

## The Heisenberg ferromagnet in spin coherent state representation

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**Abstract.** A new approach to Heisenberg ferromagnet using the spin coherent state representation is developed. The differential operator representation of spin angular momentum operators is used to derive the  $c$ -number analogs of the basic quantum mechanical equations, viz., the Schrödinger, Bloch and Liouville equations for the Heisenberg ferromagnet. As an important illustration of our formulation, which has no *ad hoc* assumptions and does not use any boson representation, the excitation spectrum for one, two and three spin waves is obtained. In these cases it is also shown that eigenvalue spectrum can be obtained by completely ignoring the kinematical interactions.

**Keywords.** Heisenberg ferromagnet; spin coherent states.

### 1. Introduction

The boson coherent states are known to play a very important role in the theory of lasers and related phenomena in quantum optics (Agarwal 1973; Haake 1973). The study of properties of the laser threshold in coherent state representation led to a study of phase transitions in systems far from equilibrium (De Giorgio and Scully 1970; Graham and Haken 1970; Scott *et al* 1975; Haken 1975). Many concepts of conventional phase transition theories such as order parameters, scaling parameters and critical exponents have been found to be very useful and important in connection with lasers, parametric oscillators, etc. (Rogovin *et al* 1973; Langer 1968, 1969; Rama Rao and Rajagopal 1977; Biswas and Rama Rao 1971, 1973). Conversely one can also ask: can we learn something new or can we understand better the usual many body systems, by using concepts known from laser theories? A few applications of coherent states to the problems of He-4 and Josephson junctions have already appeared. In an earlier work Douglass (1971 a, b) has applied boson coherent states to simplify Dyson's (1956 a, b) monumental work on the low temperature expansion of the thermodynamic properties of the Heisenberg ferromagnet. The obvious advantages of working with spin-operators directly has led to alternative approaches such as the Green's function methods (see, for example, Balakrishnan 1975). However a more natural basis to study the magnetic problem is provided by the spin coherent states introduced by Radcliffe (1971) and widely studied by many authors (Kutzner 1973; Arecchi *et al* 1972; Takahashi and Shibata 1971). These states obviate the necessity of introducing fictitious boson fields and yet offer all the advantages of the boson coherent states. For instance the mapping of operators onto functions of classical variables has been used by Lieb (1973) to obtain remarkable upper and lower

bounds to the free energy of the quantum Heisenberg hamiltonian in terms of the classical free energy. Spin coherent states also provide another interesting and yet unexplored alternative, on which we focus our attention in this work. We express spin-operators in terms of differential operators, in a manner analogous to the boson case where its utility is well known (Agarwal 1973; Haake 1973; Haken 1970).

In this paper, we treat various aspects of the Heisenberg hamiltonian in terms of the spin coherent states. The outline of the paper is as follows: In section 2, we review some of the relevant and more important properties of the coherent states, and which are used extensively in later sections. In section 3, we derive the spin coherent state (SCS) representation of Schrödinger equation, Liouville equation for the density operator and the Bloch equation for the unnormalized thermal density matrix  $e^{-\beta H}$ . In section 4, we present the interpretation of the SCS hamiltonian, in terms of magnon interactions. The excitation spectrum is treated explicitly in section 5 and in appendix. We hope to apply our formulation to a calculation of the thermodynamic averages and other problems in a future communication. The method of the present paper is quite general and can be used to discuss any spin system, for example the Heisenberg model with dipole interactions or an antiferromagnet. We hope to apply our formulation to a calculation of the thermodynamic averages and other problems in a future communication.

## 2. Spin coherent states

In this section, we present some well known properties of the spin coherent states for the convenience of the reader and self containment of the paper. We define the spin coherent state (SCS) as

$$|Z\rangle = \exp(ZS^-) |0\rangle \quad (1)$$

where  $Z$  is a complex variable,  $S^-$  the usual lowering operator ( $S^\pm = S^x \pm iS^y$ ) and  $|0\rangle$  is the zero spin deviation state or ground state. The  $n$  spin deviation by definition is an eigenstate of  $S^z$

$$S^z |n\rangle = (s - n) |n\rangle. \quad (2)$$

Equation (1) is analogous to the boson case since  $S^-$  is similar to the destruction operator in that case. The exponential can be expanded and we get

$$|Z\rangle = \sum_{n=0}^{2s} \binom{2s}{n}^{1/2} Z^n |n\rangle. \quad (3)$$

Note that the summation in (3) is restricted to  $n \leq 2s$  due to the operator identity  $(S^-)^{\gamma} = 0$ ;  $\gamma \geq 2s+1$ . The scalar product is given by

$$\langle Z' | Z \rangle = (1 + Z'^* Z)^{2s}. \quad (4)$$

Thus  $|Z\rangle$  is neither normalized nor orthogonal. The normalized states introduced by Radcliffe (1971) are scaled very simply to ours:

$$|Z\rangle_{\text{Radcliffe}} = [1 + |Z|^2]^{-s} |Z\rangle \quad (5)$$

Clearly the states  $|Z\rangle$  form an overcomplete basis of states. We can express unity in this basis as

$$1 = \int \mu(Z) |Z\rangle \langle Z| d^2Z; \quad \mu(Z) = \frac{2s+1}{\pi} (1 + |Z|^2)^{-2s-2}. \quad (6)$$

As usual  $d^2Z = d(\text{Re}Z) d(\text{Im}Z)$  and the integration in eq. (6) is over the entire complex plane.

