Dyson's Brownian Motion and Universal Dynamics of Quantum Systems

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We establish a correspondence between the evolution of the distribution of eigenvalues of a $N \times N$ matrix subject to a random Gaussian perturbing matrix, and a Fokker-Planck equation postulated by Dyson. Within this model, we prove the equivalence conjectured by Altshuler and co-workers between the space-time correlations of the Sutherland-Calogero-Moser system in the thermodynamic limit and a set of two-variable correlations for disordered quantum systems calculated by them. Multiple variable correlation functions are, however, shown to be inequivalent for the two cases.

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In a series of recent papers, an interesting generalization of the problem of eigenvalue statistics for complex quantum systems has been introduced by Altshuler, Simons, and co-workers [1]. They consider the change in the positions of the eigenvalues in response to an external perturbation; after an appropriate normalization of the perturbing potential, they show that the evolution of the eigenvalues of the system as a function of the strength of the perturbation is universal. By treating this extra parameter (the strength of the perturbation) as a timelike coordinate, they conjecture that this problem is identical to the ground state *dynamics* of an integrable one-dimensional interacting many-body quantum model, the Sutherland model [2]. The ground state *equal time* correlations of this latter model are known to correspond to the eigenvalue statistics at (any) fixed strength of the perturbation for an appropriate choice of the coupling constant, depending on the universality class of the perturbation (real, complex, or symplectic). If the equivalence between the two systems is valid, then the full time-dependent correlation functions of this model are known from Ref. [1], and constitute a dramatic progress in our understanding of this many-body problem. Although the complete spectrum of an integrable many-body problem can often be found, the time-dependent correlations involve matrix elements, and the above calculation is one of the very few for a nontrivial many-body system.

In this work, we prove the equivalence between the space-time correlators of a Sutherland type quantum model and those of a novel version of the quantum chaos problem wherein we interpolate parametrically between two random matrices of a given ensemble [3] in a *periodic* fashion. By integrating over all perturbations from a given ensemble, we show that the eigenvalues evolve according to a Fokker-Planck equation proposed by Dyson [4] in the course of his classic work on random matrices [5]. This equation is equivalent, under a Wick rotation, to the quantum mechanics of the Calogero model [6]. In the thermodynamic limit, the Calogero model has identical bulk properties as the Sutherland model, provided the constants scale properly with N, the size of the matrix [2]; this scaling is shown to be fulfilled, whereby we establish the conjecture of Ref. [1]. Our work basically demonstrates that the Dyson Fokker-Planck dynamics [4] represents the equations of motion describing the evolution of eigenvalues of random matrices *within a given ensemble.*

For the evolution of eigenvalues, Simons and co-workers consider a system with a Hamiltonian $H = H_0 + xV$, where V is the perturbation, and x is the strength of the perturbation. Units are normalized so that the mean level spacing of the eigenvalues ϵ_i of H_0 is unity, as is the rms "velocity" of the eigenvalues, defined as $\langle(\partial\epsilon_i/\partial x)^2\rangle$. The autocorrelation function of the energy eigenvalues is defined as

$$k(x;\omega) = \left\langle \sum_{ij} \delta(\epsilon - \epsilon_i(\overline{x}))\delta(\epsilon - \omega - \epsilon_j(\overline{x}+x)) \right\rangle. \quad (1)$$

By explicit calculation for disordered systems, Simons and co-workers provide strong evidence that k is universal, and depends only on whether H is in the orthogonal, unitary, or symplectic ensemble. Numerical simulations are used to argue that this is also true for quantum chaotic systems.

