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**ON THE GAUGE FOR THE NEUMAN PROBLEM**

**IN THE HALF SPACE**

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**SUMMARY.** We consider the gauge function $G$ for the Neumann problem for $\frac{1}{2} \Delta + q$ in the half space $D = \{(x, z) \in \mathbb{R}^d : x > 0\}$, where $q$ is independent of $x$ and is periodic in $x$. It is shown that if $G \neq \infty$, then $G$ is a bounded continuous function on $Cl(D)$. If $H(x) = \int_{0}^{\infty} G(x, z) dz \neq \infty$, it is shown that the corresponding Feynman-Kac semi-group decays exponentially.

The gauge function plays a central role in studying the Neumann problem for the Schrödinger operator, $\frac{1}{2} \Delta + q$, in a bounded domain. The gauge function for the Neumann problem is defined in terms of the reflected Brownian motion. If the gauge function is not identically infinite, then the so-called gauge theorem states that it is a bounded continuous function and that the corresponding Feynman-Kac semigroup exponentially decays; (and in such a case the existence of a unique solution to the Neumann problem is guaranteed). A crucial ingredient of the proof of the gauge theorem is that the transition probability density function of the reflected Brownian motion in a bounded region is bounded away from zero; see Chung and Hsu [2], Chung and Rao [3], and Hsu [4].

In this note we consider the gauge function in the half space $D = \{(x, x_2, ..., x_d) : x > 0\}$. Clearly the transition probability density function of the reflected Brownian motion in $Cl(D)$ is not bounded away from zero. We have been able to deal effectively with only the case when the potential $q$ is independent of $z$ (which is the normal direction) and is a periodic function of $(x_2, ..., x_d)$; in this case it becomes essentially a problem on $[0, \infty) \times T^{d-1}$, where $T^{d-1}$ is the $(d-1)$-dimensional torus. Since $T^{d-1}$ is compact, and as explicit computation can be done concerning the $x$-coordinate because of our assumption, the analysis can be carried through, though it is not quite trivial.

Even with these assumptions, it differs from the case of a bounded domain. As in the case of a bounded domain, if the gauge function is not identically infinite then it is a bounded continuous function. However, an additional

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\( \text{To 2-26} \)
assumption regarding integrability of the gauge function in the $\alpha$-variable is needed to establish the exponential decay of the corresponding Feynman-Kac semigroup.

Let $\{P_{(\alpha, x)} : (\alpha, x) \in C(D)\}$ denote the reflecting Brownian motion with state space $C(D)$ and with normal reflection at the boundary. Each $P_{(\alpha, x)}$ is a probability measure on $\Omega = C([0, \infty) : C(D))$; let $X(t, \omega) = (X_1(t, \omega), X_2(t, \omega), \ldots, X_d(t, \omega)) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$. Note that for each $(\alpha, x) \in C(D)$, under $P_{(\alpha, x)}$ the process $\{X(t) : t \geq 0\}$ is a reflected Brownian motion in $C(D)$ starting at $(\alpha, x)$. Let $\xi$ denote the local time of the process at the boundary. Note that we may write $P_{(\alpha, x)} = P_x \times P_{\alpha}$, where $\{P_x : x > 0\}$ is the one dimensional reflected Brownian motion on $[0, \infty)$ and $\{P_{\alpha} : \alpha = (\alpha_2, \ldots, \alpha_d) \in R^{d-1}\}$ is the $(d-1)$ dimensional Brownian motion. Also observe that $\xi$ depends only on the process $\{X_1(t)\}$; that is, $\xi$ is the same as the local time of the one dimensional reflected Brownian motion at 0.

We can now define the Brownian motion on $T^{d-1}$ by

$$
\tilde{X}(t) = (X_1(t) \bmod 1, \ldots, X_d(t) \bmod 1).
$$

Note that $\{\tilde{X}(t) : t \geq 0\}$ is a strong Markov, strong Feller process with state space $T^{d-1}$ and transition probability density function

$$
\tilde{p}(t, \tilde{x}, \tilde{z}) = \frac{(2\pi t)^{(d-1)/2}}{\pi^{d/2}} \sum_{k \in Z^{d-1}} \exp \left( -\frac{1}{2t} |\tilde{x} - \tilde{z} + k|^2 \right)
$$

for $t > 0, \tilde{x}, \tilde{z} \in T^{d-1}$; (cf. see Bhattacharya [1]). Let $\tilde{P}_x$ denote the distribution of the process $\{\tilde{X}(t)\}$ under $P_{(\alpha, x) \in T^{d-1}}$.

