THE SMALL MOTION AT THE NODES OF A VIBRATING STRING.[1]

By C. V. Raman.

IT is generally recognized that the nodes of a string which is maintained permanently in oscillation cannot be points of absolute rest, as the energy requisite for the maintenance of the vibrations is transmitted through these points. I have not however seen anywhere a discussion or experimental demonstration of some peculiar properties of this small motion. I shall therefore endeavor to give an account of some experiments and observations of mine relating to this subject.

2. Some rather striking effects are observed when the small motion at a node is viewed stroboscopically, *i. e.*, under periodic illumination. For this purpose, the frequency of intermittence of the light should be nearly twice that of the oscillation of the string. A tuning-fork maintains the string in oscillation in any convenient number of loops, by imposing a transverse obligatory motion at one point of it. Another tuning-fork, which has nearly twice the frequency of the other, forms the interrupter of a Ruhmkorff's coil, the spark from which furnishes the periodic source of illumination. Both forks are electrically self-maintaining. The string is seen in *two* slowly-moving positions, which represent opposite phases of the actual motion. If the nodes were points of absolute rest, then the two positions seen under the periodic illumination would intersect at fixed points. On account, however, of the small transverse motion at the nodes, the points of intersection or "fictitious nodes" are seen to execute a motion of *large* amplitude parallel to the string—the range of the motion being equal to the whole length of a loop. This motion, best seen under a magnifying glass, is represented in Fig. 1, in which nine successive stages at equal intervals of a complete cycle are shown.

[1] A preliminary note on this subject was published in Nature, November 4, 1909, as a letter to the Editor.

3. The "fictitious" node is in the first stage at the center of the field. It then moves to one side of the field, first slowly, then more rapidly; at the fourth stage, it is off the field; at the fifth, the two positions of the string are, at the center of field (*i. e.*, at the position of the node in the actual oscillation), sensibly parallel to each other. It then reappears on the other side of the field, moves in rapidly, then more slowly; at the ninth stage, it is back again at the center of the field.

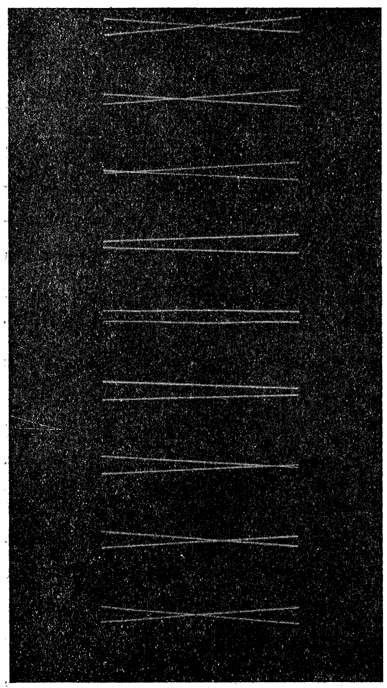

Fig. 1.

3. A motion of the type shown in Fig. 1 can be represented mathematically by the expression $A \sin \alpha x \cos pt + B \cos \alpha x \sin pt$, where x is the distance from the center of the field, t is the time, the other quantities being constants or nearly so. The two terms differ in phase by quarter of an oscillation. The significance of this is that the small motion at the node and the large motion elsewhere differ in phase by quarter of an oscillation. If the expression for the displacement were of the type $A \sin \alpha x \cos pt + B \cos \alpha x \sin (pt + E)$, the two terms differing in phase by more or less than $\pi/2$, the motion would not be of the type shown in Fig. 1. It would be unsymmetrical, the velocity of the point of intersection when at a given distance from the center of the field on one side and approaching it, being much greater than its velocity when at the same distance on the other side and receding from the center.

5. Another experiment, which was first performed by me in collaboration with Mr. V. Apparao, of the Presidency College, Madras, was found later to furnish a second method of determining the phase of the small motion at the nodes. The principle of this

method was to compound the oscillation at every point on the string with another perpendicular to it of half the frequency and to observe the compound oscillation at the nodes and elsewhere. A string can be maintained permanently in a compound oscillation of this character by attaching one end of it to the prong of an electrically-maintained tuning-fork, so that it lies in a plane perpendicular to the prongs, but in a direction inclined to their line of vibration.[1]

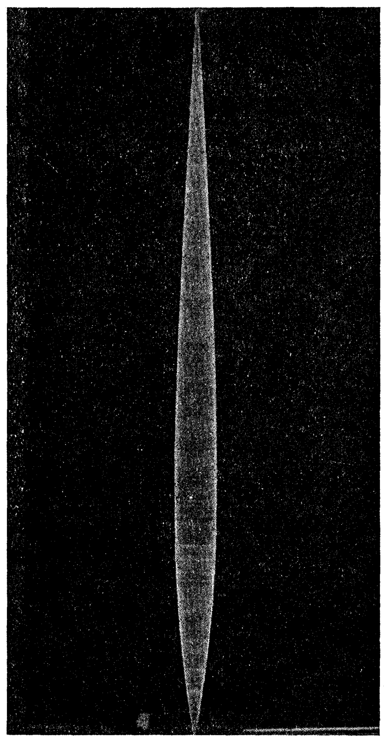

Fig. 2.

