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A basis-free approach to time-reversal for symmetry groups

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Abstract. We develop a basis-free approach to time-reversal for the quantal angular momentum group, $SU2$, and apply these methods to the physical symmetry $SU2_{\text{isospin}}$, $SU3_{\text{flavor}}$, $SU3_{\text{nuclear}}$ and the nuclear collective symmetry group $SL(3, R)$ of Gell-Mann and Tomonaga.

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1. Introduction and summary

Of all the transformations on a quantum system those transformations relating to time reversal are the most natural; whenever there is time evolution one can ask: What is the time reversed description? However since the energy is always bounded from below a purely geometric unitary time reversal is not possible. Rather time reversal must be *Bewegungsumkehr* (reversal of all motions) and must therefore involve reversal of momenta but preserve the sign of position and energy. Time reversal in this sense can be required even of irreversible processes; and one can ask for tests of time reversal invariance in particle decay phenomena. The pioneering work of Wigner [1] for time-reversal in non-relativistic quantum mechanics, determined that alone among all symmetries the quantal time-reversal symmetry operator T is *non-linear* (or more precisely semi-linear). This result is certainly true in the context of the physically important Newtonian and Einsteinian relativities (the Galilei and Poincaré symmetry groups, respectively) but it does not follow that time-reversal must necessarily be implemented for *all* physical symmetry groups in the same semi-linear fashion.

We were led to these considerations by the problem of defining time-reversal for the internal symmetries of isospin and of flavor $SU3$, a problem which we have not, so far, found to have been discussed in the literature. In the course of our investigation we were plagued by the many distinct basis conventions (often contradictory) to be found in physical treatments of group symmetries, which for complex phases can be most confusing. Here the mathematicians have pointed the way: work if possible in a coordinate-free, basis-independent manner, for this way is logically, and usually actually, simpler.

To illustrate our procedure we will first re-examine time-reversal for the quantal angular momentum group, $SU2$ using basis-free methods (§ 2). Having established the methodology, we then turn (§ 3) to the original question of time-reversal for isospin and flavor $SU3$. We then turn to two other symmetry groups of interest in nuclear physics: the nuclear $SU3$ group of Elliot [2] and the collective motion nuclear symmetry group, $SL(3, R)$ of Gell-Mann [3] and Tomonaga [4].

We have relegated to appendices several of the more detailed topics. For example, in Appendix A we detail difficulties of the basis-dependent approach to the results in § 2.

2. Time-reversal for angular momentum, $SU2$

Let us consider first the well-known case of time-reversal for $SU2$, the quantum angular momentum group. Time-reversal was defined by Wigner as reversal of motion (*Bewegungsumkehr*) based on the principle that for every physical motion there is an equally physical possible motion in reverse order. It follows that linear momentum, $\mathbf{P}$, reverses under motion-reversal, that is $T:\mathbf{P} \to -\mathbf{P}$. Since orbital angular momentum is defined by $\mathbf{L} = \mathbf{r} \times \mathbf{P}$, one sees that orbital angular momentum reverses: $T:\mathbf{L} \to -\mathbf{L}$. On grounds of uniformity, one assumes that *spin* angular momentum also reverses, [5] so that for the total angular momentum $J = \mathbf{L} + \mathbf{S}$, we have,

$$T:\mathbf{J} \to -\mathbf{J}. \tag{2.1}$$

For quantum mechanics to obey time-reversal, one postulates that the Schrödinger equation

$$H\psi = i\frac{\mathrm{d}}{\mathrm{d}t}\psi, \tag{2.2}$$

be invariant under $T:t \to -t$. This will be true if we require (as Wigner did) that the time-reversal operation T not only reverse time order ($t \to -t$), but also involves complex conjugation, denoted by K_0. Thus Wigner time-reversal is, at this stage of the discussion, the operation

$$T = \mathscr{T} K_0, \tag{2.3}$$

where $\mathscr{T}$ implies $t \to -t$.

The operation of time-reversal, (2.3), is consistent with the previous results, where T implied that $\mathbf{P} \to -\mathbf{P}$ and $\mathbf{L} \to -\mathbf{L}$, since the operators $\mathbf{P}$ and $\mathbf{L}$, as quantum observables, are Hermitian and hence formally real operators. The action of (2.3) is however problematic, since any given spin realization by matrix operators is basis dependent (so that the action by K_0 is not canonically-defined). The Hamiltonian is an observable and is required to be invariant under Wigner time-reversal. If the Hamiltonian involves electro-magnetism we see, from gauge invariance, that the combination (kinetic momentum), $\mathbf{p} - e\mathbf{A}/c$, enters so that we must have $T:\mathbf{A} \to -\mathbf{A}$. It follows that $T:\mathbf{E} \to \mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$.

We wish now to find *in a coordinate-free way* the consequences of Wigner time-reversal for a general angular momentum. Consider an arbitrary unitary irrep of $SU2$, say, $\mathbf{D}^{(j)}(g)$, where $\mathbf{D}$ is the (unitary) irrep labelled, as usual, by the total angular momentum j and g is a group element (a rotation).

The explicit matrix form of the irrep is given by

$$\mathbf{D}^{(j)}(g) \equiv e^{-i\boldsymbol{\omega}\cdot\mathbf{J}}, \tag{2.4}$$

where $\mathbf{J}$ is a $(2j+1) \times (2j+1)$ Hermitian matrix realization of the abstract operator

J, and $\boldsymbol{\omega} = \vartheta\hat{n}$ is explicitly real (ϑ = angle of rotation, $\hat{n}$ = unit vector denoting axis of rotation).

It might appear at this point that the action of time-reversal, eq. (2.3) on the irrep matrix $\mathbf{D}^{(j)}$, eq. (2.4), is now obvious: namely that under T both i and **J** reverse, so the irrep $\mathbf{D}^{(j)}$ is invariant. This conclusion is, however, not really warranted since the matrix $\mathbf{D}^{(j)}$ is *basis-dependent* and, to be precise, we must also examine the effect of time-reversal on the basis, *per se*. (The basis could be real, or, as is generally the case, complex, so that the basis itself could transform under T).

