Sankhyā : The Indian Journal of Statistics
1994, Volume 56, Series A, Pt. 2, pp. 379-384.

ON THE GAUGE FOR THE NEUMANN PROBLEM IN THE HALF SPACE

By S. RAMASUBRAMANIAN
Indian Statistical Institute

SUMMARY. We consider the gauge function G for the Neumann problem for $\frac{1}{2}\Delta+q$ in the half space $D = \{(\alpha, x) \in R^d : \alpha > 0\}$, where q is independent of α and is periodic in x. It is shown that if $G \not\equiv \infty$, then G is a bounded continuous function on $Cl(D)$. If $H(x) = \int_0^\infty G(\alpha, x)d\alpha \not\equiv \infty$, it is shown that the corresponding Feynman-Kac semi-group decays exponentially.

The gauge function plays a central role in studying the Neumann problem for the Schrödinger operator, $\frac{1}{2}\Delta+q$, in a bounded domain. The gauge function for the Neumann problem is defined in terms of the reflected Brownian motion. If the gauge function is not identically infinite, then the so called gauge theorem states that it is a bounded continuous function and that the corresponding Feynman-Kac semigroup exponentially decays ; (and in such a case the existence of a unique solution to the Neumann problem is guaranteed). A crucial ingredient of the proof of the gauge theorem is that the transition probability density function of the reflected Brownian motion in a bounded region is bounded away from zero ; see Chung and Hsu [2], Chung and Rao [3], and Hsu [4].

In this note we consider the gauge function in the half space $D = \{(\alpha, x_2, \ldots, x_d) ; \alpha > 0\}$. Clearly the transition probability density function of the reflected Brownian motion in $Cl(D)$ is not bounded away from zero. We have been able to deal effectively with only the case when the potential q is independent of α (which is the normal direction) and is a periodic function of $(x_2, \ldots, x_d)$; in this case it becomes essentially a problem on $[0, \infty)\times \boldsymbol{T}^{d-1}$, where $\boldsymbol{T}^{d-1}$ is the $(d-1)$ dimensional tarns. Since $\boldsymbol{T}^{d-1}$ is compact, and as explicit computation can be done concerning the α-coordinate because of our assumption, the analysis can be carried through, though it is not quite trivial.

Even with these assumptions, it differs from the case of a bounded domain. As in the case of a bounded domain, if the gauge function is not identically infinite then it is a bounded continuous function. However, an additional

Paper received. August 1992 ; revised February 1993.
AMS (1991) *subject classification.* Primary 60J65 ; secondary 60J45, 60H30.
Key words and phrases. Reflected Brownian motion, gauge function, Feynman-Kac semigroups.

assumption regarding integrability of the gauge function in the α-variable is needed to establish the exponential decay of the corresponding Feynman-Kac semigroup.

Let $\{P_{(\alpha,x)} : (\alpha, x) \in Cl(D)\}$ denote the reflecting Brownian motion with state space $Cl(D)$ and with normal reflection at the boundary. Each $P_{(\alpha,x)}$ is a probability measure on $\Omega = C([0, \infty) : Cl(D))$; let $X(t, \omega) = (X_1(t, \omega), X_2(t, \omega), \ldots, X_d(t, \omega)) = \omega(t), t \geqslant 0, \omega \in \Omega$. Note that for each $(\alpha, x) \in Cl(D)$, under $P_{(\alpha,x)}$ the process $\{X(t) : t \geqslant 0\}$ is a reflected Brownian motion in $Cl(D)$ starting at (α, x). Let ξ denote the local time of the process at the boundary. Note that we may write $P_{(\alpha, x)} = P_\alpha \times P_x$, where $\{P_\alpha : \alpha \geqslant 0\}$ is the one dimensional reflected Brownian motion on $[0, \infty)$ and $\{P_x : x = (x_2, \ldots, x_d) \in \boldsymbol{R}^{d-1}\}$ is the $(d-1)$ dimensional Brownian motion. Also observe that ξ depends only on the process $\{X_1(t)\}$; that is, ξ is the same as the local time of the one dimensional reflected Brownian motion at 0.

We can now define the Brownian motion on $\boldsymbol{T}^{d-1}$ by

$$\tilde{X}(t) = (X_2(t) \bmod 1, \ldots, X_d(t) \bmod 1). \qquad \ldots (1)$$

Note that $\{\tilde{X}(t) : t \geqslant 0\}$ is a strong Markov, strong Feller process with state space $\boldsymbol{T}^{d-1}$ and transition probability density function

$$\tilde{p}(t, \tilde{x}, \tilde{z}) = (2\pi t)^{-(d-1)/2} \sum_{k \in \boldsymbol{Z}^{d-1}} \exp\left(-\frac{1}{2t} |\tilde{x} - \tilde{z} + k|^2\right) \qquad \ldots (2)$$

for $t > 0, \tilde{x}, \tilde{z} \in \boldsymbol{T}^{d-1}$; (cf. see Bhattacharya [1]). Let $\tilde{P}_x$ denote the distribution of the process $\{\tilde{X}(t)\}$ under $P_x, x \in \boldsymbol{T}^{d-1}$.