For $x = 0$, k is known to be the same as the ground state equal time correlations in the positions of a collection of N particles confined to a circle, interacting with the Sutherland [2] Hamiltonian. Here we consider the alternative Calogero model [6], in which the particle positions are allowed to range over $(-\infty, \infty)$, with a confining parabolic potential:

$$H_C = -\sum_i \frac{\partial^2}{\partial\lambda_i^2} + \sum_{i>j} \frac{\beta(\beta/2-1)}{(\lambda_i - \lambda_j)^2} + \frac{1}{4a^4}\sum_i \lambda_i^2. \quad (2)$$

The width of the confining potential is chosen to give a

mean ground state interparticle separation in the center of the distribution to be unity. This requirement yields

$$a^2 = 2N/\pi^2\beta. \tag{3}$$

The coupling constant β is equal to 1, 2, and 4 when H is in the orthogonal, unitary, and symplectic ensembles, respectively. Under the mapping [7]

$$x^2 = 2it, \quad \epsilon = \lambda, \tag{4}$$

Simons and co-workers argue that $k(x;\omega)$ is equal to the corresponding *time-dependent* correlation function for the Sutherland model. This correlation function can be calculated explicitly for $\beta = 2$, where it agrees with the expression obtained by Simons and co-workers. For $\beta = 1$ and 4, while a complete calculation is not possible for the Sutherland model, the asymptotic forms of k for large and small t can be calculated; these agree with the asymptotics of the expressions derived for $k(x;q)$ [1] for the orthogonal and symplectic ensembles, respectively.

In this paper we demonstrate that, when V is a Gaussian random matrix from the appropriate ensemble, the evolution of *all* quantities such as k, involving only the eigenvalues $\epsilon_i(x)$ at two different values of x, are equal to the corresponding time-dependent correlation function for the Calogero model. However, correlation functions involving quantities at more than two values of x are in general different, so that the full dynamics for the two systems are not identical.

In order to do this, it is more convenient to work with a Fokker-Planck equation that is equivalent to the Calogero Hamiltonian. With $P(\{\lambda_i\};t) = \psi_0(\{\lambda_i\})\psi(\{\lambda_i\};t)$, where ψ_0 is the ground state wave function for the Calogero model, we obtain [4,5]

$$\frac{\partial P(\{\lambda_i\};t)}{\partial\tau} = \sum_i \frac{\partial}{\partial\lambda_i}\left[\frac{\partial P}{\partial\lambda_i} + \frac{\lambda_i}{a^2}P\right] - \sum_{i,j\neq i}\frac{\partial}{\partial\lambda_i}\left[\frac{\beta}{\lambda_i-\lambda_j}P\right]. \tag{5}$$

Equation (4) is then changed to

$$x^2 = 2\tau, \quad \epsilon = \lambda. \tag{6}$$

Equation (5) is the Fokker-Planck equation corresponding to the Langevin dynamics of a collection of N classical particles at finite temperature, with logarithmic repulsive pairwise interactions, the Wigner-Dyson Coulomb gas [5]. Because of the repulsion between the particles, no steady state distribution is achieved without the parabolic confining potential. Correlation functions involving the particle positions at two different times τ_1 and $\tau_2 > \tau_1$ can be found by considering the evolution from a general initial state for a time $\tau_2 - \tau_1$, and then averaging over initial states.

For the quantum system, we modify the parametrization of the disorder strength from $H(x) = H_0 + xV$ to

$$H(x) = H_0\cos(\Omega x) + V\sin(\Omega x)/\Omega. \tag{7}$$

Here V is taken to be of the form $V = \sum_{r=0}^{\beta-1} V_r e_r$, *à là* Dyson, where e_r are units of the appropriate algebra, so that $\langle V_{ij}^2\rangle = \sum_{r=0}^{\beta-1}\langle (V_r)_{ij}^2\rangle = (\beta/2)\langle V_{ii}^2\rangle$. This parametrization has the advantage that when V/Ω and H_0 are considered to be Gaussian random matrices from the same distribution, the distribution for $H(x)$ is stationary as a function of x. For unit mean eigenvalue spacing, we obtain

$$\langle V_{ii}^2\rangle = 2\Omega^2 N/\pi^2\beta, \quad \langle V_{ij}^2\rangle = \Omega^2 N/\pi^2. \tag{8}$$

From the additional normalization condition that the rms velocity of the eigenvalues must be unity, it is easy to see from first order perturbation theory that $\langle V_{ii}^2\rangle = 1$, so that we must choose

$$\Omega = \sqrt{\pi^2\beta/2N} = 1/a. \tag{9}$$

Since $\Omega \to 0$ for large N, the change we have made in parametrizing the disorder strength is inconsequential for finite x.