The difficulties caused by a choice of basis (see Appendix A) are made even worse by the fact that in writing eq. (2.4)—in the standard (physics) form—we are guilty of choosing a *complex* basis for the representations of a *real* Lie algebra [6] [7]. One can avoid this choice if one represents the $SU2$ Lie algebra (over the real field **R**) by generators which are *anti-Hermitian* operators. Such a choice, however, *conflicts* with a basic postulate of quantum mechanics: that generators are observables to be represented by *Hermitian* operators [8]. Nonetheless let us proceed in this explicitly real way and use anti-Hermitian generators, **K**, defined by

$$\mathbf{K} \equiv i\mathbf{J}, \tag{2.5}$$

so that in eq. (2.4),

$$\mathbf{D}(g) = e^{-\boldsymbol{\omega}\cdot\mathbf{K}}, \tag{2.6}$$

is a unitary representation with $\boldsymbol{\omega}$ real (numbers) and **K** anti-Hermitian generating a real Lie algebra ($su2$).

To answer the question as to how eq. (2.6) transforms under Wigner time-reversal, in a basis-independent way, one uses the Frobenius-Schur invariant (FSI)

$$\mathrm{FSI} \equiv \int \mathrm{d}g \,\mathrm{tr}\, \mathbf{D}^{(\lambda)}(g^2) \tag{2.7}$$

and—for the general case—finds: [9] FSI $= +1, -1$ or 0 for irreps λ that are real, quaternionic ("pseudo real" [10]) or complex, respectively. All irreps of $SU2$ are found to be [9] either *real* (j = integer) or *quaternionic* ($j = \frac{1}{2}$–integer). (The Frobenius–Schur invariant is discussed further in Appendix B).

The Frobenius–Schur invariant answers the question for the angular momentum ($SU2$) irreps. Since there are no complex irreps, it follows that under time reversal, the unitary $SU2$ irreps, labelled by $j = 0, \frac{1}{2}, 1, \ldots$, are *invariant*.

This result does not, however, answer the question about the behaviour of the representative *matrices* under time-reversal. Put differently, the Frobenius–Schur invariant being non-zero guarantees only that the complex conjugated irrep matrix is *equivalent* to the original matrix (and not necessarily equal). In symbols

$$(\mathrm{FSI} = \pm 1) \Rightarrow \mathbf{D}^{(\lambda)*} \simeq \mathbf{D}^{(\lambda)}, \tag{2.8}$$

that is,

$$\mathbf{D}^{(\lambda)*} = \mathbf{U}^{-1}\mathbf{D}^{(\lambda)}\mathbf{U}, \tag{2.9}$$

where **U** is a unitary matrix. (We cannot conclude that if FSI $= \pm 1$ then $\mathbf{U} = 1$, since, even though the irrep is real, the matrix basis itself may be complex. See Appendix A).

The basis independent approach uses the fact that complex conjugation implies the equivalence relation, eq. (2.9), which—using eq. (2.6)—may be written in terms of the (non-Hermitian) generators as the *linear* transformation

$$\mathbf{K} \to \mathbf{U}^{-1}\mathbf{K}\mathbf{U}. \tag{2.10}$$

Since this transformation (as an equivalence transformation) preserves all irreps (leaves j invariant) it must be an automorphism of the $su2$ Lie algebra. For $SU2$ there exist only two automorphisms (both inner): [11] the identity and the involutary Cartan automorphism, $\mathscr{C}$. To define the Cartan automorphism [12] in a basis-free way we use a Cartan splitting of the (complexified) Lie algebra, $\mathbf{g}$

$$\mathbf{g} = \mathbf{k} + \mathbf{p}, \tag{2.11}$$

where

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k},\ [\mathbf{p}, \mathbf{k}] \subset \mathbf{p},\ [\mathbf{p}, \mathbf{p}] \subset \mathbf{k}. \tag{2.12}$$

A Cartan automorphism is the transformation

$$\mathscr{C}: \mathbf{k} \to \mathbf{k},\ \mathbf{p} \to -\mathbf{p}, \tag{2.13}$$

which clearly leaves the commutation relation, eq. (2.12), invariant [13].

The standard, basis-dependent, choice for the (anti-Hermitian) $su2$ Lie algebra generators in the (complex) Cartan basis yields for $\mathscr{C}$ the transformation

$$\mathscr{C}: K_z \to K_z,\ K_{\pm} \to -K_{\pm}, \tag{2.14}$$

or in the Cartesian basis

$$\mathscr{C}: K_z \to K_z,\ K_x \to -K_x,\ K_y \to -K_y. \tag{2.15}$$

Let us now apply these results to the physical angular momentum operator, $\mathbf{J}$. From (2.9) we have determined that complex conjugation (K_0) implies (2.10) the automorphism U, which is precisely the Cartan involution $\mathscr{C}$, eq. (2.13). We have thereby determined the action of complex conjugation on the anti-Hermitian generators $\mathbf{K}$, to be

$$K_0: \mathbf{K} \to \mathscr{C}(\mathbf{K}), \tag{2.16}$$

and hence, since $\mathbf{K} \equiv i\mathbf{J}$ (eq. (2.5)) we obtain

$$K_0: \mathbf{J} \to -\mathscr{C}(\mathbf{J}). \tag{2.17}$$

Since the action of time-reversal on $\mathbf{J}$ has been defined from physical principles (in eq. (2.1)) to be

$$T: \mathbf{J} \to -\mathbf{J}, \tag{2.18}$$

we can conclude (from eqs (2.3) and (2.18)) that the final basis-free form for the Wigner time reversal operator is

$$T = \mathscr{T}\mathscr{C}K_0. \tag{2.19}$$

(The three operators on the RHS of (2.19) can be shown to commute).

Equation (2.19) is the abstract form of Wigner's time-reversal operator, which has now been obtained in a basis-free (coordinate independent) way.

If we now combine the operations of complex conjugation, K_0, and the Cartan automorphism, $\mathscr{C}$, we find that the combined action $\mathscr{C}K_0$ on any unitary irreducible representation $\mathbf{D}(g)$ of $SU2$ is given by

$$\mathscr{C}K_0:\mathbf{D}(g)\rightarrow\mathscr{C}(\mathbf{D}^*(g))=\mathscr{C}(\mathbf{U}^{-1}\mathbf{D}(g)\mathbf{U}) \tag{2.20}$$

$$=\mathbf{U}^{-1}(\mathbf{U}^{-1}\mathbf{D}(g)\mathbf{U})\mathbf{U}=\mathbf{D}(g), \tag{2.21}$$

since $\mathscr{C}$ is involutary and $U^2=\pm\mathbf{1}$, see Appendix B. Thus the transformation $\mathscr{C}K_0$, and hence time-reversal, acts as the identity transformation on every unitary $SU2$ representation. (It does not follow that $\mathscr{C}K_0$ is the identity, because K_0 unlike $\mathscr{C}$, is not a linear transformation).

