

THE DYNAMICAL ORIGIN OF SYMMETRY OF ELEMENTARY PARTICLES *

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1. INTRODUCTION

ONE of the most remarkable features of elementary particles is their multiplet structure. The simplest such structure is the particle-antiparticle pairing with equal mass, spin, lifetime, etc., but with opposite charge (and baryon number, hypercharge, etc.). We relate this regularity to the TCP invariance of the theory, though in earlier years we would have considered the regularity to be a consequence of charge conjugation invariance. We may say that we understand the origin of the particle-antiparticle symmetry.¹

But among the strongly interacting particles we see multiplets of particles with equal spin and parity, but only approximately equal mass. It is conventional to identify such a multiplet structure with the manifestation of an internal symmetry group, the multiplets constituting irreducible representations of this group. Charge independence of strong interactions is now well established, and it is not inconsistent to assume that the deviations from exact charge independence are due to the charge dependent coupling with the radiation field: though it is by no means true that this is the only possible mechanism of violation of charge independence. By now it is also well established that there are regularities in the particle (and resonance) spectrum which go beyond charge independence, in the sense that the multiplets can be further grouped together to constitute supermultiplets with the same spin, parity and baryon number and comparable masses, which constitute irreducible representations of the special unitary group in three dimensions.² In this case the departures from symmetry are not easily blamed on a known non-strong interaction, but have to be ascribed to a "small" part of the strong interactions themselves.

All along, the framework was one in which the symmetry group was "given". As long as the perturbations are neglected, the particles and resonances are to constitute irreducible multiplets, with the coupling constants being

proportional to the Clebsch-Gordan coefficients. But as to which multiplets occur, or as to the identification of observed particles with irreducible representations, the theory is silent. The difference between the Sakata and the Gell-Mann-Ne'eman versions of SU_3 is a case in point. A simple-minded suggestion is that the lowest dimensional representations of the group only occur. There are at least two shortcomings to this point of view: first, it does not tell which of the "smaller" representations actually occur and the order of their masses; second, one has to invoke extraneous considerations to eliminate the triplet (and sextet, etc.) representations of the unitary group. In view of these, it is worthwhile to seek a more intimate connection between the symmetry group and the dynamics of the system.

There is another line of development which makes such a connection even more desirable. In a dynamical scheme, where the particles or resonances appear in the direct channel of a two-particle scattering process as a result of the exchange of these and other particles or resonances in the crossed channels, there are self-consistency demands on the number of particles or resonances in a multiplet and on the relative magnitudes of the various coupling constants; and the multiplets that can be exchanged to give an *attractive* force are not arbitrary.³ Thus there is a possibility of seeking a dynamical origin of symmetries, *starting from the existence* of (mass-spin-parity degenerate) *multiplets of interacting particles* and requiring self-consistency. In addition to the need of self-consistency, in most such attempts to-date, one includes other conservation laws, like conservation of baryon number, electric charge, hypercharge, etc. As a result of these, with restriction to equal masses of the particles within a multiplet, the problem of dynamical self-consistency reduces to a set of algebraic non-linear equations. In case these equations lead to a symmetry group, the group comes equipped with the specific representations furnished by the interacting multiplets.

Essentially the same, but weaker, equations follow from considerations of an entirely different nature. In this, one again starts from the existence of (mass-spin-parity degenerate)

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multiplets but extends the identity of their one-particle properties to particles in interaction by requiring that the two-point (i.e., one-particle) propagators of the interacting "particles" belonging to the multiplet are identical.⁴ No questions of self-consistency are necessary in this scheme, but the equations derived by isolating and equating suitable terms in the different propagators serve more or less the same purpose as the self-consistency equations in establishing a dynamical origin of symmetries. Again, as before, the groups come equipped with the irreducible representations furnished by the interacting multiplets.