6. A beautiful and interesting type of stationary oscillation is maintained when the tension is somewhat greater than that necessary for the most vigorous maintenance. The curves described by points on the string are then parabolic arcs.[2] As the frequency of oscillation in one plane is half that in a perpendicular plane, there are *two* vibration-loops in the latter for every *one* in the former. The consequence of this is that the parabolic arcs which form the paths of points on the string have their curvatures in opposite directions in alternate halves of a big loop, *i. e.*, in alternate small loops. The surface generated by the moving string is one of great delicacy and purity, and an adequate idea of it can only be had on actually performing the experiment. The photograph herewith

[1] Under these circumstances, the motion of the prong may be resolved into two components, one perpendicular, and the other parallel to the string. The first maintains an oscillation having the same frequency as that of the fork, and the secomd maintains an oscillation having half that frequency. The two oscillations occur, or can be made to occur in perpendicular planes.

[2] A parabolic arc is one of the Lissajons figures for the 1:2 composition ratio.

published (Fig. 2) gives only a very feeble idea of the effect. It can be seen from the shading that the paths of points on the string are curved arcs, and that the curvatures are in opposite directions in the two halves of the loop (which appear unequal in the photograph as the string pointed towards the lens of the camera).

7. For the study of the small motion at the nodes of the oscillation excited by transverse obligatory motion, it is necessary that the tension of the string be adjusted so that maintenance is as vigorous as possible. When this is done, it is noticed that points on the string (except near the nodes) describe 8 curves. The curve at the node, *i. e.*, the path compounded of the small motion at the node, and the large motion of half the frequency perpendicular to it, in neither an 8 curve not a straight line, but is a flat parabolic arc. From this, the phase-difference is again seen to be $\pi/2$. The direction of the curvature of the arc, in other words, the sign of the phase-difference, was found to agree with theory.

8. The above relates to the small motion at *any* node. One particular case is of importance, as it admits of independent experimental verification. For a string to be maintained in vigorous vibration by the imposition of an obligatory motion at one point, this point should itself lie at or near a node of the oscillation. It follows that under such circumstances, the imposed obligatory motion and the general oscillation of the string should differ in phase, the difference being equal to $\pi/2$ when the obligatory motion is imposed exactly at a node. This difference of phase between the motion of the prong and the general oscillation of the string, which may have been anticipated from the general principles of resonance, may be verified experimentally in two ways: (1) By stroboscopic observation and (2) by a tilting-mirror and Lissajous's figure arrangement. Before entering into experimental details regarding these, we may first discuss the mathematics of the questions dealt with above.

Mathematical Note.

9. The result of the investigation by Donkin of the problem of forced oscillations of stretched strings (Acoustics, 2d ed., pp. 121–124) is erroneous. Starting with the assumption that the obligatory

motion at the point $x = b$ is $p \sin nt + q \cos nt$ and taking dissipation of energy into account, the final approximate result obtained by him for the motion of points not near a node is

$$\sin \theta (p \sin nt + q \cos nt)/(\sin^2 \varphi + \delta_0^2 \cos^2 \varphi)^{\frac{1}{2}}.$$

From the original, it will be seen that $\sin \theta$ and $(\sin^2 \varphi + \delta_0^2 \cos^2 \varphi)$ do not involve the time t; the phase of the general motion of the string should therefore be identical with that of the obligatory motion at the point $x = b$. This result does not agree with that given in paragraph 8 above. The exact step in the mathematical work which introduces the error is putting $\tan \Phi = q/\rho$, where

$$\tan \Phi = (q\sigma_0 \sin \varphi + \rho\delta_0 \cos \varphi)/(\rho\sigma_0 \sin \varphi - q\delta_0 \cos \varphi).$$

To show that this step is erroneous, we may, without loss of generality, put $q = 0$. Then

$$\tan \Phi = \frac{\delta_0}{\sigma_0} \cot \varphi = \beta\varphi \cot \varphi = \frac{c\varphi}{2n} \cot \varphi.$$

Donkin's approximation is therefore equivalent to putting the damping factor $c = 0$. This is inadmissible, for the coefficient of the term, *i. e.*, $\cot \varphi$, is very large, and when $\Phi = i\pi$ at the exact stage of resonance, becomes infinite. At this stage $\tan \Phi = \infty$ and $\Phi = \pi/2$, whereas Donkin would have $\tan \Phi = 0$ and therefore $\Phi = 0$.