Remarks. (a) This basis-free derivation of the time-reversal operation is logically simpler, and more general, than the original (basis-dependent) derivation. For example, Wigner's derivation of time-reversal was explicitly non-relativistic and moreover restricted to spin $\frac{1}{2}$, taking the form

$$T_{\text{Wigner}}=\mathscr{T}K_0\cdot(\text{commutation with } i\sigma_y). \tag{2.22}$$

The basis-free derivation given above, however, is founded on Schrödinger's equation—and hence is valid for relativistic as well as non-relativistic quantum mechanics—and uses an automorphism for $SU2$ (which is clearly valid for all irreps). In consequence, the basis-free result, eq. (2.18), is valid for relativistic quantum mechanics and all spins. This result for relativistic time reversal was first given by Biedenharn [14] (for the Dirac equation) correcting previous incorrect results by Racah [15] and by Pauli [16]. For completeness, we should discuss time-reversal, in a basis-free way, for the Poincaré group (and Galilei group as well), but we forego this here.

(b) An advantage of the basis-free derivation is that eq. (2.18) implies the proper behaviour of the basis vectors of angular momentum irreps under time-reversal, [9] and correspondingly the correct time-reversal behaviour of the WCG coefficients.

(c) The basis-independent form of time-reversal shows that the Hamiltonian (for a time-reversal invariant theory) is *invariant* under T. This has the consequence that, in the Fermi theory of weak interactions, the five interaction constants S, V, T, A, P are *real* in a time-reversal adapted basis [17]. Expressed in terms of the standard model for weak interactions, the Kobayashi-Maskawa mass matrix [18] in the Cartan–Weyl basis diagonalizing the observable quantum numbers must be real if time-reversal is to be obeyed.

(d) As Wigner remarked [8], the fact that T is a non-linear operation prevents its (direct) use to define quantum numbers. However, T^2 is linear and, in fact, $T^2\rightarrow(-1)^{2j}$ is a quantum number, namely the FSI invariant.

(e) The fact that for fermionic systems (FSI $=-1=(-1)^{2j}$) the time reversal operator obeys $T^4=\mathbf{1}$ means that the Hilbert space of such systems is *quaternionic* and not just complex. One consequence is Kramer's theorem (namely, energy levels for FSI $=-1$ are at least doubly degenerate in electric fields) but there are other more subtle purely topological consequences [19].

3. Time-reversal for $SU(3)_{\text{flavor}}$

With these results for $SU2$ in hand, let us now examine the extension of time reversal to unitary symmetry, $SU(3)_{\text{flavor}}$, [20]. What does *Bewegungsumkehr* do to, say, a baryon in the octet representation? Clearly a baryon reversing its motion is still a baryon, in the same state of the same $SU3$ irrep, with the same charge, (where the charge operator $\mathbf{Q}_{\text{ch}}$ is defined by the $SU3$ generators $\mathbf{Q}_{\text{ch}} = \mathbf{I}_z + \frac{1}{2}\mathbf{Y}$).

Accordingly, we see that

$$T(\mathbf{D}^{(\lambda)}) = \mathbf{D}^{(\lambda)}, \tag{3.1}$$

where λ denotes an $SU3$ irrep, and moreover the charge operator $\mathbf{Q}$ must obey

$$T(\mathbf{Q}_{\text{ch}}) = \mathbf{Q}_{\text{ch}}, \tag{3.2}$$

which implies that both the isospin $\mathbf{I}$ and the hypercharge $\mathbf{Y}$ are invariant under T. The Frobenius–Schur invariant for $SU3$ has only two values: $+1$ and 0 corresponding to real and complex irreps [11]. Unlike $SU2$, complex conjugation is no longer an inner automorphism, but an outer automorphism for $SU3$ [11]. (This can be seen from the fact that $D^{(\lambda)*}$ is inequivalent to $\mathbf{D}^{(\lambda)}$ for irreps, such as the decimet **10**, with FSI $= 0$).

In particular, eq. (3.1) implies, for unitary irreps, defined by

$$\mathbf{D}^{(\lambda)}(g) \equiv \exp\left(- \sum_{\mu=1}^{8} \omega_\mu \mathbf{X}_\mu \right), \tag{3.3}$$

(where ω_μ are real parameters and $\mathbf{X}_\mu$ are anti-Hermitian generators for irrep λ), that the time-reversal operation for $SU3_{\text{flavor}}$ cannot involve complex conjugation. Equation (3.2) shows that at least four of the eight-anti-Hermitian generators are invariant under time-reversal. Thus the simplest realization of time reversal for flavor $SU3$ is the identity transformation, not only for the irreps (as was the case also for $SU2$) but also for the individual basis-vectors (since the generators would be unchanged, again unlike $SU2$).

Remark. This is simplest realization but are there other possibilities? To answer this consider again the $SU2$ case. There we learned that T was also the identity transformation on all irreps, but only because conjugation was an inner automorphism. It was this property that allowed the identity transformation for representations, and allowed compatibility with $\mathbf{J} \to -\mathbf{J}$ (achieved by combining conjugation with the Cartan automorphism). From this we conclude that the only other possibility available for time-reversal in (flavor) $SU3$ is a Cartan involutory inner automorphism. For $SU3$ there exist, besides the identity, only two distinct automorphisms

(*a*)
$$\mathbf{g} = \mathbf{k} + \mathbf{p},\ \mathbf{k} = \{\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_5, \boldsymbol{\lambda}_7\},$$
$$\mathbf{p} = \{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4, \boldsymbol{\lambda}_6, \boldsymbol{\lambda}_8\}, \tag{3.4}$$

with

$$\mathscr{C}_a\!:\mathbf{k} \to \mathbf{k},\ \mathbf{p} \to -\mathbf{p}. \tag{3.5}$$

(Here the $\{\boldsymbol{\lambda}_i\}$ are representations of the eight Gell-Mann matrices).

Clearly the automorphism $\mathscr{C}_a$ is not acceptable for time reversal since the charge operator would not be invariant under $\mathscr{C}_a$. (The charge operator involves λ_3 and λ_8 which belong to the set **p**). Moreover $\mathscr{C}_a$ is outer.

