Replace **(***)** by **(A1)**, **(A2)** below:

(A1) $v_\delta^*(z,x) \stackrel{def}{=} v_\delta^*(du|z,x)$ is a stable Markov control for (1), (2) for $\delta \in [0, \delta_0], \epsilon \in (0, \epsilon_0)$. Furthermore, there exists a $\hat{g} \in C(\mathcal{R}^{d+s})$ satisfying:

$$\sup_u |k'(z,x,u)|^{1+a} \leq K\hat{g}(z,x), \tag{15}$$

such that the stationary distributions of (1), (2) corresponding to $\{v_\delta^*\}$, denoted by $\zeta_\delta^\epsilon(dzdx), 0 < \epsilon < \epsilon_0$, satisfy:

$$\sup_{0<\epsilon<\epsilon_0} \int \hat{g}(z,x)\zeta_\delta^\epsilon(dzdx) < \infty \tag{16}$$

for each $\delta \in [0, \delta_0]$.

Once again in view of our non-degeneracy assumption, the transition probabilities for $t > 0$ of the time-homogeneous Markov process described by (7) under Markov control $v_\delta^*, \delta \in [0, \delta_0]$, have densities w.r.t. the Lebesgue measure. Therefore so do the corresponding invariant probability measures $\hat{\eta}_\delta(dx|z)$. Let $\chi_\delta(x|z)$ denote this density. Let $\bar{\mu}_\delta(dxdu|z) \stackrel{def}{=} \hat{\eta}_\delta(dx|z)v_\delta^*(du|z,x)$ and $\bar{\varphi}_\delta$ the unique stationary distribution for (9) under the Markov control $\bar{\mu}_\delta$. Let $\zeta_\delta^0(dzdx) \stackrel{def}{=} \hat{\eta}_\delta(dx|z)\bar{\varphi}_\delta(dz)$ and $\Phi_\delta^0(dzdxdu) \stackrel{def}{=} \zeta_\delta^0(dzdx)v_\delta^*(du|z,x)$ for δ as above. Note that $\Phi_0^0 \in \mathcal{Q}_{opt}$. We also assume:

(A2) $\bar{\mu}_\delta(dxdu|z)$ is a stable Markov control for (9) for $\delta \in [0, \delta_0]$, and for $\hat{g}$ as above,

$$\sup_{\delta\in[0,\delta_0]} \int \hat{g}(z,x)\zeta_\delta^0(dzdx) < \infty. \tag{17}$$

In view of the results of [11], this implies in particular that $\zeta^0_\delta, \delta \in [0, \delta_0]$, and therefore $\bar{\varphi}_\delta, \delta \in [0, \delta_0]$, form tight sets.

Lemma 7.1 As $(\delta_n, z_n) \to (\delta, z)$ in $[0, \delta^*] \times \mathcal{R}^d$, $\hat{\eta}_{\delta_n}(dx|z_n) \to \hat{\eta}_\delta(dx|z)$ in total variation.

Proof This follows by an argument based on Harnack inequality as in the proof of Lemma 4.1, using the fact that $\chi_\delta(\cdot|z)$ will be equicontinuous pointwise bounded. □

Lemma 7.2 $\int k' d\Phi^0_\delta \to \int k' d\Phi^0_0$ as $\delta \downarrow 0$.

Proof By (17) and the results of [11], $\bar{\varphi}_\delta, \delta \in [0, \delta_0]$, are tight. Let $\bar{\varphi}$ be any limit point of $\bar{\varphi}_\delta$ as $\delta \downarrow 0$. Since $\bar{\varphi}_\delta$ is characterized by

$$\int \tilde{\mathcal{L}} f(z, \bar{\mu}_\delta(\cdot|z)) \bar{\varphi}_\delta(dz) = 0, \;\; f \in C^2(\mathcal{R}^s), \tag{18}$$

an argument based on the Harnack inequality analogous to that of Lemma 4.1 implies that this convergence is in fact in total variation. Now for $f \in C_b(\mathcal{R}^d \times \mathcal{R}^s \times A')$,

$$\int f(z, x, u) v^*_\delta(du|z, x) \to \int f(z, x, u) v^*_0(du|z, x) \;\; \text{a.e.}$$