10. To compare the facts stated in paragraphs 1 to 8 above with the results of theory, we may make use of the notation and results given on pages 197–199 of Lord Rayleigh's Theory of Sound, Vol. I., second edition. The expression for the displacement at every point of the string maintained in vibration there given is

$$\gamma \frac{R_x}{R_b} \cos (pt + \Sigma_x - \Sigma_b),$$

where

$$R_x^2 = \sin^2 \alpha x + \frac{k^2 x^2}{4a^2} \cos^2 \alpha x$$

and

$$\tan \Sigma_x = \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta x} + e^{-\beta x}} \cot \alpha x.$$

corresponding to an obligatory motion $\gamma \cos pt$ at the point $x = b$. In this expression β is a small quantity, and therefore

$$\tan \Sigma_x = \beta x \cot \alpha x,$$

and

$$(\Sigma_x - \Sigma_b) = \tan^{-1}(\beta x \cot \alpha x) - \tan^{-1}(\beta b \cot \alpha b).$$

This value of $(\Sigma_x - \Sigma_b)$ is very small and may be put equal to zero except in the two cases, where $\cot \alpha x$ or $\cot \alpha b$ is very large, *i. e.*, αx or αb is nearly equal to any integral multiple of π. In other words, the motion at any point is in the same phase as the obligatory motion unless it happens that (1) the point of observation, or (2) the point at which the obligatory motion is imposed, or (3) both points either coincide with, or are situated near nodes of the forced oscillation. In short, it may be said that there is a localized change of phase at the nodes, the difference between the phase at the node and at some point a considerable distance away from it being $\pi/2$.

11. The expression for the displacement can be written in the form

$$\frac{\gamma}{R_b}\left(\sin \alpha x \cos pT + \frac{kx}{2a} \cos \alpha x \sin pT\right),$$

which is seen to be of the type given in paragraph 4 above.

12. We now return to the experimental details referred to at the conclusion of paragraph 8 above.

Method (1).—A short length of the vibrating string is brightly illuminated, and an image of it is focused in the field of view of a stroboscopic disc. A slit held parallel to the string at some distance from it is illuminated, and the light issuing therefrom suffers reflection at a small mirror attached to the prong of the electrically-maintained tuning-fork which keeps the string in vibration, and is then focused by a second lens into a linear image. The two images are adjusted so as to be in juxtaposition. On starting the tuning-fork and the stroboscopic disc, it can be seen that the two linear images are in different phases of motion, and the gradual change of the difference with the alteration of the tension of the string can be studied.

Method (2).—The motion of the prong and that of any point on the string are parallel to each other. To apply the method of Lissajous's figures for the observation of the phase-difference, a tilting-mirror arrangement was adopted. The tuning-fork actuates a light pivoted mirror by means of a thread which is kept taut by a spring pressing against the mirror. The plane of oscillation of the tilting-mirror is perpendicular to that of the vibration of the string, and a point on the latter brightly illuminated throughout its path with the aid of a cylindrical lens, and viewed by reflection, first at a fixed mirror and then at the tilting-mirror, is seen to describe a Lissajous's figure (circle, ellipse, or straight-line). From this figure the phase-difference can be inferred at once. It was found that the phase-difference was not quite independent of the amplitude of oscillation of the prongs: the explanation of this effect probably being that a large amplitude of oscillation involves a departure from constancy of the tension of the string, the average tension being greater than the normal value.

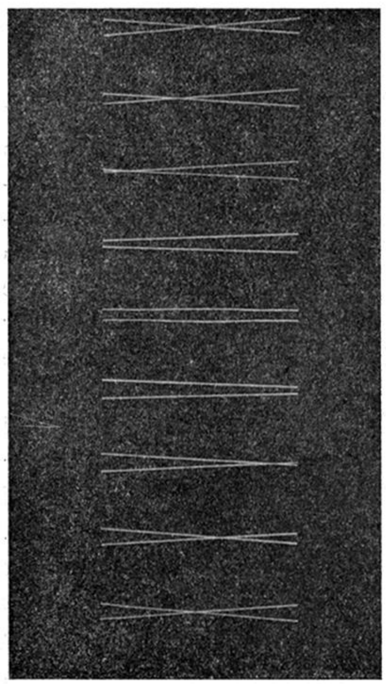

Fig. 1.

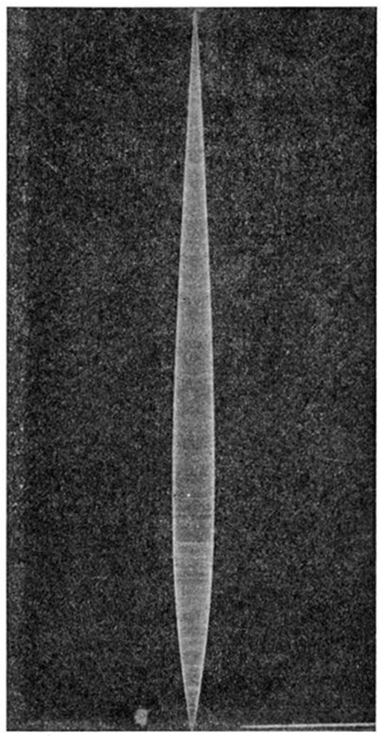

Fig. 2.