(*b*) $$\mathbf{g} = \mathbf{k} + \mathbf{p}, \mathbf{k} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_8\} \tag{3.6}$$
and
$$\mathbf{p} = \{\lambda_4, \lambda_5, \lambda_6, \lambda_7\},$$

This automorphism, $\mathscr{C}_b$, is quite acceptable for time reversal since

$$\mathscr{C}_b(\mathbf{D}^{(\lambda)}) \cong \mathbf{D}^{(\lambda)}, \quad \text{and} \quad \mathscr{C}_b(\mathbf{Q}) = (\mathbf{Q}), \tag{3.7}$$

Moreover, all of the state vectors of every irrep are preserved under $\mathscr{C}_b$. ($\mathscr{C}_b$ is inner). [To see this last point, let us observe that the ket vector $|[M], (m)\rangle$ is defined (to within a complex constant) by the relations:

$$\mathbf{X}\cdot\mathbf{X}|[M],(m)\rangle = I_2([M]]\ |[M], (m)\rangle,$$
$$\mathbf{X}\cdot\mathbf{X}\circ\mathbf{X}|[M], (m)\rangle = I_3([M])\ |[M], (m)\rangle,$$
$$\sum_{\mu=1}^{3} \mathbf{I}_\mu^2|[M],(m)\rangle = I_2(m_{12}m_{22})|[M], (m)\rangle,$$
$$\mathbf{I}_z|[M],(m)\rangle = \left(m_{11} - \frac{m_{12}+m_{22}}{2}\right)|[M], (m)\rangle,$$

and

$$\mathbf{Y}_8|[M], (m)\rangle = \left(\frac{m_{12}+m_{22}}{2} - \frac{M_{13}+M_{23}}{3}\right)|[M], (m)\rangle. \tag{3.8}$$

Here the states are defined by Gel'fand patterns: $|[M], (m)\rangle = \left|\begin{pmatrix} M_{13}M_{23}0 \\ m_{12}m_{22} \\ m_{11} \end{pmatrix}\right\rangle$, and $I_2([M])$ and $I_3([M])$ are eigenvalues of the two invariant operators of $SU3$, with $I_2(m_{12}, m_{22})$ being the eigenvalue of the Casimir invariant of $SU2$. The invariant operators of $SU3$ are denoted by $\mathbf{X}\cdot\mathbf{X}$ for the quadratic (Casimir) invariant and $\mathbf{X}\cdot\mathbf{X}\circ\mathbf{X}$ for the cubic invariant (with $\circ$ denoting the symmetric octet product). Clearly the automorphism $\mathscr{C}_b$ leaves every eigenvalue invariant, so that the irrep vectors are themselves invariant under $\mathscr{C}_b$].

The above remark shows that we have two distinct possibilities for time-reversal in $SU_{\text{flavor}}(3)$: either (a) the identity automorphism, or (b) the Cartan involuntary automorphism $\mathscr{C}_b$. It would be interesting to see whether or not there is a physical reason for choosing between these two options. (We hope to discuss this question in the near future).

The result that we have obtained for time-reversal in $SU3_{\text{flavor}}$ is that the $SU3$ irreps, as well as the carrier space vectors, are invariant under time-reversal. This is actually quite plausible *a priori* since flavor-symmetry is clearly not a space-time symmetry and should be therefore unaffected by space-time transformations. The only reason for supposing otherwise is that the lesson of Wigner time-reversal has been over-learned, and complex conjugation is not necessarily a general feature after all.

Such a result for non-spacetime (internal) symmetries such as $SU3_{\text{flavor}}$ is quite acceptable and plausible, but despite this there are grounds for worry. How is one to distinguish, in the operation of complex conjugation, the imaginary unit used in describing quantal space-time states from the imaginary unit used in, say, flavor irrep vectors? Actually there is no real problem here since the flavor symmetry group must occur as an element in a direct product for the complete symmetry group, so that the space-time ket vectors ψ are distinct (and distinguishable) from flavor symmetry kets X in the tensor product states $\psi \otimes X$.

4. Time-reversal for nuclear $SU(3)$

The nuclear shell model of Mayer and Jensen has an approximate Hamiltonian symmetry, $SU3_{\text{nuclear}}$, the symmetry of the isotropic three-dimensional harmonic oscillator. This $SU3$ symmetry becomes more nearly exact in the limit that the spin-orbit splitting becomes zero. Since spin is neglected in this limit, one sees that the $SU2$ rotational symmetry consists of the orbital angular momentum $SO3$—a sub-group of $SU3$—and the separate spin symmetry, $SU2$. Thus one deals with the symmetry $(SU2_{\text{spin}}) \times (SU3_{\text{nucl}})$, which may be embedded in the larger symmetry $SU6$, somewhat reminiscent of, but actually quite distinct from, the Radicati-Gürsey baryonic $SU6$ symmetry.

Elliot [2] suggested that a feasible model for certain nuclear mass regions—the rotational nuclei—is the nuclear rotational symmetry $SU3$ generated by $\mathbf{L}$, the orbital angular momentum, and $\mathbf{Q}$, the mass quadrupole operator. Adding a quadrupole-quadrupole interaction to the $SU3$ invariant Hamiltonian, H_{symm}, leads to a total Hamiltonian with an $SU3$-splitting term, $\Delta H \propto \mathbf{L}^2$, as befits rotational nuclei.

To examine the time-reversal properties of this physical model, we begin by noting that the generators $\mathbf{L}$ and $\mathbf{Q}$ must have the physical time-reversal properties

$$T{:}\mathbf{L} \rightarrow -\mathbf{L} \text{ since } \mathbf{L} \text{ is an angular momentum,} \tag{4.1}$$

and

$$T{:}\mathbf{Q} \rightarrow +\mathbf{Q} \tag{4.2}$$

(since $\mathbf{Q}$ is interpreted in the Elliott model as a mass quadrupole operator).

The Frobenius–Schur invariant for $SU3$ has, as noted in § 3, only two values: $+1$ and 0, so the only real and complex (finite-dimensional) unitary irreps occur.

Explicitly real representations of the real Lie group $SU3$ may be generated from the adjoint realization using *real anti-symmetric* 8×8 generators and real parameters. Hence

$$K_0{:}\mathbf{D}(g) \rightarrow \mathbf{D}^*(g) = \mathbf{D}(g). \tag{4.3}$$

It follows from (4.3) that for the Hermitian generators, $\mathbf{L}$ and $\mathbf{Q}$, we have

$$K_0{:}\mathbf{L} \rightarrow -\mathbf{L},\ \mathbf{Q} \rightarrow -\mathbf{Q}. \tag{4.4}$$

In order to obtain the physical time-reversal properties, (4.1, 2), we must therefore use, in addition to K_0, the Cartan automorphism $\mathscr{C}_a$

$$\mathscr{C}_a{:}\mathbf{L} \rightarrow \mathbf{L},\ \mathbf{Q} \rightarrow -\mathbf{Q}, \tag{4.5}$$

which is associated to the Cartan splitting of $SU3$ with $\mathbf{k} = \{\mathbf{L}\}$ and $\mathbf{p} = \{\mathbf{Q}\}$ exactly as in (3.5) for the automorphism $\mathscr{C}_a$.