We first note that, for any perturbation V, the evolution of the eigenvalues $\epsilon_i(x)$ as a function of x can be expressed in the form of a set of first order differential equations in the continuously changing eigenbasis of $H(x)$ perturbed by $H'(x) \equiv \partial H(x)/\partial x$:

$$\frac{d\epsilon_i}{dx} = H'_{ii}, \quad \frac{dH'_{ii}}{dx} = -\Omega^2\epsilon_i + \sum_{j\neq i}\frac{2|H'_{ij}|^2}{\epsilon_i-\epsilon_j},$$

$$\frac{dH'_{ij}}{dx} = \sum_{k\neq i,j} H'_{ik}H'_{kj}\left(\frac{1}{\epsilon_i-\epsilon_k} + \frac{1}{\epsilon_j-\epsilon_k}\right) + H'_{ij}(H'_{jj}-H'_{ii})\frac{1}{\epsilon_i-\epsilon_j}. \tag{10}$$

Equation (10) can be viewed as a classical Newtonian system, with N degrees of freedom corresponding to the eigenvalues ϵ_i, and $\nu \equiv N + \beta N(N-1)/2$ degrees corresponding to the diagonal and off-diagonal elements of H', obeying appropriate Poisson bracket relations. This classical system can be shown to be *integrable* [8]. The integrability of this system implies that it is *essential* to make assumptions about the nature of the matrix V, either of the form we have made above or otherwise [9].

It is possible to integrate these equations of motion formally; irreversible equations in x arise when $V_{i,j}$ are averaged over. In the present work, we find it convenient to follow a different strategy to obtain a Fokker-Planck equation for the eigenvalues. Our method follows closely along the lines of a beautiful proof by Dyson [4] for the behavior of the eigenvalues of a matrix subject to random thermal noise; although our problem does *not* have a source of thermal noise, this will be seen to be unim-

portant for two time correlation functions. Since V is a Gaussian random matrix, for any given initial H_0 the matrix $H(x)$ has the distribution at a given x:

$$\mathcal{P}(H,x|H_0) = \frac{\Omega^\nu}{\sin^\nu(\Omega x)} \exp\left(-\frac{1}{2}\mathrm{Tr}[H(x) - H_0\cos(\Omega x)]^2 \frac{\Omega^2}{\sin^2(\Omega x)}\right). \tag{11}$$

We recognize this distribution as the solution to the equation [4]

$$\frac{\partial \mathcal{P}}{\partial \hat{\tau}} = \sum_{ij}\left[g_{ij}\frac{\partial^2 \mathcal{P}}{\partial H_{ij}^2} + \frac{1}{\hat{a}^2}\frac{\partial}{\partial H_{ij}}(H_{ij}\mathcal{P})\right] \tag{12}$$

with the initial condition $H = H_0$, where $\hat{a} = 1/\Omega$,

$$\hat{\tau} = -\hat{a}^2 \ln\left[\cos(\Omega x)\right],$$
$$g_{ij} = \delta_{ij} + (1-\delta_{ij})\beta/2. \tag{13}$$

This is the Fokker-Planck equation for a system in which all the matrix elements H_{ij} undergo *independent* Langevin dynamics [starting from inital values $(H_0)_{ij}$] in a parabolic confining well, with an appropriate width and time coordinate. The temperature at which the Langevin motion takes place is unity for the diagonal elements of the matrix and $\beta/2$ for the off-diagonal elements. Equation (11) thus implies that, for any given H_0, *all* moments of the different elements of $H(x)$ will be exactly the same at any x *as if* the elements of H were moving independently in parabolic confining wells at a finite temperature. But for such a thermal motion, as shown by Dyson [4], it is possible to go to the eigenbasis of H at any time, and obtain to second order in perturbation theory for an infinitesmal increase in time $\delta\hat{\tau}$ a simplified form of Eqs. (10):

$$\delta\epsilon_i = \delta H_{ii} + \sum_{j\neq i}(\delta H_{ij})^2/(\epsilon_i - \epsilon_j). \tag{14}$$