2. SMUSHKEVICH EQUATIONS FROM FIELD THEORY

Consider three multiplets of interacting particles E, F, ϕ with m , n and ν members. Then the dynamical principle of equal propagators require that

$$G^{E}_{rr'}(x-y) = \delta_{rr'} G^E(x-y) \quad 1 \leq r, r' \leq m \quad (1a)$$

$$G^{F}_{ss'}(x-y) = \delta_{ss'} G^F(x-y) \quad 1 \leq s, s' \leq n \quad (1b)$$

$$G^{\phi}_{aa'}(x-y) = \delta_{aa'} G^{\phi}(x-y) \quad 1 \leq a, a' \leq \nu \quad (1c)$$

Here G^E , G^F , G^{ϕ} are appropriate Green functions. The essential point is the appearance of the Kronecker delta on the right-hand side, and it is a consequence of the requirement that the propagators be equal for any two members of the multiplet, since we have the freedom to redefine the components using any unitary transformation :

$$E_r(x) \rightarrow E'_r(x) = \sum_{r'} U_{rr'} E_{r'}(x) \quad (2a)$$

$$F_s(y) \rightarrow F'_s(y) = \sum_{s'} W_{ss'} F_{s'}(y) \quad (2b)$$

$$\phi(\xi) \rightarrow \phi'_a(\xi) = \sum_{a'} V_{aa'} \phi_{a'}(\xi) \quad (2c)$$

We now assume that there exists a non-vanishing trilinear vertex (Fig. 1) coupling the particle multiplets E, F, ϕ which is of the form

$$\Gamma^{\alpha}_{r,s}(x,y,\xi) = g^{\alpha}_{r,s} \Gamma(x,y,\xi) \quad (3)$$

By isolating the contribution from two-particle intermediate states (Fig. 2) to the propagators (1) we get the bilinear relations

$$\sum_{\alpha,s} g^{\alpha}_{r,s} (g^{\alpha}_{r',s})^* = A_1 \delta_{rr'} \quad (4a)$$

$$\sum_{\alpha,r} g^{\alpha}_{r,s} (g^{\alpha}_{r',s})^- = B_1 \delta_{ss'} \quad (4b)$$

$$\sum_{r,s} g^{\alpha}_{r,s} (g^{\alpha}_{r',s})^* = C_1 \delta_{\alpha\alpha'} \quad (4c)$$

It is convenient to introduce a matrix notation at this point, identifying $g^{\alpha}_{r,s}$ as the (r,s) matrix element of the matrix g^{α} . Then we can rewrite (4) in the form

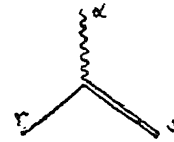


FIG. 1

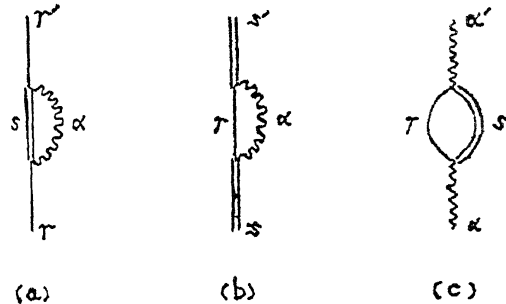


FIG. 2

FIGS. 1-2 Fig. 1. The primitive vertex. Fig. 2. The second order diagrams.

$$\sum_{\alpha} g^{\alpha} g^{+\alpha} = A_1 \mathbf{1} \quad (5a)$$

$$\sum_{\alpha} g^{+\alpha} g^{\alpha} = B_1 \mathbf{1} \quad (5b)$$

$$\text{tr}(g^{-\alpha} g_{\alpha,+}) = C_1 \delta_{\alpha\alpha'} \quad (5c)$$

It is to be noticed that the unit matrices in (5a) and (5b) are respectively $m \times m$ and $n \times n$.

To proceed further we must appeal to a diagrammatic expansion of the propagator. Such an expansion follows most naturally in a perturbation theory: there are, however, some grounds for believing that the higher order relations derived by equating these quantities to have a large range of validity than perturbation theory itself. In the general case,⁵ when no two of the multiplets E, F, ϕ are identical the next irreducible contribution to the propagator comes from a sixth order diagram (Fig. 3). These take the form:

$$\sum_{\alpha F \gamma} g^{\alpha} g^{+\beta} g^{\gamma} g^{+\alpha} g^{\beta} g^{+\gamma} = A_2 \mathbf{1} \quad (6a)$$

$$\sum_{\alpha \beta \gamma} g^{+\alpha} g^{\beta} g^{+\gamma} g^{\alpha} g^{+\beta} g^{\gamma} = B_2 \mathbf{1} \quad (6b)$$

$$\sum_{\beta \gamma} \text{tr}(g^{\alpha} g^{+\beta} g^{\gamma} g^{+\alpha} g^{\beta} g^{+\gamma}) = C_2 \delta_{\alpha\alpha'} \quad (6c)$$