Let $q$ be a measurable function on $R^d$ such that

(i) $q$ is a function of $(\alpha_2, \ldots, \alpha_d)$ only and is periodic with period one in each variable;

(ii) $q \in K_\alpha$.

See Simón [3] or Hau [4] for the definition of $K_\alpha$. Under the hypothesis (i) note that $q \in K_\alpha$ if and only if

$$
\lim_{t \to 0} \sup_{\alpha \in T^{d-1}} E_\alpha \left( \int_0^t |q(\tilde{X}(r))| dr \right) = 0
$$

where $E_\alpha$ denotes expectation with respect to $\tilde{P}_x$.

Define the semigroups $\{T^{(\alpha)}\}$ and $\{R^{(\alpha)}\}$ by

$$
(T^{(\alpha)}f)(x) = E_x \left[ e_{\alpha}(0) f(\tilde{X}(t)) \right], \quad (\alpha, x) \in C(D)
$$

$$
(R^{(\alpha)}f)(x, x) = E_{(\alpha, x)} \left[ e_{\alpha}(0) f(\tilde{X}(t)) \right], \quad (\alpha, x) \in C(D)
$$

where $f, f'$ are defined in $L^1(C(D))$.

whenever the latter exist.

Lemma

(a) $F^{(\alpha)} \in L^1(C(D))$;

(b) $F^{(\alpha)}$ is measurable for any $\alpha \in C(D)$.

Proof.

Khasminskii's result can be shown by similar arguments. By self-adjointness $\{L^{(\alpha)}\}$ is irreducible. As before, let $R^{(\alpha)}$ be the strong Feller operator in $L^1(C(D))$ and $F^{(\alpha)}$ defined similarly.

(b) $\mathfrak{N}(\alpha) < \infty$ for any $\alpha$ and

$$
\sup_{\alpha \in C(D)} E_{(\alpha, x)} |e_{\alpha}(t)| < \infty
$$

for all $x, x'$ in terms of Chung and Moser [7].

The general solution

$$
G(\alpha, x) = \int_{R^d} e^{it \cdot \alpha} \phi(x + t \cdot \alpha) dt
$$

where $e_{\alpha}$ is defined in $L^1(C(D))$.

$C(D)$
where $f, f$ respectively are functions on $T^{d-1}$, $\mathcal{C}(D)$, and
\[
eq_q(t) = \exp \left( \int_0^t q(X(r)) \, dr \right),
\]
whenever the r.h.s. of (4), (5) make sense.

**Lemma 1.** Let $q$ satisfy (i), (ii).

(a) For $t > 0$, $T_t q$ (resp. $R_t q$) is a bounded operator from $L^1(T^{d-1})$ (resp. $L^1(\mathcal{C}(D))$) into $C_0(T^{d-1})$ (resp. $C_0(\mathcal{C}(D))$).

(b) For $t > 0$, there exists a constant $C(t)$ such that for any nonnegative measurable function $f$ on $T^{d-1}$,
\[
\int_{T^{d-1}} f(x) \, dx \leq C(t) (T_t f)(x),
\]
for any $x \in T^{d-1}$.

**Proof.** (a) The proof is as in the case of bounded domains. Using Khasminskii's lemma (see p. 461 of Simon [6]) and Schwartz inequality, it can be shown that $R_t q$ is a bounded operator from $L^2(\mathcal{C}(D))$ into $L^\infty(\mathcal{C}(D))$. By self-adjointness it is now clear that $R_t q$ is a bounded operator from $L^1(\mathcal{C}(D))$ into $L^2(\mathcal{C}(D))$. Consequently by the semigroup property it follows that $R_t q$ is a bounded operator from $L^1(\mathcal{C}(D))$ into $L^\infty(\mathcal{C}(D))$. As $R_t q$ is strong Feller, once again by the semigroup property it follows that $R_t q$ maps $L^1(\mathcal{C}(D))$ into $C_0(\mathcal{C}(D))$. The assertion concerning $T_t q$ can also be proved similarly. See Hsu [4] and Simon [6].