The normalization of Eq. (13) then implies that

$$\langle \delta\epsilon_i\rangle = -\frac{\epsilon_i}{\hat{a}^2}\delta\hat{\tau} + \beta\sum_{j\neq i} 1/(\epsilon_i - \epsilon_j)\delta\hat{\tau},$$
$$\langle \delta\epsilon_i^2\rangle = 2\delta\hat{\tau}. \tag{15}$$

The distribution of eigenvalues $P(\{\epsilon_i\},\hat{\tau}|\{\epsilon_i^0\})$ at any time $\hat{\tau}$ then satisfies a Fokker-Planck equation like Eq. (5), with λ_i replaced by ϵ_i, and τ and a replaced by $\hat{\tau}$ and $\hat{a}$, respectively. But comparing Eqs. (6) to Eqs. (9) and (13), we see that, in the large N limit, $a = \hat{a}$ and $\tau = \hat{\tau}$, yielding Eq. (5). Thus the distribution of eigenvalues of the quantum system evolves in the same way as the distribution of particle positions in a Wigner-Dyson gas.

While the equivalence between the time coordinate defined in Eq. (13) and Eq. (6) is true only in the large N limit, the form given in Eq. (13) is true for *any* N. From the form of Eq. (7), we see that $2\pi/\Omega$ is the Poincaré recurrence "time" interval in x for all the eigenvalues to return to their initial values. From Eq. (13) we see that the interval $(0,\pi/2)$ in x is stretched out to $(0,\infty)$ in $\hat{\tau}$, so that the system continues to lose memory of its initial configuration for all $\hat{\tau}$, with complete equilibration achieved only in infinite time.

Note that we have found that the two distributions evolve in the same way for *arbitrary* initial conditions. Quantities such as $k(x;q)$ can be seen from Eq. (1) to involve a sum over various moments of ϵ, weighted suitably and then averaged over *equilibrium* initial conditions, requiring only a weaker equivalence. Under a self-averaging assumption, such equilibrium averages will be the same as for a generic choice of initial conditions taken from the equilibrium distribution.

Since our results are independent of initial conditions, it is also possible to dispense with the parabolic confining well: Although there is no longer any steady state, one can follow the transient dynamics. This actually corresponds to the original parametrization of $H = H_0 + xV$; although conceptually slightly subtle, the algebra is actually simpler. Although time translational invariance is now broken, for any fixed finite time the correction terms vanish at large N.

The result obtained above has been for explicit averaging over a Gaussian random perturbing potential. Apart from the issue of self-averaging, which is relevant for quantum chaos, it is necessary to verify that the additional non-Gaussian terms in the distribution of V (properly scaled with N) do not affect the result in the large N limit, in order to claim universality [9]. We hope to return to this problem in the future.

The ground state correlation function of Eq. (1) can be expressed for Gaussian random matrices H_0 and V in the closed form expression

$$k(x;\omega) = \frac{\Omega^\nu}{|\sin^\nu(\Omega x)|}\int dH\, dH_0\, \mathrm{Tr}[\delta(\epsilon - \omega - H)]\mathrm{Tr}[\delta(\epsilon - H_0)]$$
$$\times \exp\left\{ -\frac{1}{2a^2\sin^2(\Omega x)}\mathrm{Tr}\left[H_0^2 + H^2 - 2H_0 H\cos(\Omega x)\right]\right\}. \tag{16}$$

All the dependence on time is explicitly present in the $\cos(\Omega x)$ and $\sin(\Omega x)$ factors. It is sometimes possible to evaluate generalized two matrix Gaussian integrals [10], and it would be interesting to apply these techniques to calculating the above.