For certain applications, it is useful to parameterize the complex variable  $Z$  as  $Z = \tan \theta/2 \exp(i\phi)$ . Thus we can label the points on the complex plane by  $\Omega \equiv (\theta, \phi)$  which is of course, the usual mapping on to the Riemann sphere. We define (normalised) states as

$$|\Omega\rangle = \cos^{2s} \theta/2 |Z\rangle \quad (7)$$

These are referred to as Bloch states by Arecchi *et al* (1972). In terms of these states we get the resolution of unity (eq. (6)) as

$$1 = \frac{2s+1}{4\pi} \int d\Omega |\Omega\rangle \langle \Omega|. \quad (8)$$

We can associate two functions corresponding to every operator  $A$  in the Hilbert space:

$$Q_A(\Omega) = \langle \Omega | A | \Omega \rangle \quad (9)$$

and

$$A = \frac{2s+1}{4\pi} \int P_A(\Omega) |\Omega\rangle \langle \Omega| d\Omega. \quad (10)$$

Equation (10) is the analog of the diagonal representation in the theory of boson coherent states. Arecchi *et al* (1972) have pointed out that  $P_A$  cannot be unique. This is a consequence of the result

$$\int Y_{lm}(\Omega) |\Omega\rangle \langle \Omega| d\Omega = 0; \quad l \geq 2s + 1. \quad (11)$$

Thus if we expand  $P_A$  in spherical harmonics  $P_A(\Omega) = \sum_{lm} P_A^{lm} Y_{lm}(\Omega)$  only terms with  $l \leq 2s$  contribute to (10), and hence any two  $P$ 's which have the same coefficients for terms with  $l \leq 2s$  (but differ otherwise) give equally valid representation of  $A$ . For completeness we note that the expansion coefficients of  $Q_A(\Omega)$  and  $P_A(\Omega)$  in spherical harmonics are related through (Gilmore 1976; Shastry 1977).

$$P_A^{lm} = Q_A^{lm} \frac{(2s+1)!(2s)!}{(2s+l+1)!(2s-l)!} \quad (12)$$

In a similar way, we can express the operator  $A$  in terms of the  $Z$  basis:

$$Q_A(Z) = \langle Z | A | Z \rangle \quad (13)$$

and

$$A = \int \mu(Z) P_A(Z) | Z \rangle \langle Z | d^2Z. \quad (14)$$

It is easy to see that

$$\left. \begin{array}{l} Q_A(Z) \\ P_A(Z) \end{array} \right\} \xrightarrow{Z = \tan \theta/2 \exp(i\phi)} \begin{array}{l} \cos^{-4s} \theta/2 Q_A(\Omega) \\ P_A(\Omega) \end{array} \quad (15)$$

In practice, the  $z$  basis is more convenient to work for certain applications, and using (15) one can reconstruct quantities in the  $\Omega$  basis. Also, note that

$$T_r(AB) = \frac{2s+1}{4\pi} \int P_A(\Omega) Q_B(\Omega) d\Omega = \frac{2s+1}{4\pi} \int Q_A(\Omega) P_B(\Omega) d^2\Omega \quad (16)$$

This equation is of much importance in practice since one can calculate expectation values of operators by setting  $A$  or  $B$  as the density matrix of the system. Note that (16) can also be written as

$$\begin{aligned} T_r(AB) &= (2s+1) \sum_{l=0}^{2s} \sum_{m=-l}^l P_A^{lm} Q_B^{lm} \\ &= (2s+1) \sum_{l=0}^{2s} \sum_{m=-l}^l Q_A^{lm} P_B^{lm}. \end{aligned} \quad (17)$$

Here only terms with  $l \leq 2s$  can contribute because the  $Q$  function only contains spherical harmonics with  $l \leq 2s$ .

### 3. Differential representation: Quantum mechanics of the Heisenberg ferromagnet in the SCS basis

As in the usual theory of quantum mechanics we can use the  $|Z\rangle$  basis to yield a representation for states and operators. States are of course represented by wave functions obtained by taking scalar product with  $\langle Z|$ . In order to represent operators, consider

$$\langle Z | S^+ | Z' \rangle = \frac{2s Z'}{(1 + Z^* Z')} \left(1 + Z^* Z'\right)^{2s} = \frac{\partial}{\partial Z^*} \langle Z | Z' \rangle. \quad (18)$$