This automorphism $\mathscr{C}_a$ is an *outer* automorphism, a symmetry of the Coxeter–Dynkin diagram for $SU3$. (To see that $\mathscr{C}_a$ is outer, note that the unimodular condition for $SU3$ is $\det(D(g)) = 1$. The transformation (4.5) however has determinant -1, and thus cannot belong to the $SU3$ group).

Explicitly real representations, eq. (4.3), of $SU3$ (more properly representations of the adjoint group $SU3/Z3$) can be fully reduced (brought to block-diagonal form) only for the self-conjugate irreps (FSI = 1), (since we are using the real field **R**). Over the complex field any not-fully-reduced explicitly-real representation with FSI = 0 can be reduced to a direct sum of pairs of conjugate irreps (each with FSI = 0).

Since quantum mechanics requires the use of the complex field, irreducible complex representations necessarily will occur, so that this analysis of $SU3_{\text{nuclear}}$ using explicitly real structures must be extended to the complex case.

Let us consider then the defining 3×3 irrep of $SU3$ which has FSI = 0. The Hermitian generators of this irrep are the Gell–Mann matrices, $\{\lambda_i\}$, which divide into two distinct sets under complex conjugation:

(a) five real, symmetric, Hermitian generators

$$\{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4, \boldsymbol{\lambda}_6, \boldsymbol{\lambda}_8\} \equiv \{\mathbf{Q}\} \tag{4.6}$$

(b) three purely imaginary, anti-symmetric, Hermitian generators

$$\{\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_5, \boldsymbol{\lambda}_7\} \equiv \{\mathbf{L}\}. \tag{4.7}$$

This splitting is clearly basis-dependent for irreps of $SU3/Z3$ (since we gave *eight* purely imaginary, anti-symmetric, Hermitian generators in (4.3) and (4.4)), but for irreps of $SU3$ not belonging to $SU3/Z3$ this splitting is generic and basis-independent.

For this realization of the Hermitian $SU3$ generators, eqs (4.6 and 4.7), it follows that

$$K_0 : \mathbf{L} \to -\mathbf{L} \text{ and } \mathbf{Q} \to +\mathbf{Q}. \tag{4.8}$$

This is exactly the desired time-reversal property of the physical $SU3_{\text{nuclear}}$ generators. It follows that time-reversal for $SU3_{\text{nuclear}}$ is given by

$$T = \mathscr{T} K_0. \tag{4.9}$$

To find out what happens to the representations, we use the technique of generating unitary representations of a real Lie algebra by anti-Hermitian generators: $\{\mathbf{L}, i\mathbf{Q}\}$. The representations thus have the form:

$$\mathbf{D}(g) = \exp(\boldsymbol{\alpha}\cdot\mathbf{L} + i\boldsymbol{\beta}\cdot\mathbf{Q}), \tag{4.10}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are explicitly real parameters

Clearly under time-reversal, eq. (4.9), we obtain

$$T : \mathbf{D}(g) = \mathbf{D}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \mathbf{D}(\boldsymbol{\alpha}, -\boldsymbol{\beta}). \tag{4.11}$$

The representation $\mathbf{D}(\boldsymbol{\alpha}, -\boldsymbol{\beta})$ is, in general, *inequivalent* to the representation $\mathbf{D}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The effect of time-reversal on a (unitary) representation can be seen from (4.11), (4.12) and (4.5) to be the same as the action of the Cartan automorphism $\mathscr{C}_a$.

The automorphism, $\mathscr{C}_a$, can be shown from eq. (3.8) to effect on the $SU3_{\text{nuclear}}$

invariant operators the transformation

$$\mathscr{C}_a: I_2 \to I_2, I_3 \to -I_3. \tag{4.12}$$

(This result, (4.12), is not obvious since one needs to know that the symmetric product in I_3 leads to an invariant form containing cubic terms with an *odd* number of quadrupole generators). It follows that for unitary $SU3_{\text{nuclear}}$ irreps the operation: $\mathscr{C}_a T$ is equivalent to the identity transformation. (Just as in § 2, the two operations are not equal because T is non-linear).

Remarks. It would be of interest to see how time-reversal affects the basis vectors carrying an $SU3_{\text{nuclear}}$ irrep. The $SU3_{\text{nuclear}}$ ket vectors are uniquely labelled by five quantum numbers: $\mathbf{I}_2 \to I_2$, $\mathbf{I}_3 \to I_3$, $\mathbf{I}_2(SO3) \to L(L+1)$, $\mathbf{L}_z \to m$ and a multiplicity index ε (labelling the multiple occurrences of L). This last (fifth) index is canonically determined [21] (that is, without any arbitrary choice whatsoever.) Under time-reversal, the labels: I_2 and L are unchanged, whereas both I_3 and m reverse (change sign) [22]. That is

$$T: I_2 \to I_2,\ I_3 \to -I_3,\ L \to L,\ L_z \to -L_z. \tag{4.13}$$

From our experience with $SU2$ in § 2 we see that the transformation $L \to L, L_z \to -L_z$ induces the $SU2$ transformation given by

$$T: \left| \begin{matrix} I_2, I_3 \\ L, m \end{matrix} \right\rangle \to (-1)^{L-M} \left| \begin{matrix} I_2, -I_3 \\ L, -m \end{matrix} \right\rangle. \tag{4.14}$$

For ket vectors without multiplicity (vectors for which the label ε is unnecessary), eq. (4.14) gives a unique prescription. For ket vectors requiring ε-labels, the transformation induced by time-reversal must be diagonal, but the appropriate sign change, that may occur, is not fully known.

A physically important conclusion follows from the results given above. We see from eq. (4.12), and the discussion there, that under time-reversal, eq. (4.10), the irrep labels, (4.13) are not invariant. Expressed differently, but equivalently, the Elliott nuclear $SU3$ symmetry does not have a time-reversal invariant significance.

This basic inadequacy, along with the failure of the $SU3_{\text{nuclear}}$ symmetry to incorporate spin intrinsically, shows that the Elliott nuclear $SU3$ symmetry can be neither a fundamental symmetry nor an approximate (time-reversal invariant) symmetry in physics.