There are tenth, fourteenth, ... order relations generalizing this structure. The constants A, B, C in these equations are not independent; they are related by

$$m A_1 = n B_1 = \nu C_1 \quad (5')$$

$$m A_2 = n B_2 = \nu C_2 \quad (6')$$

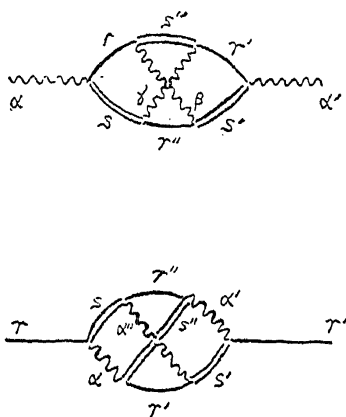


FIG. 3

FIG. 3. The sixth order diagrams.

and so on. All these equations including (5) are invariant under the automorphism

$$g^\alpha \rightarrow \sum_{\alpha'} V_{\alpha\alpha'} U g^{\alpha'} W^+ \quad (7)$$

Considerable simplifications result if two of the multiplets are the same, say $E = F$. In this case,⁴ we could choose ϕ to be hermitian without any essential loss of generality. This implies that the coupling matrices g^α could be chosen hermitian so that we could drop the hermitian conjugation sign in these equations; (5 a) and (5 b) are no longer independent, and similarly

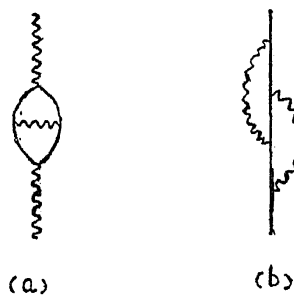


FIG. 4

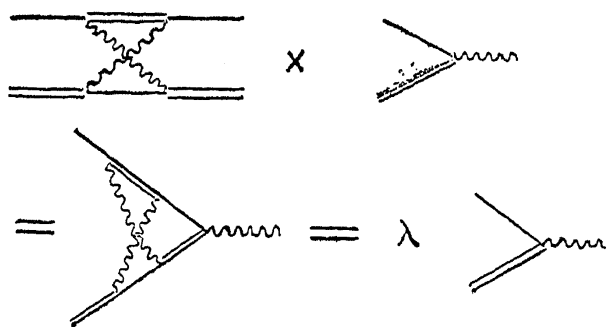


FIG. 5

FIGS. 4-5. Fig. 4. The fourth order diagrams. Fig. 5. Diagram illustrating the self-consistency relation—Eq. (14).

(6 a) and (6 b). In addition, there are irreducible diagrams of fourth order, eighth order, etc. The fourth order diagrams (Fig. 4) give the contributions

$$\sum_{\alpha\beta} g^\alpha g^\beta g^\alpha g^\beta = A_2 \mathbf{1} \quad (8)$$

$$\sum_{\beta} \text{tr} (g^\alpha g^\beta g^{\alpha'} g^\beta) = C_2 \delta_{\alpha\alpha'} \frac{n A_2}{\nu} \delta_{\alpha\alpha'} \quad (9)$$

These relations are not, in general, invariant under the automorphisms (7) but only under the restricted automorphisms

$$g^\alpha \rightarrow \sum_{\alpha'} V_{\alpha\alpha'} U g^{\alpha'} U^\dagger \quad (10)$$

We shall refer to the relations satisfied by the coupling matrices in either case as the Smushkevich relations.⁴

3. SMUSHKEVICH EQUATIONS FROM SELF-CONSISTENT DYNAMICS

Essentially the same relations can be deduced from the self-consistent dynamical model. In this case, the interaction potential (or the N function of an N/D method) is due to the exchange of particles in the crossed channel, while the wave function of, say, the ϕ particle viewed as being composed of an E particle and an \bar{F} particle (an F antiparticle) is to be considered an eigenfunction of this potential (corresponding to a low-lying bound state). Equality of the masses within a multiplet enables us to factorize the wave function⁶ in a manner completely analogous to (3):

$$\psi^{\alpha}_{(r,s)} = g^{\alpha}_{rs} \psi \quad (11)$$

where ψ is a suitable function of momentum variables but independent of α, r, s . Normalization of the various components of ψ in (11) demands:

$$\sum_{s',s} g^{\alpha}_{rs} (g^{\alpha}_{rs'})^* |\psi|^2 = \delta_{\alpha\alpha'}$$

which is equivalent to (4 c) with $C_1 = 1/|\psi|^2$. Similar relations follow from viewing an E(ϕ) particle as a bound state of an F particle and a $\phi(E)$ particle, thus completing the set of equations (4) or (5).