Thus symbolically we have (noting that  $Z'$ , is arbitrary)

$$\langle Z | S^+ = \frac{\partial}{\partial Z^*} \langle Z | \quad (19)$$

Similarly, we calculate  $\langle Z | S^- | Z' \rangle$  and  $\langle Z | S^z | Z' \rangle$  and find

$$\langle Z | S^z = \left( s - Z^* \frac{\partial}{\partial Z^*} \right) \langle Z | \quad (20)$$

$$\langle Z | S^- = Z^* \left( 2s - Z^* \frac{\partial}{\partial Z^*} \right) \langle Z | \quad (21)$$

Equations (19)–(21) are the differential operator representations for spin operators, which are analogous to the Schrödinger representation [ $p \rightarrow -i\hbar (\partial/\partial x)$ ] in quantum mechanics. We should note that if  $|Z\rangle$  were normalized, the resulting representation for operators is quite messy: this in fact motivates our use of unnormalized states. In this paper we will apply the spin coherent states to the study of the Heisenberg model of magnetism which is described by the Hamiltonian

$$H = - \sum_{ij} J_{ij} \left[ S_i^z S_j^z + \frac{\gamma}{2} \{ S_i^+ S_j^- + S_i^- S_j^+ \} \right]. \quad (22)$$

Here  $J_{ij}$  is the 'exchange integral' which we leave unspecified and  $\gamma$  is an anisotropy parameter. The Hilbert space under consideration is the direct product space over the  $N$  spin spaces. We introduce SCS's corresponding to each spin  $i$  and introduce the notation

$$\mu(Z) = \prod_{i=1}^N \mu(Z_i), \quad Z = (Z_1, Z_2, \dots, Z_N) \text{ and } d^2Z = \prod_{i=1}^N d^2Z_i.$$

The resolution of unity is then in the same form as in eq. (6).

### 3.1. Schrödinger equation in SCS representation

The time independent equation  $H|\psi\rangle = E|\psi\rangle$  can be written in this representation by taking scalar product with  $\langle Z|$ . Thus we get

$$\mathcal{H} \psi(Z^*) = E \psi(Z^*) \quad (23)$$

$$\psi(Z^*) = \langle Z | \psi \rangle \quad (24)$$

where  $\psi$  depends only on  $Z^*$  and where

$$\begin{aligned} \mathcal{H} = E_0 + 2s \sum_{ij} J_{ij} (Z_i^* - \gamma Z_j^*) \frac{\partial}{\partial Z_j^*} \\ + \sum_{ij} J_{ij} \left\{ \frac{\gamma}{2} (Z_i^{*2} + Z_j^{*2}) - Z_i^* Z_j^* \right\} \frac{\partial^2}{\partial Z_i^* \partial Z_j^*} \end{aligned} \quad (25)$$

$E_0$  is the (ground state) energy  $-\sum_{ij} J_{ij} s^2$ . We should note that that  $\psi(Z^*)$  suffices to determine  $|\psi\rangle$  since from the completeness relation (6) we have

$$|\psi\rangle = \int \mu(Z) d^2Z \psi(Z^*) |Z\rangle. \quad (26)$$

It should be noted that we have presented a method for going over *from* the abstract Schrödinger equation *to* the differential eq. (23). If, however, we wish to start from (23) and calculate the eigenvalues and eigenfunctions of  $H$ , we must supplement the differential equation (23) with boundary conditions. In analogy with the similar problem in quantum mechanics, we demand that  $|\psi\rangle$  should be normalized, i.e. that

$$\int \mu(Z) |\psi(Z^*)|^2 d^2Z = 1. \quad (27)$$

This immediately places the restriction that  $\psi(Z^*)$  should be polynomial bounded: the maximum power of  $Z^*$  allowed in  $\psi$  is  $2s$ . Such a restriction is recognizable as a *kinematical* one, preventing a flipping of a (physical) spin more than  $2s$  times, and defines the domain of physical states.

### 3.2. Liouville's equation in SCS representation

The equation  $i\hbar (\partial\rho/\partial t) = [H, \rho]$  for the time dependent density matrix can be written on taking diagonal SCS matrix elements as

$$i\hbar \frac{\partial}{\partial t} Q_\rho(Z, Z^*) = [\mathcal{H} Q_\rho(Z, Z^*) - \text{C.C.}] \quad (28)$$

$$Q_\rho(Z, Z^*) = \langle Z | \rho | Z \rangle \quad (29)$$

where in contrast with  $\psi$ ,  $Q_\rho$  depends on both  $Z$  and  $Z^*$ . The differential eq. (28) must be supplemented with boundary conditions analogous to those for the wave function. The condition we require is that  $\text{Tr}\rho$  be time independent, leading to

$$\text{Im} \int \mu(Z) d^2Z \mathcal{H} Q_\rho(Z, Z^*) = 0. \quad (30)$$

Similar type of consistency conditions and their implications (referred usually to as 'surface terms') have been recently analysed by Lugiato *et al* (1977).