Hence by Lemma 7.1,

$$\int \int f(z, x, u) v^*_\delta(du|z, x) \hat{\eta}_\delta(dx|z) \to \int \int f(z, x, u) v^*_0(du|z, x) \hat{\eta}_0(dx|z)$$

a.e., which in turn leads to

$$\begin{aligned} & \int \int \int f(z, x, u) v^*_\delta(du|z, x) \hat{\eta}_\delta(dx|z) \bar{\varphi}_\delta(dz) \\ \to \; & \int \int \int f(z, x, u) v^*_0(du|z, x) \hat{\eta}_0(dx|z) \bar{\varphi}_0(dz). \end{aligned}$$

In particular, letting $\delta \downarrow 0$ along an appropriate subsequence in (18), we have

$$\int \tilde{\mathcal{L}} f(z, \bar{\mu}_0(\cdot|z)) \bar{\varphi}(dz) = 0, \;\; f \in C^2(\mathcal{R}^s), \tag{19}$$

i.e., $\bar{\varphi} = \bar{\varphi}_0$. Thus $\Phi^0_\delta = \mu_\delta(dxdu|z)\bar{\varphi}_\delta(dz) \to \Phi^0_0 = \mu_0(dxdu|z)\bar{\varphi}_0(dz)$ as $\delta \downarrow 0$. (15), (17) ensure uniform integrability of k' under these, which in turn implies the claim. □

Going back to (1), (2), let $u(\cdot) = v^*_\delta(z^\epsilon(\cdot), x^\epsilon(\cdot))$ and $\Phi^\epsilon_\delta \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A')$ the corresponding ergodic occupation measure for $\delta > 0$.

Lemma 7.3 $\int k' d\Phi^\epsilon_\delta \to \int k' d\Phi^0_\delta$ as $\epsilon \downarrow 0$.

The proof goes along similar lines using (16) in place of (17), and is omitted.

Theorem 7.1 $\lim_{\epsilon\downarrow 0} \int k' d\Phi^*_\epsilon = \int k' d\Phi^*_0$.

Proof Fix $\alpha > 0$ and take $\delta > 0$ small enough such that

$$|\int k' d\Phi^0_\delta - \int k' d\Phi^0_0| < \frac{\alpha}{2}.$$

Then pick $\epsilon > 0$ small enough so that

$$|\int k' d\Phi^\epsilon_\delta - \int k' d\Phi^0_\delta| < \frac{\alpha}{2}.$$

Thus

$$\begin{aligned} \limsup_{\epsilon\downarrow 0} \int k' d\Phi^*_\epsilon &\leq \limsup_{\epsilon\downarrow 0} \int k' d\Phi^\epsilon_\delta \\ &\leq \int k' d\Phi^0_0 + \alpha. \end{aligned}$$

Since $\alpha > 0$ is arbitrary, the claim follows in view of Corollary 5.1. □

We conclude this section by pointing out a routine extension of the condition (4): it can be replaced by the weaker requirement

$$\lim_{||z||\to\infty} \inf_{x,u} k(z, x, u) > \beta^* \stackrel{def}{=} \sup_{0\leq\epsilon<\epsilon_0} \beta^\epsilon, \tag{20}$$

for some $\epsilon_0 > 0$, where $\beta^\epsilon, \epsilon > 0$, is the optimal cost for the ergodic control problem ($\epsilon = 0$ corresponds to the same for the averaged problem). This goes exactly along the lines of Chap. VI, [9]. Since in particular this presupposes that β^ϵ are uniformly bounded for $\epsilon \in (0, \epsilon_0)$, we may replace the '$\sup_{0\leq\epsilon<\epsilon_0} \beta^\epsilon$' above by '$\sup_{0<\epsilon<\epsilon_0} \beta^\epsilon$' in view of Theorem 5.1.