The potential is again defined in terms of the coupling constants g^{α}_{rs} introduced by (11). Again, two distinct cases arise depending upon whether we are considering two identical multiplets or all these multiplets being distinct. In the former case we have a one-particle exchange contribution to the potential, while in the second case the potential has only contributions corresponding to two (or four, six, eight, etc.) particle exchanges. Continuing to assume the degeneracy of the masses, we can write down the factorized forms

$$\sum_{\beta\gamma r' s'} g^{\beta r s} (g^{\gamma r' s'})^* g^{\gamma r' s} (g^{\beta r' s'})^* \cdot N$$

$$= \sum_{\beta\gamma} (g^{\beta} g^{\gamma+})_{r r'} (g^{\beta+} g^{\gamma})_{s' s} \cdot N \quad (12)$$

for the scattering of an E particle and an \bar{F} particle (an F antiparticle). The requirement (see Fig. 5) that (11) is an eigenfunction of (12) leads to the relation⁵

$$\sum_{\beta\gamma r' s'} (g^{\beta} g^{\gamma+})_{r r'} (g^{\beta+} g^{\gamma})_{s' s} g^{\alpha r' r} = \lambda_3 g^{\alpha r s} \quad (13)$$

where λ_3 is a constant, since the D matrix is diagonal. This may be rewritten

$$\sum_{\beta\gamma} g^{\beta} g^{\gamma+} g^{\alpha} g^{\beta+} g^{\gamma} = \lambda_3 g^{\alpha}. \quad (14)$$

By similar considerations of the wave function of the E and F particle in terms of the other two particles we get exactly the same equation. Combining (14) with (5) we can deduce (6) with the parameters satisfying (5') and (6') in the form

$$\frac{A_3}{A_1} = \frac{B_3}{B_1} = \frac{C_3}{C_1} = \lambda_3.$$

For the case of two multiplets being identical, we have a one-particle exchange potential of the form⁶

$$\sum_{\beta} g^{\beta r r'} g^{\beta s' s} N \quad (15)$$

which (see Fig. 6) leads to

$$\sum_{\beta} g^{\beta} g^{\alpha} g^{\beta} = \lambda_2 g^{\alpha} \quad (16)$$

which, in turn, leads to (8) and (9) with

$$\frac{A_2}{A_1} = \frac{C_2}{C_1} = \lambda_2.$$

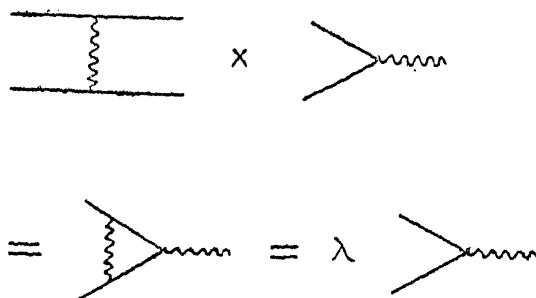


FIG. 6

FIG. 6. Diagram illustrating the derivation of Eq. (16).

This completes the derivation of the Smushkevich equations from dynamical considerations. We shall now seek the conditions under which the Smushkevich equations imply a symmetry group. We remark here that if the coupling constants transform as invariant three index symbols (generalized Clebsch-Gordan coefficients) then we know that these relations are all satisfied automatically, provided there

is only one unique invariant three index symbol (multiplicity-free coupling); the problem facing us is to determine the conditions under which the converse is true.

4. UNITARY SYMMETRY

The simplest case to consider is one in which $m = n = \sqrt{1 + \nu}$ but $E \neq F$. In this case, we can deduce SU_n symmetry⁵ with E, F identified with the self-representations and ϕ with the adjoint representation of the group, making use of only (5) which follow, without the use of any diagrammatic expansion. To show this, we note that the matrices g^{α} which according to (5) satisfy the trace orthogonality

$$tr (g^{\alpha} g^{\alpha'+}) = A_1 \delta_{\alpha\alpha'}, \quad 1 \leq \alpha, \alpha' \leq n^2 - 1 \quad (17)$$

can be augmented by a matrix g^0 which satisfies

$$tr (g^0 g^{\mu+}) = A_1 \delta_{3\mu} \quad 0 \leq \mu \leq n^2 - 1 \quad (18)$$