### 3.3. Bloch equation in SCS representation

The (unnormalized) density matrix  $e^{-\beta H}$  obeys the equation of motion  $\partial\rho/\partial\beta = -H\rho$ . Taking diagonal matrix element in SCS we get

$$\frac{\partial}{\partial\beta} Q_\rho(Z, Z^*) = -\mathcal{H} Q_\rho(Z, Z^*); \quad Q_\rho = \langle Z | \rho | Z \rangle. \quad (31)$$

A knowledge of the density matrix contains much of the relevant information regarding the system. The partition function for an equilibrium problem is reduced to a classical integral

$$Z = \text{Tr} e^{-\beta H} = \int d^2Z \mu(z) Q_\rho(Z, Z^*) \quad (32)$$

#### 4. Interpretation of Heisenberg hamiltonian in SCS representation

In this section we briefly discuss the physical interpretation of various terms in the hamiltonian (25) in terms of magnons. For this purpose it is convenient to introduce a new set of  $N$  (momentum space) variables

$$Z_{\mathbf{k}}^* = \left(\frac{2s}{N}\right)^{1/2} \sum_i [\exp(-i\mathbf{k} \cdot \mathbf{r}_i)] Z_i^* \quad (33)$$

The inverse relation reads

$$Z_i^* = \frac{1}{(2sN)^{1/2}} \sum_{\mathbf{k}} [\exp(i\mathbf{k} \cdot \mathbf{r}_i)] Z_{\mathbf{k}}^* \quad (34)$$

The summation in (34) runs over the  $N$  vectors in the Brillouin zone. The rules for expressing  $\partial/\partial Z_i^*$  in terms of  $\partial/\partial Z_{\mathbf{k}}^*$  follow trivially and we can write the hamiltonian (25) as

$$\begin{aligned} \mathcal{H} = E_0 + \sum_{\mathbf{k}} \omega_{\mathbf{k}} Z_{\mathbf{k}}^* \frac{\partial}{\partial Z_{\mathbf{k}}^*} + \frac{1}{2N} \sum_{\substack{\mathbf{k}_1, \mathbf{k}_2 \\ \mathbf{K}}} V_{\mathbf{K}}(\mathbf{k}_1, \mathbf{k}_2) Z_{\mathbf{K}/2 + \mathbf{k}_2}^* \times \\ Z_{\mathbf{K}/2 - \mathbf{k}_2}^* \frac{\partial^2}{\partial Z_{\mathbf{K}/2 + \mathbf{k}_1}^* \partial Z_{\mathbf{K}/2 - \mathbf{k}_1}^*} \end{aligned} \quad (35)$$

where

$$\omega_{\mathbf{k}} = 2s [J(\mathbf{0}) - J(\mathbf{k})]; J(\mathbf{k}) = \frac{1}{N} \sum_{ij} \exp(-i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)) J_{ij} \quad (36)$$

and

$$V_{\mathbf{K}}(\mathbf{k}_1, \mathbf{k}_2) = \gamma \{J(\mathbf{K}/2 + \mathbf{k}_1) + J(\mathbf{K}/2 - \mathbf{k}_1)\} - J(\mathbf{k}_1 - \mathbf{k}_2) - J(\mathbf{k}_1 + \mathbf{k}_2). \quad (37)$$

In this form the different terms in the hamiltonian are recognized easily. The second term corresponds to free spinwaves and the last term to an interaction between spinwaves. The interaction is such that total momentum is conserved. We will see in next sections that the magnon excitation spectrum is determined by the eigenvalues of an integral equation whose kernel is essentially  $V$ . In appendix A, we discuss in detail the relation and difference between the present approach based on SCS and the approach based on the *boson* coherent states.

#### 5. Excitation spectrum of the Heisenberg hamiltonian

The ground state wave function is found by inspection to be a constant. Thus

$$(\mathcal{H} - E_0) \text{const} = 0. \quad (38)$$

The first excited states can also be found easily. We have from (35)

$$(\mathcal{H} - E_0) Z_k^* = \omega_k Z_k^*. \quad (39)$$

Thus  $Z_k^*$  is the eigenfunction of  $\mathcal{H}$  with eigenvalue  $\omega_k$  which is the usual spin wave result.

The two spin deviation problem is the first nontrivial problem that we encounter in the Heisenberg model. In analogy with the one spin wave function, we could try the product wave function.

$$\phi_K(k) = Z_{K/2+k}^* Z_{K/2-k}^*. \quad (40)$$

This function, however, is not a proper basis to work in since it contains terms like  $Z_i^{*2}$  which are unphysical for  $s=\frac{1}{2}$  (since they correspond to two spin flips at the same site). Such terms also lead to a  $\psi'$  which is not normalizable. Some examples of states having this type of structure are treated in appendix B. The physical basis (see however appendix C) must be built up from the two spin deviation state (the pair wave function)

$$\psi_{ij} = \langle Z | S_i^- S_j^- | 0 \rangle = \text{const} \left( 1 - \frac{\delta_{ij}}{2s} \right) Z_i^* Z_j^*. \quad (41)$$

Thus, taking fourier transform of  $\psi_{ij}$ , we have a physical basis

$$\psi_K(\mathbf{k}) = \phi_K(\mathbf{k}) - \frac{1}{2sN} \sum_{\mathbf{q}} \phi_K(\mathbf{q}) \quad (42)$$