Let q be a measurable function on $\boldsymbol{R}^d$ such that

(i) q is a function of $(x_2, \ldots, x_d)$ only and is periodic with period one in each variable ;

(ii) $q \in K_d$.

See Simon [6] or Hsu [4] for the definition of K_d. Under the hypothesis (i) note that $q \in K_d$ if and only if

$$\lim_{t \downarrow 0} \sup_{x \in \boldsymbol{T}^{d-1}} \tilde{E}_x\left(\int_0^t |q(\tilde{X}(r))| \, dr\right) = 0 \qquad \ldots (3)$$

where $\tilde{E}_x$ denotes expectation with respect to $\tilde{P}_x$.

Define the semigroups $\{T_t^{(q)}\}$ and $\{R_t^{(q)}\}$ by

$$(T_t^{(q)} \tilde{f})(x) = \tilde{E}_x [e_q(t) \tilde{f}(\tilde{X}(t))], \quad x \in \boldsymbol{T}^{d-1} \qquad \ldots (4)$$

$$(R_t^{(q)} f)(\alpha, x) = E_{(\alpha,x)} [e_q(t) f(X(t))], \quad (\alpha, x) \in Cl(D) \qquad \ldots (5)$$

where $\bar{f}, f$ respectively are functions on T^{d-1}, $Cl(D)$, and

$$e_q(t) = \exp\left(\int_0^t q(X(r))\,dr\right), \qquad \dots (6)$$

whenever the r.h.s. of (4), (5) make sense.

Lemma 1. *Let q satisfy (i), (ii).*

(*a*) *For $t > 0$, $\bar{T}_t^{(q)}$ (resp. $R_t^{(q)}$) is a bounded operator from $L^1(T^{d-1})$(resp. $L^1(Cl(D))$) into $C_b\,(T^{d-1})$(resp. $C_b\,(Cl(D))$).*

(*b*) *For $t > 0$, there exists a constant $C(t)$ such that for any nonnegative measurable function f on T^{d-1},*

$$\int_{T^{d-1}} f(z)\,dz \leqslant C(t)\,(T_t^{(q)} f)\,(x), \qquad \dots (7)$$

for any $x \in T^{d-1}$.

Proof. (a) The proof is as in the case of bounded domains. Using Khasminskii's lemma (see p. 461 of Simon [6]) and Schwartz inequality, it can be shown that $R_t^{(q)}$ is a bounded operator from $L^2\,(Cl(D))$ into $L^\infty(Cl(D))$. By self-adjointness it is now clear that $R_t^{(q)}$ is a bounded operator from $L^1(Cl(D))$ into $L^2(Cl(D))$. Consequently by the semigroups property it follows that $R_t^{(q)}$ is a bounded operator from $L^1(Cl(D))$ into $L^\infty(Cl(D))$. As $\{R_t^{(q)}\}$ is strong Feller, once again by the semigroup property it follows that $R_t^{(q)}$ maps $L^1(Cl(D))$ into $C_b(Cl(D))$. The assertion concerning $T_t^{(q)}$ can also be proved similarly. See Hsu [4] and Simon [6].

(b) Next, as T^{d-1} is compact and $\tilde{p}$ given by (2) is continuous, for any $t > 0$, there exist constants $c_1(t)$, $c_2(t)$ such that

$$0 < c_1(t) \leqslant \tilde{p}(t, x, z) \leqslant c_2(t) < \infty, \qquad \dots (8)$$

for all $x, z \in T^{d-1}$. Consequently, assertion (b) can be proved like Lemma 2 of Chung and Hsu [2]. □

The gauge function for the Neumann problem is defined by

$$G(\alpha, x) = E_{(\alpha,x)}\left(\int_0^\infty e_q(t)\,d\xi(t)\right) \qquad \dots (9)$$

where e_q is defined by (6). Since ξ and $\tilde{X}$ are independent and as q depends only on x, using the occupation density formula (or otherwise), we get

$$G(\alpha, x) = \lim_{T\to\infty} E_\alpha\left[\int_0^T \tilde{E}_x(e_q(t))\right]$$

$$= \frac{1}{2}\lim_{T\to\infty}\int_0^T \left[\tilde{E}_x(e_q(t))\right](2\pi t)^{-1/2}\exp\,(-\alpha^2/2t)dt. \qquad \dots (10)$$

Theorem 1. *Let q satisfy* (*i*), (*ii*). *If $G \not\equiv \infty$, then G is a bounded continuous function on $Cl(D)$.*

Proof. By (10), $G(\alpha, x) \leqslant G(0, x)$. Consequently we may assume without loss of generality that $G(\alpha, x) < \infty$, for some $\alpha > 0$, $x \in T^{d-1}$.