These n^2 matrices constitute a complete set of $n \times n$ matrices and, hence, satisfy the completeness relation

$$\sum_{\mu=0}^{n^2-1} g^{\mu r s} (g^{\mu r' s'})^* = a \delta_{r r'} \delta_{s s'} \quad (19)$$

where a is a suitable constant. Hence, in particular

$$\sum_{\mu=0}^{n^2-1} g^{\mu} g^{\mu+} = a \mathbf{1}$$

But we had, from (5),

$$\sum_{\alpha=1}^{n^2-1} g^{\alpha} g^{\alpha+} = \left(n - \frac{1}{n}\right) A_1 \mathbf{1}$$

This implies

$$g^0 g^{0+} = \left\{ a - A_1 \left(n - \frac{1}{n}\right) \right\} \mathbf{1}$$

so that g^0 is a multiple of a unitary matrix. We now make use of redefinition (2) of the particles of the multiplet which generates an automorphism of the type (7). If we now choose

$$U = \mathbf{1}, \quad W = \sqrt{\frac{n}{na + (n^2 - 1) A_1}} g^0 \quad (20)$$

then it follows that

$$g^{\alpha} \rightarrow \sum_{\alpha'} V_{\alpha\alpha'} g^{\alpha'} W^+ \quad (21)$$

with

$$tr (g^{\alpha}) \rightarrow \sum_{\alpha'} V_{\alpha\alpha'} tr (g^{\alpha'} W^+) = 0$$

according to (18) and (20). But $(n^2 - 1)$ traceless matrices satisfying (18) can be chosen to be proportional to the $(n^2 - 1)$ generators of SU_n in the $n \times n$ representation. We have thus deduced the SU_n invariance of the system with E, F transforming as the n dimensional representations of the group and ϕ transforming as the $(n^2 - 1)$ dimensional adjoint representa-

tion. Once we have deduced the group structure we can be assured that the higher order equations like (6) are automatically satisfied.

For the case of $E = F$, $m = n = \sqrt{1 + \nu}$ the derivation of unitary symmetry⁴ is more complicated since the automorphisms (10) do not allow the transformation of the type (20). To prove tracelessness, one has to use the fourth order equations. We use (9) and (5) to deduce

$$\sum_B \text{tr} \{ [g^\alpha, g^\beta] [g^{\alpha'}, g^{\beta'}] \} \\ = 2i(C_2 - A_1 C_1) \delta_{\alpha\alpha'}$$

From this equation, making use of the automorphisms (10) it is possible to show that the g^α must be traceless and, hence, that the interaction is invariant under SU_n with the multiplet $E = F$ and ϕ transforming as the n dimensional self-representation and the $(n^2 - 1)$ dimensional adjoint representation respectively.

These results are of interest in connection with the recent interest in the existence of particles belonging to the three-dimensional representation of SU_3 and the possible role of these particles in the realization of SU_3 as the relevant symmetry group. The models found here, in so far as the coupling of these "quarks" with octets is concerned, are similar to the constructions⁷ of Zweig and of Gell-Mann for the $E = F$ case and to the construction of Bacry, Nuyts and Van Hove for the $E \neq F$ case.

We must emphasize here that the derivation of unitary symmetry from the Smushkevich equations in these two cases involved the assumption neither of isotopic spin conservation nor of the electric charge. With proper identification of the generators, we deduce the conservation of isospin and of electric charge.⁴

5. OCTET-OCTET-OCTET COUPLING

Another case of practical interest is the coupling of two identical octets with another octet in accordance with SU_3 . In this case, the Smushkevich equations by themselves cannot yield the SU_3 invariant coupling. But if we assume that all three multiplets are the same and if their coupling matrices are completely antisymmetric (appropriate, for example, for the "gauge" coupling of vector mesons), we can deduce⁸ that the coupling constants constitute the structure constants of some semi-simple Lie group with the particles transforming as the adjoint representation; if we further restrict the set of particles to be irreducible (*i.e.*, cannot be separated into mutually non-interacting sub-multiplets), it must be a simple Lie group. If we now seek the solution corresponding to $n = 8$, we can single out the simple group SU_3 . Hence,

a completely antisymmetric trilinear coupling of an octet must be SU_3 invariant⁹ as a consequence of the Smushkevich equations.

