Exact Result for the Magnetic Susceptibility of an Electron Gas

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The magnetic susceptibility of an electron gas at high density is found to be $\chi=\chi_P\{1-(\alpha r_s/\pi)+\frac{1}{2}(\alpha r_s/\pi)^2[0.306-\ln(\alpha r_s/\pi)]+O(r_s^3)\}^{-1}$.

The magnetic susceptibility of an electron gas has an asymptotic development in powers of r_s and $r_s \ln r_s$ at high densities [where $r_s=(3/4\pi n)^{1/3}a_0^{-1}$, n is the density, and a_0 the Bohr radius]. The classic work of Brueckner and Sawada[1] (BS) evaluates two correlation correction terms of $O(r_s^2)$ beyond the Hartree-Fock term which is exact to $O(r_s)$ [the symbol $O(r_s^2)$ is understood to include terms like $r_s^2 \ln r_s$]. One of the two terms has subsequently been rederived by simpler methods,[2] but the other term is far more complicated and involves a careful analysis of the "ring diagram" series. The purpose of this Letter is to point out and correct a subtle error in the BS analysis for this term, and thereby establish the exact high-density expansion of the susceptibility to $O(r_s^2)$.

The susceptibility can be written as

$$\chi=\chi_P[1-(\alpha r_s/\pi)+\tfrac{3}{4}(\alpha r_s)^2\alpha_c]^{-1}, \tag{1}$$

where χ_P is the noninteracting Pauli susceptibility, and $\alpha_c=[2(d^2/dp^2)\epsilon_{\rm corr}(p)]_{p=0}$, where $\epsilon_{\rm corr}$ is the correlation energy per electron in Rydbergs and p is the fractional polarization (I follow the notations of BS). As $r_s\to 0, \alpha_c$ is the sum of a constant and a $\ln r_s$ term which BS evaluated by means of a subtraction integral. Their result is compactly expressed by

$$\alpha_c \underset{r_s\to 0}{=} \frac{2}{3\pi^2}\left[0.534-\ln\left(\frac{4\alpha r_s}{\pi}\right)\right]+\frac{4}{g}\left[\left(\frac{\partial^2}{\partial x^2}-\frac{2\partial}{\partial x}\right)f(x,y)\right]_{x=1,\,y=1}, \tag{2}$$

with

$$f(x,y)=\left(-\frac{3}{2\pi^4}\right)\lim_{\beta\to 0}\int_\beta^\infty \frac{dq}{q}[h(x,y;q)-\theta(1-q)h(x,y;0)], \tag{3}$$

$$h(x,y;q)=\frac{1}{q}\iint\frac{d^3k_1\,d^3k_2}{\vec{q}\cdot(\vec{k}_1+\vec{k}_2+\vec{q})}\theta(x-k_1)\theta(|\vec{k}_1+\vec{q}|-x)\theta(y-k_2)\theta(|\vec{k}_2+\vec{q}|-y). \tag{4}$$

It is readily seen that the function h has the nice scaling property: $h(x,x;q)=x^3\varphi(q/x)$, which can be used in Eq. (3) to give

$$f(x,x)=\left(-\frac{3}{2\pi^4}\right)x^3\lim_{\beta\to 0}\left[\varphi(0)\ln\beta+\int_{\beta/x}^\infty\frac{dt}{t}\varphi(t)\right]. \tag{5}$$

In the second term, the integrand can be expanded in the neighborhood of the lower limit; and on careful evaluation of the limit, one gets

$$f(x,x)=(-3/2\pi^4)[x^3K+\varphi(0)x^3\ln x], \tag{6}$$

where K is some constant independent of x. The $x^3 \ln x$ term is a manifestation of the fact that the two integrals in (3) are not separately convergent and only their difference is. This term has been missed by BS and hence the identity (A5) is inapplicable. However by using Eq. (6), one can derive another identity,

$$\left[\left(\frac{\partial^2}{\partial x^2}-\frac{2\partial}{\partial x}+\frac{\partial^2}{\partial x\partial y}\right)f(x,y)\right]_{x=1,\,y=1}=\frac{3}{2}\left(-\frac{3}{2\pi^4}\right)\varphi(0), \tag{7}$$

which is the same as (A5) of BS if $\varphi(0)$ is set at zero. This identity can be used in conjunction with Eq. (2), and the relations[3] $\varphi(0)=\frac{8}{3}\pi^2(1-\ln 2)$ and[1]

$$\left[\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\right]_{x=1,\,y=1}=-\frac{3}{\pi^2}(\ln 2+\tfrac{1}{2})$$

to give

$$\alpha_c = \frac{2}{3\pi^2}\left[0.534 - \ln\left(\frac{4\alpha r_s}{\pi}\right) + 6\ln 2 - 3\right]. \tag{8}$$

This gives the exact expression for the susceptibility to $O(r_s^2)$ as

$$\chi = \chi_P\left\{1 - \frac{\alpha r_s}{\pi} + \frac{1}{2}\left(\frac{\alpha r_s}{\pi}\right)^2\left[0.306 - \ln\left(\frac{\alpha r_s}{\pi}\right)\right] + O(r_s^3)\right\}^{-1}. \tag{9}$$

The BS result for the susceptibility [obtained from Eq. (9) by replacing 0.306 by 1.534] is smaller than mine for all densities. I have verified Eq. (9) by two other methods. These and other results pertaining to the relevance of the Brueckner-Sawada approach in the context of recent experiments will be published elsewhere.[4]

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[1]K. A. Brueckner and K. Sawada, Phys. Rev. 112, 328 (1958).

[2]C. Herring, *Magnetism,* edited by G. T. Rado and H. Suhl (Academic, New York, 1966), Vol. IV. Section III contains two simple derivations of the $r_s^2 \ln r_s$ coefficient.

[3]See, e.g., A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971), p. 32, problem 1.5(a).

[4]B. S. Shastry, Tata Institute for Fundamental Research Report No. TH/76-44 (to be published), and Ph.D. thesis, Bombay University, 1976 (unpublished).

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ELECTRON-SURFACE-PLASMON SCATTERING USING A PARABOLIC NONTOUCHING TRAJECTORY. J. Lecante, Y. Ballu, and D. M. Newns [Phys. Rev. Lett. 38, 36 (1977)].

In Fig. 3, the labels (a) and (b) should be interchanged.

EVIDENCE FOR SURFACE-INTERACTION EFFECTS VIA A NUCLEAR HYPERFINE-INTERACTION EXPERIMENT. M. Hass, J. M. Brennan, H. T. King, T. K. Saylor, and R. Kalish [Phys. Rev. Lett. 38, 218 (1977)].

The received date should read as follows: (Received 2 December 1976).

Reference 16 on page 221 should read, "...preferential population of $m_I = +1$... ."

RESOLUTION OF AN AMBIGUITY IN ALTERNATIVE SOFT-PION APPROACHES TO $\pi N \rightarrow \pi\pi N$ NEAR THRESHOLD. M. G. Olsson, E. T. Osypowski, and Leaf Turner [Phys. Rev. Lett. 38, 296 (1977)].

On page 296, the first author's name was incorrectly spelled. It should read M. G. Olsson.