8 The stable case

We briefly indicate the corresponding developments when a blanket stability condition is available. We do not assume (4) or its generalization (20), but require that k' be bounded from below. Suppose for $\epsilon \in (0, \epsilon_0)$ there exist $\Delta_\epsilon, a_\epsilon > 0, B \subset \mathcal{R}^{d+s}$ bounded and $V_\epsilon^{(i)} \in C^2(\mathcal{R}^{d+s}), i = 1, 2$, such that $V_\epsilon^{(i)} \geq 0, \lim_{||(z,x)|| \to \infty} V_\epsilon^{(i)}(z,x) = \infty$ for $i = 1, 2$, and,

$$\hat{\mathcal{L}} V_\epsilon^{(1)}(z,x,u) \leq -\Delta, \tag{21}$$

$$\hat{\mathcal{L}} V_\epsilon^{(2)}(z,x,u) \leq -a_\epsilon V_\epsilon^{(1)}(z,x), \tag{22}$$

for $(z,x) \notin B$. Let $\tau \stackrel{def}{=} \inf\{t \geq 0 : (z^\epsilon(t), x^\epsilon(t)) \in B\}$ and $\bar{\tau}_N \stackrel{def}{=} \inf\{t \geq 0 : ||(z^\epsilon(t), x^\epsilon(t))|| > N\}, N \geq 1$. Then $\bar{\tau}_N \uparrow \infty$ and for $(z_0, x_0) \notin B$, the Ito-Dynkin formula and (21) lead to

$$E[V_\epsilon^{(1)}(z^\epsilon(\tau \wedge \bar{\tau}_N), x^\epsilon(\tau \wedge \bar{\tau}_N))] - V_\epsilon^{(1)}(z_0, x_0) \leq -\Delta_\epsilon E[\tau \wedge \bar{\tau}_N].$$

Letting $N \uparrow \infty$ and rearranging terms, we have

$$E[\tau] \leq \frac{V_\epsilon^{(1)}(z_0, x_0)}{\Delta_\epsilon}.$$

Similarly from (22) we get

$$E[\int_0^{\tau \wedge \bar{\tau}_N} V_\epsilon^{(1)}(z^\epsilon(t), x^\epsilon(t)) dt] \leq \frac{V_\epsilon^{(2)}(z_0, x_0)}{a_\epsilon},$$

and therefore

$$\begin{aligned}
&\frac{1}{2} E[(\tau \wedge \bar{\tau}_N)^2] = \\
&= E[\int_0^{\tau \wedge \bar{\tau}_N} (\tau \wedge \bar{\tau}_N - t) dt] \\
&= E[\int_0^{\tau \wedge \bar{\tau}_N} E[\tau \wedge \bar{\tau}_N - t | \mathcal{F}_t] dt] \\
&\leq \frac{1}{\Delta_\epsilon} E[\int_0^{\tau \wedge \bar{\tau}_N} V_\epsilon^{(1)}(z^\epsilon(t), x^\epsilon(t)) dt] \\
&\leq \frac{1}{a_\epsilon \Delta_\epsilon} V^{(2)}(z_0, x_0).
\end{aligned}$$

Letting $N \uparrow \infty$,

$$E[\tau^2] \leq \frac{2}{a_\epsilon \Delta_\epsilon} V^{(2)}(z_0, x_0).$$

In view of this, one can argue as in Ch. VI, [9], to conclude Theorem 3.1. Conditions similar to (21), (22) imposed on (9) ensure Theorem 4.1. Next, for obtaining the counterparts of the results of section 6 above for the affine case, assume the additional conditions stipulated in section VI.4 of [9] to ensure the existence of C^2 value functions for the two ergodic control problems that feature in the proof of Lemma 6.1. The rest remains as before.

9 Some future directions

We conclude by pointing out some further possibilities. We have not allowed σ to depend on the control variable u or γ to depend on either u or x. This is because such dependence would lead to a diffusion matrix which is measurable but not necessarily continuous under a Markov control (for the averaged system in the latter case). Even in the non-degenerate case, only the existence of weak solutions is known for this level of generality, not their uniqueness [24], [27]. It may be possible to work with 'set of all weak solutions' in place of the unique weak solution and extend the foregoing. In the degenerate case, even with the existing form of (1), (2), there are problems. The results of [8] extend the characterization of ergodic occupation measures from Lemma 1.1, p. 144, [9], which allows us to prove Theorem 5.1 under suitable hypotheses. But Theorem 6.1 is a more difficult proposition due to lack of ergodicity and other problems.

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