Let $\tau = \inf\{t > 0 : X(t) \in \partial D\} = \inf\{t > 0 : X_1(t) = 0\}$. Since ξ increases only when $X(t) \in \partial D$, by the strong Markov property we have

$$\infty > G(\alpha, x) = E_{(\alpha,x)} [e_q(\tau)\, G(X(\tau))].$$

As $\alpha > 0$, note that the hitting measure $P_{(\alpha,x)}\, X_\tau^{-1}$ on ∂D is equivalent to the $(d-1)$ dimensional Lebesgue measure ; (see Karatzas and Shreve [5], Chap. 4). Hence the above implies that $G(0, z) < \infty$ for a.a. $(0, z) \in \partial D$; (and hence $G(\beta, z) < \infty$ for a.a. $(\beta, z) \in Cl(D)$).

Put $g(z) = G(0, z)$, $z \in T^{d-1}$. Then as τ is independent of $\widetilde{X}$, we get

$$\infty > G(\alpha, x) = E_{(\alpha,x)}[g(\widetilde{X}(\tau)\exp\Big(\int_0^\tau q(\widetilde{X}(s))ds)]$$

$$= \int_0^\infty (T_t^{(q)}g)\,(x)\; dP_a\tau^{-1}(t).$$

As $\alpha > 0$, $P_a\, \tau^{-1}$ is equivalent to the Lebesgue measure on $[0, \infty)$; (see Karatzas and Shreve [5], Chap. 2). Hence from the above $(T_t^{(q)}\, g)\,(x) < \infty$ for some $t > 0$. Therefore by (7), we obtain $g \in L^1\, (T^{d-1})$.

Since $R_t^{(q)}\, G \leqslant T_t^{(q)}\, g$, by the semigroup property and Lemma 1(a), it now follows that $R_t^{(q)}\, G$ is a bounded continuous function on $Cl(D)$ for any $t > 0$.

It is easy to verify that for any $t_0 > 0$, there exists $c > 0$ such that

$$\sup\,\{E_{(\beta,z)}\xi^2(t) : (\beta, z) \in Cl(D)\} \leqslant ct \tag{11}$$

for all $t \leqslant t_0$. As in Chung and Hsu [2], using the Markov property and (11) it can now be shown that $R_t^{(q)}\, G \rightarrow G$ uniformly over $Cl(D)$ as $t \rightarrow 0$, whence the theorem follows. □

Theorem 2. *Let q satisfy* (*i*), (*ii*), *and G be given by* (9). *For $x \in T^{d-1}$ define*

$$H(x) = \int_0^\infty G(\alpha, x)d\alpha. \qquad \ldots \quad (12)$$

If $H \not\equiv \infty$, then there exist constants $a, b > 0$ such that

$$\sup\{E_{(\alpha,x)}(e_q(t)\} : (\alpha, x) \in Cl(D)\} \leqslant be^{-at}. \qquad \ldots \quad (13)$$

Proof. By (10) and Fubini's theorem note that

$$H(x) = \frac{1}{2} \int_0^\infty \tilde{E}_x(e_q(t))dt.$$

Consequently by our hypothesis and the Markov property of the process $\tilde{X}$, for some $x \in T^{d-1}$,

$$\infty > H(x) = \frac{1}{2} \tilde{E}_x\left(\int_0^t e_q(s)ds\right) + (T_t^{(q)}H)(x). \quad \ldots \quad (14)$$

By (14) and (7), note that $H \in L^1(T^{d-1})$, and hence $T_t^{(q)} H \in C_b(T^{d-1})$ for any $t > 0$. As $q \in K_d$, by Khasminskii's lemma and (14), it follows that $T_t^{(q)} H \to H$ uniformly as $t \to 0$. Thus H is a bounded continuous function on T^{d-1}, and hence there is a constant c such that $H(z) \geqslant c > 0$ for all $z \in T^{d-1}$.