(Note that the terms  $i=j$  are suppressed for  $s=\frac{1}{2}$  in eq. (41) and hence in (42)). Since the hamiltonian (35) is an operator in terms of  $Z_k$ 's, we start by operating with  $\mathcal{H}$  on  $\phi$ . Thus we get

$$(\mathcal{H} - E_0) \phi_K(k) = E_K(k) \phi_K(\mathbf{k}) + \frac{1}{N} \sum_{\mathbf{k}, \mathbf{q}} V_K(\mathbf{k}, \mathbf{q}) \phi_K(\mathbf{q}) \quad (43)$$

where

$$E_K(\mathbf{k}) = \omega_{K/2+\mathbf{k}} + \omega_{K/2-\mathbf{k}}. \quad (44)$$

$E_k$  is the sum of two spin wave energies and the second term in (43) corresponds to scattering of the spin waves. We can readily get the action of  $\mathcal{H}$  on  $\psi$  as

$$(\mathcal{H} - E_0) \psi_K(\mathbf{k}) = E_K(\mathbf{k}) \psi_K(\mathbf{k}) + \frac{1}{N} \sum_{\mathbf{q}} V_K(\mathbf{q}, \mathbf{k}) \psi_K(\mathbf{q}) \quad (45)$$

where we have used the identities

$$\frac{1}{2s} \{E_K(\mathbf{k}) - E_K(\mathbf{q})\} + V_K(\mathbf{k}, \mathbf{q}) = V_K(\mathbf{q}, \mathbf{k}) \quad (46)$$



and

$$\frac{1}{N} \sum_{\mathbf{k}} J_{\mathbf{k}-\mathbf{q}} = 0 \quad \forall \mathbf{q}. \quad (47)$$

Note that (45) differs from (43) only in that  $V_{\mathbf{K}}(\mathbf{k}, \mathbf{q})$  is replaced by  $V_{\mathbf{K}}(\mathbf{q}, \mathbf{k})$ . A two spin wave eigenstate of  $\mathcal{H}$  can be obtained as a linear combination of  $\psi'_s$  as follows. Let

$$\psi_{\mathbf{K}}^{(2)} = \sum_{\mathbf{k}} U_{\mathbf{K}}(\mathbf{k}) \psi_{\mathbf{K}}(\mathbf{k}) \quad (48)$$

such that

$$(\mathcal{H} - E_0) \psi_{\mathbf{K}}^{(2)} = \omega_{\mathbf{K}}^{(2)} \psi_{\mathbf{K}}^{(2)}. \quad (49)$$

Thus  $\omega_{\mathbf{K}}^{(2)}$  is the two spin wave eigenvalue labelled by  $\mathbf{K}$  (which is conserved by  $\mathcal{H}$ ). From (48), (49) and (45) we have

$$\sum_{\mathbf{k}} \psi_{\mathbf{K}}(\mathbf{k}) \left[ (\omega_{\mathbf{K}}^{(2)} - E_{\mathbf{K}}(\mathbf{k})) U_{\mathbf{K}}(\mathbf{k}) - \frac{1}{N} \sum_{\mathbf{q}} V_{\mathbf{K}}(\mathbf{k}, \mathbf{q}) U_{\mathbf{K}}(\mathbf{q}) \right] = 0. \quad (50)$$

Now  $\psi$ 's are arbitrary complex variables and hence (50) can be satisfied identically if

$$(\omega_{\mathbf{K}}^{(2)} - E_{\mathbf{K}}(\mathbf{k})) U_{\mathbf{K}}(\mathbf{k}) = \frac{1}{N} \sum_{\mathbf{q}} V_{\mathbf{K}}(\mathbf{k}, \mathbf{q}) U_{\mathbf{K}}(\mathbf{q}). \quad (51)$$

This integral equation gives both the eigenvalues  $\omega^{(2)}$  and the coefficients  $U$ 's (and hence the wave function) and is essentially the same as that of Fukuda and Wortis (1963) in the case of nearest neighbour interactions and that of Majumdar (1969) for the case of nearest plus next nearest neighbours. The resulting spectrum of scattering and bound states is well known in literature and will not be repeated here.

## 6. Discussions and conclusions

We have presented a new description of Heisenberg ferromagnet in spin coherent state representation. The approach has no *ad-hoc* assumptions and avoids the boson representation of spin operators. The advantages of SCS representation are clear: for example one does not have to use any 'projection operators' in the calculation of expectation values (cf appendix A). We have shown how the eigenfunctions and eigenvalues can be obtained. The Bloch equation has been transformed into phase space and the conventional perturbation theory, such as that used by Langer (1968, 1969), can be used to calculate the free-energy. The Liouville equation for density operator is also transformed into a differential equation, the solution of which can be used to calculate a variety of time dependent properties such as the Green's functions, by using the methods familiar from phase space methods for boson systems

(cf. Agarwal 1969). Central to our formulation is the representation of spin angular momentum operators in terms of differential operators (cf Gilmore *et al* 1975; Vilenkin 1968) which assume a particularly simple form in terms of unnormalized coherent states.