Remark. There is an interesting application of these results to the topological Skyrme-Witten model for hadrons [23]. There is an alternative procedure for injecting the static minimal energy soliton of this model into the $SU3_{\text{flavor}}$ group which involves using $\mathbf{L}$ (orbital angular momentum) and $\mathbf{Q}$ (the mass quadrupole) as $SU3_{\text{flavor}}$ generators [24]. This imbedding is far less satisfactory in its physical consequences than that used by Witten, but cannot (so far) be excluded. We see from the results above that this imbedding violates time-reversal invariance and is accordingly to be excluded. The imbedding discussed in § 5 below would appear to be satisfactory (since it is time-reversal invariant) but again it is excluded since the soliton is static (and accordingly the time-derivative $\dot{\mathbf{Q}}$ operator, eq. (5.1), vanishes).

5. Time-reversal for the Gell-Mann–Tomonaga nuclear collective symmetry group $SL(3,\mathscr{R})$

At about the same time that $SU3_{\text{nuclear}}$ was proposed for nuclear physics, non-compact internal symmetry groups were proposed as dynamical hadronic symmetries, generating Regge sequences for hadrons [25].

One such non-compact group, $SL(3,\mathscr{R})$, was also proposed as a nuclear symmetry [26]. This $SL(3,R)$ symmetry group is physically the group of rotations and volume-preserving deformations of three-space. Since nuclear matter has, as a rough approximation, an energy independent of shape, the suggestion that $SL(3,\mathscr{R})$ might be a useful symmetry for nuclear physics is certainly reasonable.

Gell–Mann had a very ingenious way to realize this symmetry. The group $SL(3,\mathscr{R})$ has two sets of Hermitian generators: the total angular momentum, $\mathbf{J}$, and a quadrupolar deformation generator. For the quadrupolar generator, Gell-Mann proposed the time-derivative of the mass quadrupole operator $\mathbf{Q}$. Since the Hamiltonian must contain $\mathbf{J}^2$, one can evaluate this time-derivative, at least approximately, as

$$\dot{\mathbf{Q}} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q} = i[H,\mathbf{Q}] \cong i[\mathbf{J}^2,\mathbf{Q}]. \tag{5.1}$$

The operators $\mathbf{J}$ and $\dot{\mathbf{Q}}$ have a commutator algebra that closes on $SL(3,\mathscr{R})$, that is, $\dot{\mathbf{Q}}$ transforms under $\mathbf{J}$ as a quadrupole and $[\dot{\mathbf{Q}},\dot{\mathbf{Q}}] = \lambda^2\mathbf{J}$ (with λ a length scale, $\hbar = c = 1$).

At roughly the same time, Tomonaga had been developing collective models for diverse physical problems including a two-dimensional nuclear collective model [3]. Applied to three-dimensions his techniques would have led him to precisely the Gell-Mann collective nuclear model [27].

This nuclear collective model has been developed further in the nuclear physics literature and subsumed in larger non-compact groups [28] [29]. Our purpose here is to examine the time-reversal properties of the model.

It is clear from the physical meaning of $\dot{\mathbf{Q}}$, as the time-derivative of a time-reversal invariant object $\mathbf{Q}$, that we must have

$$T{:}\dot{\mathbf{Q}} \to -\dot{\mathbf{Q}}, \tag{5.2}$$

and moreover, from (2.1), we must have

$$T{:}\mathbf{J} \to -\mathbf{J}. \tag{5.3}$$

As observables, both $\mathbf{J}$ and $\dot{\mathbf{Q}}$ are Hermitian. To obtain a unitary representation we exponentiate the anti-Hermitian generators $i\mathbf{J}$ and $i\dot{\mathbf{Q}}$. For the anti-Hermitian generators we have

$$\begin{aligned} T{:}i\mathbf{J} &\to +i\mathbf{J} \\ i\dot{\mathbf{Q}} &\to +i\dot{\mathbf{Q}}. \end{aligned} \tag{5.4}$$

We see that, from (5.4) (since the group parameters are real), the unitary representation $\mathbf{D}(g)$ must obey.

$$T{:}\mathbf{D}(g) \to \mathbf{D}'(g) \cong \mathbf{D}(g). \tag{5.5}$$

that is to say, the irrep labels are invariant under time-reversal.

To be more specific (and thus specify whether or not an automorphism enters in the definition of the operator T) we must discuss the properties of $SL(3,\mathscr{R})$ representations in more detail [11]. The non-compact covering group; $\overline{SL(3,\mathscr{R})}$ has the topology $S^3 \times \mathscr{R}^5$, and, since the centre of this group is Z_2, there are spin representations with $g(Z_2) \neq 1$.

The defining irrep is given by the 3×3 matrix group over **R**. Clearly this representation is real. The generators are **L** given by the three 3×3 anti-symmetric matrices $(\mathbf{L})_{ik} = -ie_{ijk}$ and $\dot{\mathbf{Q}}$ by five 3×3 real, symmetric, and traceless matrices. Under complex conjugation we find

$$K_0 : \mathbf{L} \to -\mathbf{L},\ \dot{\mathbf{Q}} \to +\dot{\mathbf{Q}}. \tag{5.6}$$

Thus to achieve the correct time-reversal properties we must augment K_0 by the Cartan outer automorphism

$$\mathscr{C}_a : \mathbf{L} \to +\mathbf{L}, \quad \dot{\mathbf{Q}} \to -\dot{\mathbf{Q}}. \tag{5.7}$$

It is the unitary irreps that are of physical interest, and all of these (except the trivial identity irrep) are of infinite dimensionality. It is a general result of abstract group theory [11] that all irreps of $SL(3,\mathscr{R})$ are real, as was the case for the defining non-unitary irrep given above.

Since the center of the group is Z_2, there are irreps having half-integer spin, as noted above. Such irreps were first constructed in [30], using a novel boson realization (the quadrupole generators are of *fourth* degree in the bosons).

The Hermitian generators of this boson realization are found to have the properties

$$\text{compact: } \mathbf{J}, \text{ non-compact: } \dot{\mathbf{Q}}$$
$$T = \mathscr{C}_a K_0 : \mathbf{J} \to -\mathbf{J},\ \dot{\mathbf{Q}} \to -\dot{\mathbf{Q}}. \tag{5.8}$$

The representation generated by these boson operators, **L** and $\dot{\mathbf{Q}}$, acting on the space of boson polynomials (ket-vectors) is unitary and splits into three irreps

(i) $j = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$ the so-called "quarkel",

(ii) $j = 0, 2, 4, \ldots,$

(iii) $j = 1, 3, 5, \ldots.$

There is only one irrep of type (i), which is, in fact, a discrete irrep. The irreps of type (ii) and (iii) are labelled by a continuous parameter. All three types of irrep are real and self-conjugate.