As T^{d-1} is compact, from (14) it follows that $T_t^{(q)} H$ converges uniformly to 0 as $t \to \infty$. Therefore

$$\begin{aligned} \text{l.h.s. of (13)} &= \sup\{\tilde{E}_x(e_q(t)) : x \in T^{d-1}\} \\ &\leqslant c^{-1} \sup\{T_t^{(q)} H(x) : x \in T^{d-1}\} \\ &\to 0, \quad \text{as } t \to \infty. \end{aligned} \quad \ldots \quad (15)$$

Now (13) follows from (15). □

Remark 1. Consider the Neumann problem

$$\left.\begin{aligned} &\frac{1}{2} \Delta u(\alpha, x) + q(\alpha, x)\, u(\alpha, x) = 0, (\alpha, x) \in D, \\ &\qquad \frac{\partial}{\partial \alpha} u(0, x) = -\phi(x), (0, x) \in \partial D, \end{aligned}\right\} \quad \ldots \quad (16)$$

where ϕ is a bounded measurable function on ∂D and $\Delta = \partial^2/\partial\alpha^2 + \sum_{i=2}^{d} \partial^2/\partial x_i^2$. A bounded measurable function u on $Cl(D)$ is said to be a stochastic solution to (16) if for each $(\alpha, x) \in Cl(D)$,

$$u(X(t)) - u(X(0)) + \int_0^t q(X(s))u(X(s))ds + \int_0^t \phi(X(s))d\xi(s)$$

is a continuous $P_{(\alpha,x)}$-martingale w.r.t. the natural filtration.

Define, for $(\alpha, x) \in Cl(D)$,

$$u(\alpha, x) = E_{(\alpha,x)}\left[\int_0^\infty e_q(s)\phi(X(s))d\xi(s)\right].$$

Suppose q satisfies (i), (ii). If $G \not\equiv \infty$, where G is defined by (9), then u is a stochastic solution to (16); also u is a bounded continuous function on $Cl(D)$. In addition, if $H \not\equiv \infty$ with H defined by (12), then u is the unique bounded stochastic solution to the problem (16). In view of Theorems 1 and 2, these assertions can be proved as in Hsu [4] with the necessary modifications; so we omit the details. $\square$

Note. Even with our seemingly strong assumptions on q, the problem (16) does not reduce to the case of a bounded domain or to a lower dimension. To see this, take $q \not\equiv$ constant, $\phi \equiv 1$ and suppose there is a solution of the form $u(\alpha, \boldsymbol{x}) = u_1(\alpha)u_2(\boldsymbol{x})$. Then the boundary condition implies that $u_2(\boldsymbol{x}) \equiv (u_1'(0))^{-1}$. Consequently u_1 should satisfy $u_1''(\alpha)+q(x)u_1(\alpha) = 0$ for $\alpha > 0$ and all values of $\boldsymbol{x}$. This is not possible for nonconstant q unless $u_1 \equiv 0$; but this would contradict $u_1'(0) \neq 0$. $\square$

Remark 2. It is possible to extend the analysis to diffusions of the form $Q_{(\alpha,\boldsymbol{x})} = Q_\alpha \times Q_{\boldsymbol{x}}$, where $\{Q_\alpha\}$ is a reflecting Brownian motion on $[0, \infty)$ and $\{Q_{\boldsymbol{x}}\}$ is a $(d-1)$-dimensional diffusion process with periodic drift and diffusion coefficients. In this case also $\widetilde{X}$ defined by (1) gives a strong Markov, strong Feller process with state space $\boldsymbol{T}^{d-1}$; (see Bhattacharya [1]). Hence the problem gets shifted to $[0, \infty) \times \boldsymbol{T}^{d-1}$. However to prove the analogue of Lemma 1 one has to consider also the adjoint of the semigroups $T_t^{(q)}$, $R_t^{(q)}$.

Acknowledgement. Thanks are due to a referee for some helpful comments.

REFERENCES

[1] BHATTACHARYA, R. N. (1985). On the central limit theorem for diffusions with periodic coefficients. *Ann. Probab.*, **13**, 385-396.

[2] CHUNG, K. L. and HSU P. (1986). Gauge Theorem for the Neumann Problem. In *Seminar on Stochastic Processes*, 1984. 63-70, Birkhauser, Boston,

[3] CHUNG, K. L. and RAO, K. M. (1988). General gauge theorem for multiplicative functionals. *T.A.M.S.* **306**, 819-836.

[4] HSU, P. (1985). Probabilistic approach to the Neumann problem. *Comm. Pure Appl. Math.* **38**, 445-472.

[5] KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus*, Springer Verlag,

[6] SIMON, B. (1982). Schrodinger semigroups. *B.A.M.S.* **7**, 447-526.

STAT. MATH. UNIT
INDIAN STATISTICAL INSTITUTE
8TH MILE, MYSORE ROAD
BANGALORE-560059
INDIA.