We have shown how the two spin wave and three spin wave spectra (see appendix C) can be obtained using the present formalism. We have done the calculations using both the 'subtracted' and 'unsubtracted' basis for the two spin wave case. Of particular interest is our demonstration that the eigenvalue spectra for two and three spin waves can be obtained using the unsubtracted basis. The difference, however, is that the Kernels of the integral equations obtained using subtracted basis are the adjoints of those using the unsubtracted basis. This also implies that the eigenfunctions obtained by using the unsubtracted basis are *left* eigenfunctions of the kernel whereas the *right* eigenfunctions of this kernel are obtained by using subtracted basis. From these results, it appears that the eigenvalue spectrum can be obtained by completely ignoring the kinematical problems. Some further calculations will most probably confirm this hypothesis. The irrelevance of the kinematical interaction in the above context seems to be related to Dyson's result that these contribute vanishing ( $\exp(-T_c/T)$ ) contributions in the low temperature limit.

It should also be pointed out that the SCS basis provides a natural basis in which the classical limit is transparent (Radcliffe 1971) Thus one should be able to study solitary waves (e.g. Tjon and Wright 1977; Pushkarov and Pushkarov 1977) and other 'classical' phenomena in quantum systems using SC states. We hope to return to the actual calculation a thermodynamics averages, and others in a future publication.

## Appendix A

Distinction between the treatment based on the SCS and the boson coherent state representation.

The problem of the Heisenberg ferromagnet has been treated previously in terms of the boson coherent state representation. For example Douglass (1971 a, b) expressed the spin operators in terms of boson operators via the Dyson-Maleev transformation.

$$S^+ \rightarrow a; S^z \rightarrow s - a^+ a; S^- \rightarrow 2sa^+(1 - a^+ a/2s) \quad (\text{A } 1)$$

and then used the boson coherent states to calculate the free energy in a series form analogous to the form developed by Langer (1968) for the case of interacting bosons. Using (A1) the Heisenberg Hamiltonian becomes

$$(H - E_0) \rightarrow 2s \sum_{ij} J_{ij} a_i^+ (\gamma a_j - a_i) + \sum_{ij} J_{ij} a_j^+ a_j [a_i^+ a_i - \gamma a_j^+ a_i] \quad (\text{A } 2)$$

where Dyson-Maleev transformation has been used for each spin  $i$ . At this stage, we might argue that the dynamical equations can be transformed into phase space equations by using the well known methods of boson coherent states. For example (where the state  $\langle Z |$  refers to unnormalised boson states).

$$\begin{aligned} \langle Z|a = \sum_{n=0}^{\infty} \frac{Z^{*n}}{(n!)^{1/2}} \langle n|a = \sum_{n=0}^{\infty} \frac{Z^{*n-1}}{[(n-1)!]^{1/2}} n^{1/2} \langle n| \\ = \frac{\partial}{\partial Z^*} \langle Z| \end{aligned} \quad (\text{A } 3)$$

Similarly

$$\langle Z|a^+ = Z^* \langle Z|; \quad (\text{A } 4)$$

Hence if we start from the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle \quad (\text{A } 5)$$

we find using (A 3), (A 4) the eigenvalue equation

$$\mathcal{H}_B \psi_B(Z^*) = E \psi_B(Z^*); \quad \psi_B(Z^*) = \langle Z|\psi\rangle \quad (\text{A } 6)$$

where

$$\begin{aligned} (\mathcal{H}_B - E_0) = 2s \sum_{ij} J_{ij} (Z_i^* - \gamma Z_j^*) \frac{\partial}{\partial Z_j^*} \\ + \sum_{ij} J_{ij} \left\{ \frac{\gamma}{2} (Z_i^{*2} + Z_j^{*2}) - Z_i^* Z_j^* \right\} \frac{\partial^2}{\partial Z_i^* \partial Z_j^*}. \end{aligned} \quad (\text{A } 7)$$

It is interesting to note that (A 7) has a form identical to that of eq. (25) obtained by using the spin coherent states; however the two are entirely different in practice for a variety of reasons. We have for example shown that the expectation values of the operators are obtained by using the integrals over the spin coherent space (eq. (16)) whereas if we use the boson representation, the expectation values are not so easily obtained. In order to calculate a trace over spin variable. We need a trace over boson space with an additional projection operator  $P$  such that  $P|n\rangle = 0$  if  $n > 2s$  and  $P|n\rangle = |n\rangle$  if  $n \leq 2s$ . Thus

$$\text{Tr}_{\text{Spin}}(\rho G) = \text{Tr}_{\text{Boson}}(\tilde{\rho} \tilde{G} P) \quad (\text{A } 8)$$

where  $\tilde{\rho}$  and  $\tilde{G}$  are the boson representatives of  $\rho$  and  $G$ . This leads to

$$\text{Tr}_{\text{Spin}}(\rho G) = \int d^2Z m(|Z|) \tilde{P}_\rho(Z) \langle Z|\tilde{G}P|Z\rangle \quad (\text{A } 9)$$

where  $\tilde{P}$  is analogous to  $P$  for the SCS case and  $m(|Z|)$  is a suitable measure of integration. By inserting a complete set of boson coherent states we get

$$\text{Tr}_{\text{Spin}}(\rho G) = \int d^2Z d^2Z' m(|Z|) m(|Z'|) \tilde{P}_\rho(Z) \langle Z|\tilde{G}|Z'\rangle \langle Z'|P|Z'\rangle \quad (\text{A } 10)$$

Thus the expectation values are more complicated in this approach, requiring off-diagonal coherent state matrix elements of operators and projection operators.