(b) Next, as $T^{d-1}$ is compact and $p$ given by (2) is continuous, for any $t > 0$, there exist constants $c_l(t), c_r(t)$ such that
\[
0 < c_l(t) \leq \tilde{p}(t, x, z) \leq c_r(t) < \infty,
\]
for all $x, z \in T^{d-1}$. Consequently, assertion (b) can be proved like Lemma 2 of Chung and Hsu [2].

The gauge function for the Neumann problem is defined by
\[
G(\alpha, x) = E_\alpha(x, x) \left( \int_0^\infty e_q(t) \, dt \right)
\]
where $e_q$ is defined by (6). Since $\xi$ and $\tilde{X}$ are independent and as $q$ depends only on $x$, using the occupation density formula (or otherwise), we get
\[
G(\alpha, x) = \lim_{T \to \infty} E_\alpha \left[ \frac{1}{T} \int_0^T \tilde{E}_\alpha(e_q(t)) \right]
= \frac{1}{2} \lim_{T \to \infty} \int_0^T \left[ \tilde{E}_\alpha(e_q(t)) \right] (2n\tau)^{-1/2} \exp \left( -\tau^2/2 \right) dt.
\]
Theorem 1. Let \( q \) satisfy (i), (ii). If \( G \equiv \infty \), then \( G \) is a bounded continuous function on \( Cl(D) \).

Proof. By (10), \( G(x, x) \leq G(0, x) \). Consequently we may assume without loss of generality that \( G(x, x) < \infty \), for some \( x > 0, x \in T^{d-1} \).

Let \( \tau = \inf\{t > 0 : X(t) \in \partial D\} = \inf\{t > 0 : X(t) = 0\} \). Since \( \xi \) increases only when \( X(t) \in \partial D \), by the strong Markov property we have

\[
\infty > G(x, x) = E_{(x, x)}[\xi(q) G(X(\tau))].
\]

As \( x > 0 \), note that the hitting measure \( P_{(x, x)} X_{\tau}^{-1} \) on \( \partial D \) is equivalent to the \((d-1)\) dimensional Lebesgue measure; (see Karatzas and Shreve [5], Chap. 4). Hence the above implies that \( G(0, x) < \infty \) for a.a. \((0, x) \in \partial D \); (and hence \( G(0, x) < \infty \) for a.a. \((0, x) \in Cl(D) \)).

Put \( g(x) = G(0, x) x \in T^{d-1} \). Then as \( \tau \) is independent of \( T^{d-1-x} \), we get

\[
\infty > G(x, x) = E_{(x, x)}[g(T^{d-1-x})G(X(\tau))].
\]

As \( x > 0 \), \( P_{(x, x)} \tau^{-1} \) is equivalent to the Lebesgue measure on \([0, \infty)\); (see Karatzas and Shreve [5], Chap. 2). Hence from the above \( (T^{d-1}) g \) \((x) < \infty \) for some \( t > 0 \). Therefore by (7), we obtain \( g \in L_{1}(T^{d-1}) \).

Since \( R_{t}^{0} G \leq T_{t}^{0} g \), by the semigroup property and Lemma 1(a), it now follows that \( R_{t}^{0} G \) is a bounded continuous function on \( Cl(D) \) for any \( t > 0 \).

It is easy to verify that for any \( t_{0} > 0 \), there exists \( c > 0 \) such that

\[
\sup\{E_{(x, x)}[\xi(q)] : (0, x) \in Cl(D)\} < ct \tag{11}
\]

for all \( t < t_{0} \). As in Chung and Hsu [2], using the Markov property and (11) it can now be shown that \( R_{t}^{0} G \to G \) uniformly over \( Cl(D) \) as \( t \to 0 \), whence the theorem follows. \( \square \)

Theorem 2. Let \( q \) satisfy (i), (ii), and \( G \) be given by (9). For \( x \in T^{d-1} \) define

\[
H(x) := \int_{0}^{\infty} G(x, x)dx. \tag{12}
\]

If \( H \equiv \infty \), then there exist constants \( a, b > 0 \) such that

\[
\sup\{E_{(x, x)}(\xi(q)) : (x, x) \in Cl(D)\} \leq be^{-at}. \tag{13}
\]
Proof. By (10) and Fubini's theorem note that

\[ H(x) = \frac{1}{2} \int_0^\infty E_x (e_q(t)) dt. \]