Remark. One might expect a fourth irrep:

(iv) $j = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots,$

but this set of states does not strictly define a unitary irrep. The operators do indeed obey the correct commutation relations on this set of states but the invariant operators I_2, I_3 have fixed eigenvalues on all but one of the states.

In our view, it appears remarkable that the collective nuclear symmetry model $(\overline{SL(3\mathscr{R})})$ overcomes the two basic objections to Elliott's $SU3_{\text{nuclear}}$ symmetry, that is,

(a) The $\overline{SL(3,\mathscr{R})}$ model contains half-integer spin *intrinsically*, unlike the $SU(3)_{\text{nuclear}}$ model where half-integer spin is forbidden, and

(b) time-reversal preserves all the irrep labels of the $\overline{SL(3,\mathscr{R})}$ irreps again unlike $SU(3)_{\text{nuclear}}$.

The principal objection to $\overline{SL(3,R)}$ as a fundamental nuclear symmetry is that this symmetry predicts unlimitedly large rotational excitations whereas any real composite nucleus must surely break up eventually. For baryons the situations is quite different. Quark confinement (as schematized in the bag model) results in a deformable composite indecomposable system with finite volume at any energy. The symmetry $\overline{SL(3,R)}$ could very well be fundamental for such a structure.

6. Concluding remarks

We have shown in the foregoing discussion that there is no universal realization of time reversal for a generic symmetry group, but rather any time-reversal realization is conditioned essentially by the physical properties of the system. Thus we note that the spin and isospin symmetries behave differently under time reversal. This reflects the different nature of spin and isospin; the first one changes sign under time reversal and the second is invariant. In fact, all "internal" symmetries are unchanged while space-time symmetries exhibit the expected behavior under time reversal. We have also shown that the coordinate-free approach has the great advantages of both simplicity and clarity.

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Appendix A

The advantages of a basis-free approach can be most easily seen by comparing the procedure used in §2 to the complications, and vagaries, in the basis-dependent approach that is discussed below.

Consider $j=1$. We may choose to realize this irrep of the real Lie algebra of *su*2 by purely real, anti-Hermitian matrices, $\{\mathbf{K}_i\}$

$$(\mathbf{K}_i)_{jk} \equiv e_{ijk} \quad \text{with} \quad \mathbf{K}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\mathbf{K}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

obeying the real Lie algebra,

$$[\mathbf{K}_i, \mathbf{K}_j] = -e_{ijk}\mathbf{K}_k. \tag{A2}$$

(Here $e_{ijk} = \pm 1$ for positive/negative permutations of 123 and 0 otherwise).

Time reversal for the physical angular momentum $\mathbf{J}^{(1)} = i\mathbf{K}$ is then simply complex conjugation, K_0

$$T = \mathscr{T} K_0 : \mathbf{J}^{(1)} \to -\mathbf{J}^{(1)}. \tag{A3}$$

Comparing this with the basis-free result

$$T = \mathscr{T}\mathscr{C}K_0, \tag{A4}$$

we see that the automorphism in (7.4) is now the identity automorphism.

Now let us consider this same $j = 1$ representation of $SU2$ using this time the standard (complex) basis of quantum physics

$$\mathbf{J}_1^{(1)} = 2^{-1/2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{J}_1^{(1)} = i2^{-1/2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{J}_3^{(1)} = 2^{-1/2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{A5}$$

obeying,

$$[\mathbf{J}_i, \mathbf{J}_j] = ie_{ijk}\mathbf{J}_k. \tag{A6}$$

Time reversal for this $j = 1$ realization is now given by the product of two (commuting) operations

$$K_0: \begin{array}{l} J_1^{(1)} \to J_1^{(1)} \\ J_2^{(1)} \to -J_2^{(1)} \\ J_3^{(1)} \to J_3^{(1)}, \end{array} \tag{A7}$$

followed by:

$$\mathscr{C}: \begin{array}{l} J_1 \to -J_1 \\ J_2 \to J_2 \\ J_3 \to -J_3. \end{array} \tag{A8}$$

The resulting time reversal operator is

$$T = \mathscr{T}\mathscr{C}K_0, \tag{A9}$$

in agreement with the basis-free result, but now the automorphism is the Cartan automorphism with J_y as the $\mathbf{k}$ subset in the Cartan split.

These two realizations of the same abstract $j = 1$ representation show an important point: the particularities of the choice of basis can completely change the form of the time-reversal result, even to the point of concealing important general features (for example, the existence of the non-trivial automorphism necessary in the general case).

The basis-free approach has a further major advantage: it avoids all arbitrary phase conventions, and hence the frequent annoyance of inconsistent conventions in different places in the literature.

Let us illustrate this by using the (basis-dependent) WCG realization of the angular momentum operators

$$(J_M^{(j)})_{m',m} = (j(j+1))^{1/2} C^{j1j}_{mMm'}. \tag{A10}$$

The reader may, or may not, notice that the indices (m', m) on the LHS and on the RHS of (7.10) appear in reverse order. Let us show that this is correct using standard techniques

$$J_M|jm\rangle = \sum_{m'} |jm'\rangle\langle jm'|J_M|jm\rangle, \tag{A11}$$

where the matrix element in (7.11) is given in the standard way by the WCG coefficient

$$(J_M)_{m'm} = \langle jm'|J_M|jm\rangle = (j(j+1))^{1/2} C^{j1j}_{mMm'}, \tag{A12}$$

using the convention that the three pairs of indices (jm) in the WCG coefficient are read off from the matrix element in (7.11) from right to left.

This is only the beginning of the problems with angular momentum conventions! We face now the problem that the WCG coefficients use a "spherical tensor" notation, when we seek to obtain the usual (symmetric) cartesian realization. The problem here is a built-in clash of standard conventions:

(a) The Cartan complexification (which is standard in the literature) uses the operator choice:

$$J_\pm = J_x \pm iJ_y,\ J_0 = J_z, \tag{A13}$$

whereas,

(b) The WCG coefficients are based on the standard convention that angular momentum operators are phased (and normed) to accord with their rôle as a vector space carrying the adjoint ($j = 1$) irrep:

$$\begin{aligned} J_+ &= -2^{-1/2}(J_x + iJ_y),\\ J_- &= 2^{-1/2}(J_x - iJ_y),\\ J_0 &= J_z. \end{aligned} \tag{A14}$$

(The conventions in (7.14) are the standard "time-reversal" phase conventions for the basis vectors, $|j, m\rangle$, namely:

$$T|j,m\rangle = (-1)^{j-m}|j, -m\rangle. \tag{A15}$$

With this convention the WCG are explicitly real. (To see that (7.14) and (7.15) are consistent recall that $T:\mathbf{J} \to -\mathbf{J}$).