## Appendix B

### Physical and some unphysical states of a model hamiltonian

In this appendix we consider the eigenfunctions and eigenvalues of two spin particles interacting through

$$H = \mathbf{S}_1 \cdot \mathbf{S}_2; \quad |S_i| = \frac{1}{2}. \quad (\text{B } 1)$$

In the SCS representation the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{4} + \frac{1}{2}(Z_2^* - Z_1^*) \left( \frac{\partial}{\partial Z_2^*} - \frac{\partial}{\partial Z_1^*} \right) - \frac{1}{2}(Z_1^* - Z_2^*)^2 \frac{\partial^2}{\partial Z_1^* \partial Z_2^*}. \quad (\text{B } 2)$$

The physical energies are clearly  $\frac{1}{4}$  and  $-\frac{3}{4}$ . The wave functions in SCS representation are as follows

$$\begin{aligned} \uparrow\uparrow &\rightarrow \text{const}; \quad (\epsilon = \frac{1}{4}) \\ (\uparrow\downarrow + \downarrow\uparrow) &\rightarrow Z_2^* + Z_1^*; \quad (\epsilon = \frac{1}{4}) \\ \downarrow\downarrow &\rightarrow Z_1^* Z_2^*; \quad (\epsilon = \frac{1}{4}) \\ (\uparrow\downarrow - \downarrow\uparrow) &\rightarrow Z_2^* - Z_1^*; \quad (\epsilon = -\frac{3}{4}). \end{aligned} \quad (\text{B } 3)$$

The eigenvalues are indicated in brackets. We can also consider polynomials of degree  $\geq 2$ . In fact it is quite simple to see that a close of eigenfunctions is provided by  $(Z_1^* - Z_2^*)^m$  for  $m$  arbitrary. We have

$$\mathcal{H}(Z_1^* - Z_2^*)^m = \left[ \frac{1}{4} + \frac{m}{2}(m-3) \right] (Z_1^* - Z_2^*)^m. \quad (\text{B } 4)$$

Various cases follow for different values of  $m$

1. Completely physical states: Cases  $m=0$  and  $m=1$  clearly reduce to (B 3) and there is no more to be said.
2. Partly unphysical states: The case  $m=2$  gives energy  $m = -\frac{3}{4}$  which is indeed one of the physical eigenvalues of  $H$ . The wave function for this case is partly physical and partly unphysical.
3. Completely unphysical: The case  $m \geq 3$  corresponds to completely unphysical states. All the eigenvalues (except for  $m=3$ ) are unphysical.

Thus we see that the operator  $\mathcal{H}$  admits infinite number of eigenfunctions (for different  $m$ ) of which only a few are physical. We observe that the state which have a mixture of physical and unphysical components (the case  $m=2$ ) corresponds to an eigenvalue which is *physical*.

It is interesting to note that under operation of the 'Hamiltonian  $\mathcal{H}$ ', the physical and unphysical states separately form invariant subspaces. The 'mixed' states however, are transformed into other 'mixed' states only i.e.  $\mathcal{H}$  does not take mixed states into either completely physical or completely unphysical states. This property is clearly shared by a  $N$  spin generalization of this simple problem, which is of course the Heisenberg model.

## Appendix C

### Two and three magnon spectra in unsubtracted basis

In this appendix we re-examine the two and three magnon problems working with partly unphysical states such as the function  $\phi_{\mathbf{K}}$  (eq. (40)). As explained in the text, these contain terms such as  $Z_i^{*2}$  which are unphysical for  $s=\frac{1}{2}$  and which must be removed (i.e. subtracted) at the very outset. However, these certainly form an interesting basis for the operator  $\mathcal{H}$  (but not one for the Heisenberg model) and we investigate the eigenvalue problem in this basis below. We start from eq. (43)

$$(\mathcal{H} - E_0) \phi_{\mathbf{K}}(\mathbf{k}) = E_{\mathbf{K}}(\mathbf{k}) \phi_{\mathbf{K}}(\mathbf{k}) + \frac{1}{N} \sum_{\mathbf{q}} V_{\mathbf{K}}(\mathbf{k}, \mathbf{q}) \phi_{\mathbf{K}}(\mathbf{q}).$$