Consequently by our hypothesis and the Markov property of the process \( \widetilde{X} \), for some \( x \in T^{d-1} \),

\[ \infty > H(x) = \frac{1}{2} \int_0^\infty E_x \left( \int_0^t e_q(s) ds \right) + (T^{(q)}H)(x). \ldots \quad (14) \]

By (14) and (7), note that \( H \in L^1(T^{d-1}) \), and hence \( T^{(q)} H \in C_b(T^{d-1}) \) for any \( t > 0 \). As \( q \in K_Z \), by Khasminskii's lemma and (14), it follows that \( T^{(q)} H \rightarrow H \) uniformly as \( t \rightarrow 0 \). Thus \( H \) is a bounded continuous function on \( T^{d-1} \), and hence there is a constant \( c \) such that \( H(z) \geq c > 0 \) for all \( z \in T^{d-1} \).

As \( T^{d-1} \) is compact, from (14) it follows that \( T^{(q)} H \) converges uniformly to \( 0 \) as \( t \rightarrow \infty \). Therefore

\[
\text{l.h.s. of (13)} = \sup \{ E_x (e_q(t)) : x \in T^{d-1} \} \\
\leq c^{-1} \sup \{ T^{(q)} H(x) : x \in T^{d-1} \} \\
\rightarrow 0, \quad \text{as } t \rightarrow \infty. \ldots \quad (15)
\]

Now (13) follows from (15).

\[ \Box \]

Remark 1. Consider the Neumann problem

\[
\begin{cases}
\frac{1}{2} \Delta u(x, x) + q(x, x) u(x, x) = 0, (x, x) \in D, \\
\frac{\partial}{\partial x} u(0, x) = -\phi(x), (0, x) \in \partial D,
\end{cases}
\]

where \( \phi \) is a bounded measurable function on \( \partial D \) and \( \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x^2} \). A bounded measurable function \( u \) on \( Cl(D) \) is said to be a stochastic solution to (16) if for each \( (x, x) \in Cl(D) \),

\[
u(X(t)) - u(X(0)) + \int_0^t q(X(s)) u(X(s)) ds + \int_0^t \phi(X(s)) d\xi(s)
\]

is a continuous \( P_{(x,x)} \)-martingale w.r.t. the natural filtration.

Define, for \( (x, x) \in Cl(D) \),

\[ u(x, x) = E_{(x,x)} \left[ \int_0^\infty e_q(s) \phi(X(s)) d\xi(s) \right]. \]

For \( x \in T^{d-1} \)

\[ \ldots \quad (12) \]

\[ \ldots \quad (13) \]

GAUGE FOR THE NEUMANN PROBLEM 383
Suppose $q$ satisfies (i), (ii). If $G = \infty$, where $G$ is defined by (9), then $u$ is a stochastic solution to (16); also $u$ is a bounded continuous function on $Cl(D)$. In addition, if $H = \infty$ with $H$ defined by (12), then $u$ is the unique bounded stochastic solution to the problem (16). In view of Theorems 1 and 2, these assertions can be proved as in Hsu [4] with the necessary modifications; so we omit the details. □

Note. Even with our seemingly strong assumptions on $q$, the problem (16) does not reduce to the case of a bounded domain or to a lower dimension. To see this, take $q \equiv 1$ constant, $\phi \equiv 1$ and suppose there is a solution of the form $u(x, x) = u_1(x)u_2(x)$. Then the boundary condition implies that $u_1(x) \equiv (u_1(0))^{-1}$. Consequently $u_1$ should satisfy $u_1''(x) + q(x)u_1(x) = 0$ for $x > 0$ and all values of $x$. This is not possible for nonconstant $q$ unless $u_1 \equiv 0$; but this would contradict $u'_1(0) \neq 0$. □

Remark 2. It is possible to extend the analysis to diffusions of the form $Q(x, x) = Q_x \times Q_x$, where $\{Q_x\}$ is a reflecting Brownian motion on $[0, \infty)$ and $\{Q_x\}$ is a $(d-1)$-dimensional diffusion process with periodic drift and diffusion coefficients. In this case also $\tilde{X}$ defined by (1) gives a strong Markov, strong Feller process with state space $T^{d-1}$; (see Bhattacharya [1]). Hence the problem gets shifted to $[0, \infty) \times T^{d-1}$. However to prove the analogue of Lemma 1 one has to consider also the adjoint of the semigroups $T^q_0, P^q_0$.

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