Instead of requiring complete antisymmetry, we may substitute other requirements. For example, by requiring conservation of isospin and hypercharge as well as charge conjugation invariance in the coupling of two (pseudoscalar) octets with a (vector) octet we can again derive unitary symmetry for their coupling.

6. NON-RELATIVISTIC MODELS; SU_6

So far, our discussion was carried out in a manner appropriate to a relativistic theory. But the notion of symmetry, the particle exchange mechanism of generating forces and the notion of antiparticles, etc., are not restricted to a relativistic theory. We can, for example, consider a Galilei invariant theory (or even a Euclidean theory); in this case, the particle-antiparticle correspondence is not inevitable but is consistent with Galilei or Euclidean invariance.¹¹ By a parallel development, we can again deduce the self-consistency equations (14) and (16) for the self-consistent dynamical Smushkevich equations.

In the case of the Galilei group, the spin is a more or less independent quantity; and it is possible to consider special kinds of interaction in which the spin is conserved by itself. By a natural generalization, it is possible to consider a theory in which we form multiplets in which the multiplet labels may include the spin labels. We can then again deduce, for the $m = n = (1 + \nu)^{\frac{1}{2}}$ case a unitary symmetry scheme. Thus, using a triplet of spin half particles for the $E = F$ multiplet, we can deduce SU_6 invariance in the interaction of these particles with a 35-component boson multiplet.¹² Since the SU_6 transformations treat the spin and the additional particle label on the same footing, the 35-component multiplet will contain particles with different spins. Gürsey, Radicati and Pais, and Sakita have shown¹² that in such a scheme, the 35-component multiplet breaks up into a pseudoscalar (scalar) octet and a vector (pseudovector) nonet. The baryon supermultiplet corresponding to the third rank symmetric tensor has 56 components, and breaks up into a $1/2 +$ baryon octet and a $3/2 +$ baryon resonance decuplet. (Incidentally, the Thomas term arising as a relativistic correction, or any other spin-orbit force produces automatically a breakdown of this SU_6 symmetry.)

7. BROKEN SYMMETRY

The particles in nature do not fall into mass degenerate super-multiplets, the masses are only

approximately equal. It is true that the mass deviations from the unitary symmetric limit can be quantitatively understood in terms of a simple mass formula. But the question arises as to how such perturbations of symmetry can be reconciled with the dynamical scheme we have been considering. There are two possible ways in which this can arise: first, the mass deviations that are observed destroy the factorizability of the potential, wave function, or the propagator contributions; the problem can then no longer be studied as an algebraic problem. The other mechanism of symmetry breakdown is to have all the Smushkevich equations satisfied and yet the solutions not exhibiting a group structure⁵; this feature of the solutions of field equations was emphasized already by Heisenberg in connection with his theory of elementary particles. The general symmetry violation is composed of both kinds.

The symmetry violations of the second type have been studied only in the special case $m = n = 3$; $E = F \neq \phi$. In this case, one finds⁵ three solutions: one corresponding to invariance under SU_2 ; another one corresponding to a 3-parameter Abelian group (this is a reducible case); and a third case in which there is no continuous symmetry group, but all the Smushkevich equations are satisfied. Clearly more work is needed to understand these cases. But it appears that for the equal mass case these violations of the symmetry are always large. This is to be contrasted with the case of the first type of symmetry violation where the deviation could be as small as we like.

In case the mass degeneracy within a multiplet is split by small amounts, we can study the modifications in a self-consistent theory by considering the deviations as expanded in terms of quantities transforming irreducibly under the symmetry group.⁶ If the coupling transforms as an invariant plus a small term transforming as an irreducible representation, then the Smushkevich equations (5) and (6) have their right-hand sides replaced by matrices transforming as this irreducible representation. The equations so obtained may not fully specify the deviations of the coupling from its unitary symmetry limit. In the self-consistent dynamical model the corresponding modification is to alter the self-consistency relations (14) and (16) by having a "small" arbitrary linear combination of matrices G^α which transform as the specific irreducible representation added to the matrix g^α on the right-hand side. The structure and stability of this system has not been investigated in any great detail; but it is possible to

have some kind of symmetry violations preferred over other kinds.

There have been a variety of attempts to understand the apparent breakdown of the higher symmetries, say SU_3 , in terms of a lack of commutability of the Lorentz group and internal symmetry group.¹³ But to-date, the results have been disappointing in that within a reasonable purely group-theoretic scheme of non-commutability it seems¹⁴ impossible to break the symmetry.

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