We can seek an eigenstate of  $\mathcal{H}$  of the form

$$\phi_{\mathbf{K}}^{(2)} = \sum_{\mathbf{k}} \Omega_{\mathbf{K}}(\mathbf{k}) \phi_{\mathbf{K}}(\mathbf{k}) \quad (\text{C } 1)$$

such that

$$(\mathcal{H} - E_0) \phi_{\mathbf{K}}^{(2)} = \tilde{\omega}_{\mathbf{K}}^{(2)} \phi_{\mathbf{K}}^{(2)}. \quad (\text{C } 2)$$

Combining (43), (C 1) and (C 2) we get

$$\sum_{\mathbf{k}} \left[ (\tilde{\omega}_{\mathbf{K}}^{(2)} - E_{\mathbf{K}}(\mathbf{k})) \Omega_{\mathbf{K}}(\mathbf{k}) - \frac{1}{N} \sum_{\mathbf{q}} V_{\mathbf{K}}(\mathbf{q}, \mathbf{k}) \Omega_{\mathbf{K}}(\mathbf{q}) \right] \phi_{\mathbf{K}}(\mathbf{k}) = 0. \quad (\text{C } 3)$$

Since  $\phi_{\mathbf{k}}$ 's are arbitrary complex variables, we get the integral equation

$$(\tilde{\omega}_{\mathbf{K}}^{(2)} - E_{\mathbf{K}}(\mathbf{k})) \Omega_{\mathbf{K}}(\mathbf{k}) = \frac{1}{N} \sum_{\mathbf{q}} V_{\mathbf{K}}(\mathbf{q}, \mathbf{k}) \Omega_{\mathbf{K}}(\mathbf{q}) \quad (\text{C } 4)$$

Note that (C 4) is simply the adjoint of (51) i.e.  $V_{\mathbf{K}}(\mathbf{k}, \mathbf{q})$  is replaced by  $V_{\mathbf{K}}(\mathbf{q}, \mathbf{k})$ . This enables us to infer that

$$\tilde{\omega}_{\mathbf{K}}^{(2)} = \omega_{\mathbf{K}}^{(2)} \quad (\text{C } 5)$$

(This result is easily obtained by viewing (51) and (C 4) as the eigenvalue equation of a non-hermitian operator and its adjoint). Thus we get the same eigenvalues as in the case of the physical eq. (51).

In a similar way we can examine the three spin wave problem. We start with the operation of  $\mathcal{H}$  on a product state

$$\phi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = Z_{\mathbf{p}_1}^* Z_{\mathbf{p}_2}^* Z_{\mathbf{p}_3}^* \quad (\text{C } 6)$$

It is easy to see that

$$\begin{aligned} (\mathcal{H} - E_0) \phi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) &= E(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \phi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\ &+ \frac{1}{N} \sum_{\mathbf{q}} \left[ V_{\mathbf{p}_1 + \mathbf{p}_2} \left( \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}, \mathbf{q} \right) \phi \left( \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} + \mathbf{q}, \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} - \mathbf{q}, \mathbf{p}_3 \right) + \mathcal{P} \right] \end{aligned} \quad (\text{C } 7)$$

Where  $E = \omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2} + \omega_{\mathbf{p}_3}$  and  $\mathcal{P}$  stands for the two terms obtained by cyclically permuting  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$  in the preceding term. We can look for eigenfunctions of  $\mathcal{H}$  in the form

$$\phi^{(3)} = \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3} a(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \phi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \quad (\text{C } 8)$$

with eigenvalue  $\omega^{(3)}$ . By following the same argument as for the two spin wave case, we get an integral equation for  $a(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$

$$\begin{aligned} [\omega^{(3)} - E(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)] a(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) &= \frac{1}{N} \sum_{\mathbf{q}} \left[ V_{\mathbf{p}_1 + \mathbf{p}_2} \left( \mathbf{q}, \frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right) \right. \\ &\left. a \left( \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} + \mathbf{q}, \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} - \mathbf{q}; \mathbf{p}_3 \right) + \mathcal{P} \right] \end{aligned} \quad (\text{C } 9)$$

This equation is precisely the adjoint of the equation found by Majumdar *et al* (1973). The eigenvalues are then the same as theirs for the same reason as in the two spin wave case. We have in fact verified this by actually solving the Fadeev equations in a manner similar to Majumdar *et al* (1973). This leads to a single variable integral equation which is the adjoint of a similar one (eq. 3.6) of Majumdar *et al* (1973).

For completeness, one should go back and look at the wave functions. This is quite easy to do in the two spin wave case. We have examined the tractable case of  $S = \frac{1}{2}$  linear chain for which the wave function (eq. (C 1)) can be easily computed. If we write in it the 'co-ordinate space' as

$$\phi_{\mathbf{K}}^{(2)} = \sum_{ij} C_{ij} Z_i^* Z_j^* \quad (\text{C } 10)$$

we find that  $C_{ii} = 0$ . Hence in this particular case, the Hamiltonian  $\mathcal{H}$  combines  $\phi$ 's in such a way as to get rid of the unphysical terms! Of course, there is no reason to expect this to happen in general.

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