If one recognizes [31] these conventional pitfalls, then it is an easy task to verify that the general angular momentum $(2j+1) \times (2j+1)$ matrix realization determined by (7.10) and (7.14) fits the time-reversal pattern of the $j = 1$ case, (7.7, 8, 9). (This includes the $j = \frac{1}{2}$ (Pauli matrix) realization as originally used by Wigner [1]).

Remark. Another clash of conventions is concealed in the above results: although the abstract automorphism depends on the Cartan split ($\mathbf{k} = J_z$) in ((7.13) above) the specific phase choices actually used for the WCG coefficients require that the automorphism in (7.8) single out the $\mathbf{k} = J_y$ Cartan split! The basis-free approach avoids such phase and label dependent "paradoxes".

Appendix B

Relation of the conjugation matrix U to the WCG coefficients

We may determine the relationship of the complex conjugation matrix transformation U to the Wigner–Clebsch–Gordan (WCG) coefficients by analyzing the identity

$$\mathbf{D}^{(\lambda)}(g)\mathbf{D}^{(\lambda)}(g^{-1}) = \mathbf{1}, \tag{B1}$$

for unitary irreps where

$$\mathbf{D}^{\dagger}(g) \equiv \tilde{\mathbf{D}}^{*}(g) = \mathbf{D}(g^{-1}). \tag{B2}$$

Inserting (A2) in (A1) and using the definition of the conjugation matrix **U**,

$$\mathbf{D}^{(\lambda)*}(g) \equiv \mathbf{U}^{-1}\mathbf{D}^{(\bar{\lambda})}(g)\mathbf{U}, \tag{B3}$$

where $\bar{\lambda}$ is the conjugate irrep to the irrep $\lambda (\bar{\lambda} = \lambda$ if, as for $SU(2)$, the irreps are self-conjugate). Thus we find

$$(\mathbf{D}^{(\lambda)}(g)\mathbf{D}^{(\lambda)}(g^{-1}))_{ij} = \sum_{k,l,m} D^{(\lambda)}_{ik}(g)\, U^{-1}_{jl}\, D^{(\bar{\lambda})}_{lm}(g)\, U_{mk} = (\mathbf{1})_{ij} \tag{B4}$$

Now we use the (generalized) Wigner product law for matrix irreps of compact groups

$$D^{(\lambda)}_{ij}(g) D^{(\mu)}_{i'j'}(g) = \sum_{\Gamma,\gamma,k,k'} \overset{\Gamma}{\underset{ijk}{C^{\lambda\mu\gamma}}}\, \overset{\Gamma}{\underset{i'j'k'}{C^{\lambda\mu\gamma}}}\, D^{(\gamma)}_{kk'}(g), \tag{B5}$$

where the $\overset{\Gamma}{\underset{ijk}{C^{\lambda\mu\gamma}}}$ are (generalized) WCG coefficients, with Γ a multiplicity label.

Substitute (A5) in (A4), multiply both sides by the (normalized) group measure dg and integrate over the group G. One finds then

$$\begin{aligned}(\mathbf{1})_{ij} &= \sum_{k,l,m} \overset{\Gamma}{\underset{il0}{C^{\lambda\bar{\lambda}0}}}\, \overset{\Gamma}{\underset{km0}{C^{\lambda\bar{\lambda}0}}}\, U^{-1}_{jl}\, U_{mk} \\ &= \overset{\Gamma}{\underset{i\bar{i}0}{C^{\lambda\bar{\lambda}0}}}\, U^{-1}_{j\bar{i}} \underbrace{\left(\sum_k \overset{\Gamma}{\underset{k\bar{k}0}{C^{\lambda\bar{\lambda}0}}}\, U_{\bar{k}k} \right)}_{\text{numerical constant} \equiv A},\end{aligned} \tag{B6}$$

where the label $\bar{i}$ is conjugate to the label i, therefore

$$U_{j\bar{i}} = A C^{\lambda\bar{\lambda}0}_{ij0} \cdot \delta^{\bar{i}}_{j}. \tag{B7}$$

(Note that the multiplicity label Γ drops out for the coupling to the identity). Using (A7) and the unitary conditions for the conjugation matrix U we find

$$\delta_{ij} = \sum_k U_{ik} U_{jk}, = AA^* \cdot \delta_{ij} \cdot (C^{\lambda\bar{\lambda}0}_{i\bar{i}0})^2 = AA^* \cdot \delta_{ij} \cdot (\dim \lambda)^{-1}. \tag{B8}$$

Thus the constant A has the value

$$A = e^{i\psi} \cdot (\dim \lambda)^{1/2}, \tag{B9}$$

showing that a phase of modulus one is arbitrary. The standard choice is $e^{i\psi} = 1$, with the result that

$$U_{ij} = (\dim \lambda)^{1/2} \cdot C^{\lambda\bar{\lambda}0}_{ij0}, = (\dim \lambda)^{1/2} \cdot C^{\lambda\bar{\lambda}0}_{i\bar{i}0} \cdot \delta^{\bar{i}}_{j}. \tag{B10}$$

The WCG coefficient in (A10) is often called the metric for the irrep λ since it couples the ket-vectors of the irrep λ to the bra-vectors of the irrep $\bar{\lambda}$ to produce an invariant.

As an example of these relations let us consider the $SU2$ case. Here the representations are self-conjugate and the WCG coefficient has the value

$$C^{jj0}_{mm'0} = \delta^{-m}_{m'} \cdot (2j+1)^{1/2} \cdot (-1)^{j-m}. \tag{B11}$$

Thus the conjugation matrix is

$$U^{(j)}_{mm'} = \delta^{-m}_{m'} \cdot (-1)^{j-m}, \tag{B12}$$

so that

$$T|jm\rangle = (-1)^{j-m}|j, -m\rangle. \tag{B13}$$

From (A13) we see that

$$T^2|jm\rangle = (-1)^{2j}|j, m\rangle, \tag{B14}$$

so that for $SU2$ the Frobenius–Schur invariant is: FSI $= (-1)^{2j}$ as used in § 2 above.

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