It is important to realize that, while Eq. (11) implies that all moments of the eigenvalues calculated at any x will be equal to the corresponding moments of the particle positions of a Wigner-Dyson gas undergoing Brownian motion, this does *not* mean that the motion of the eigenvalues is indeed Brownian. The randomness in the dynamics of the eigenvalues comes from the matrix V, which acts like *quenched* disorder. As a simple illustration of the result of the disorder being quenched, we consider the case of $N = 1$. Equation (10) yields $\epsilon(x) - \epsilon(0) = V \sin(\Omega x)/\Omega + \epsilon(0)[\cos(\Omega x) - 1]$. Averaging over V, for small x we find that, with $\overline{\epsilon}(x) = \epsilon(x) - \epsilon(0)$,

$$\langle \overline{\epsilon}(x_1)\overline{\epsilon}(x_2)\rangle = x_1 x_2 + \epsilon^2(0)[\Omega^4 x_1^2 x_2^2/4]. \quad (17)$$

For the Wigner-Dyson gas, on the other hand, for $N = 1$ we have a particle in a parabolic well with thermal noise, so that with $\overline{\lambda}(\tau) = \lambda(\tau) - \lambda(0)$,

$$\langle \overline{\lambda}(\tau_1)\overline{\lambda}(\tau_2)\rangle = 2\min[\tau_1, \tau_2] + \lambda^2(0)[\tau_1\tau_2/a^4]. \quad (18)$$

When $\tau_1 = \tau_2$, with $2\tau = x^2$, Eqs. (17) and (18) are identical [since $a = 1/\Omega$ from Eq. (9)]. However, when $\tau_1 \neq \tau_2$, the two equations are different; this difference persists even for large N.

It is precisely such multiple time averages that are involved in three point (and higher order) correlation functions. For instance, the density correlation function that is an extension of Eq. (1) is

$$k(x_1, x_2; \omega_1, \omega_2) = \left\langle \sum_{ijl} \delta(\epsilon - \epsilon_i(\overline{x}))\delta(\epsilon - \omega_1 - \epsilon_j(\overline{x} + x_1))\delta(\epsilon - \omega_2 - \epsilon_j(\overline{x} + x_2)) \right\rangle. \quad (19)$$

This involves moments of the eigenvalues at *two* values of x, x_1, and x_2 (averaged over initial conditions). For the Wigner-Dyson gas, the n point correlation functions can be expressed in terms of the two point functions, since each measurement "rezeros" time, yielding an n matrix integral. For the matrix version of quantum chaos, the n point function *remains* a two matrix integral.

In this paper, we have proved the equivalence of the dynamics of the Wigner-Dyson gas (or, alternatively, the Sutherland model) with the evolution of the eigenvalues of a Hamiltonian under a perturbation drawn from a Gaussian ensemble for two time correlation functions, as argued by Simons, Altshuler, and co-workers [1]. Multiple time correlation functions are not the same in general for the two systems, because of the difference between annealed and quenched randomness. It is also possible to prove the equivalence of the two time velocity-velocity correlation function, $c(\omega; x)$, that measures the correlations in the rate of change of the eigenvalues [1], by a perturbative method, to all orders in perturbation theory.

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Note added.—A recent Letter [11] addresses similar equivalences between the two time correlators, but only within a hydrodynamical limit.

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[9] In general, any distribution that we choose for V that is a function only of the constants of motion of Eq. (10) is invariant under the evolution. (See, e.g., Yukawa, in Ref. [8].) However, many of these constants of motion can be eliminated on *physical* grounds—for instance, that the normalization of the distibution for V should not depend on H_0—that are not apparent from a purely dynamical systems viewpoint of Eq. (10). Whether all distributions for V that satisfy these criteria—for example, $\exp[-\mathrm{Tr}V^2 - \lambda \mathrm{Tr}V^4]$—lead to the same dynamics as the Gaussian ensembles we consider here is a more difficult question. Since both for $x = 0$ and for $x = \pi/2\Omega$ (when $H = V$) the eigenvalue statistics of H are independent of the details of the distribution of V, and since we shall see that Eq. (10) can be expressed as a diffusive evolution with $x = \pi/2\Omega$ equivalent to $t \to \infty$, we think it is plausible that the dynamics will indeed be universal for all t in the large N limit. Note that the question of an appropriate distribution for the matrices is distinct from the issue of self-averaging, which deals with whether a single choice of matrices from a *particular* distribution is representative in the large N limit of the *entire* distribution. (For a single matrix, eigenvalue statistics would have to be computed as an average over the N eigenvalues, rather than an ensemble average.) We think that physical systems are very likely to be self-averaging.

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