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The Nature of the Axioms of Relativistic Quantum Field Theory. II*

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This paper continues the study of the nature and interdependence of the axioms of relativistic field theory; attention is focused on the notion of relativistic invariance. The central result of the present paper is the derivation of necessary and sufficient conditions for a representation of the covariant free field to admit a unitary representation of the inhomogeneous Lorentz group associated with the field operator. It is shown that only the standard Fock-Cook representation has this property. The relevance of the requirement that the Lorentz group is represented by a unitary family associated with the field operator is exhibited by an analysis of the covariant representations of Shale and Segal. These representations involve extremal states which are not pure, and group representations by intertwining operators.

I. INTRODUCTION

In relativistic quantum field theory[1] it is natural to assume, in accordance with the principle of relativity, that the correspondence $\phi(x) \to \phi(\Lambda x + a)$ of the field operators, imaging the change of frame represented by (a, Λ), must leave the algebra of the field operators unaltered, i.e., it must be an automorphism. These automorphisms of the operator algebra preserve Hermiticity properties and constitute a realization of the (inhomogeneous) Lorentz group. In conventional treatments of relativistic quantum field theory, it is also assumed that there must exist a family of unitary operators $U(a, \Lambda)$ furnishing a (unitary, true) representation of the (inhomogeneous) Lorentz group, and implementing the local automorphism:

$$\phi(x) \to U(a, \Lambda)\phi(x)U^{-1}(a, \Lambda) = \phi(\Lambda x + a).$$

Since the representations are assumed to be continuous, one can assert the existence of the ten Hermitian infinitesimal generators of $U(a, \Lambda)$, and impose additional spectral conditions on these generators. It is usually assumed that there exists a unique invariant state called the vacuum, and that all other states belong to the continuous positive energy spectrum of the Hamiltonian (time-translation) operator. Of course there is no *a priori* reason to insist that the unitary family $U(a, \Lambda)$ must implement the local automorphism. We may, then, distinguish several distinct concepts that enter the characterization of a relativistic field:

(i) The local automorphism $\phi(x) \to \phi(\Lambda x + a)$.

(ii) A unitary family $U(a, \Lambda)$ furnishing representation of the Lorentz group.

(iii) The local automorphism $\phi(x) \to \phi(\Lambda x + a)$ being implemented by the unitary family $U(a, \Lambda)$.

(iv) The existence of the (unique) vacuum.

(v) The nonnegative spectrum of the Hamiltonian.

While it is usual to include the requirements of a unique vacuum and of a nonnegative energy spectrum as well as of the local automorphism implemented by a unitary family under the postulate of relativistic invariance, they are by no means essential. We know of models in which the vacuum is not unique,[1,2] and nontrivial interacting models of quantum field theory exist in which the unitary family $U(a, \Lambda)$ fulfills the representation and spectrum conditions, but does not lead to the local automorphism.

It is also generally assumed that the field operator ring contains the unitary family $U(a, \Lambda)$; loosely speaking this implies that the ten generators can be "built up" using the field operators. This apparently innocent axiom has the consequence that all theories with an invariant cyclic vacuum are in fact direct integrals of theories with a unique invariant vacuum.[3] In the major part of the investigations in the sequel we shall assume that the family $U(a, \Lambda)$ is contained in the operator ring generated by the

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[1] For part I, see E. C. G. Sudarshan and K. Bardakci, J. Math. Phys. 2, 767 (1961).

[2] It is amusing to note at this point that not only is the uniqueness of the vacuum not essential, but the vacuum itself may be dispensed with. The simplest example is provided by starting with the conventional free neutral scalar field $\varphi(x)$ in the standard Fock-Cook representation and constructing the Wightman polynomial $\phi(x) = :\varphi^2(x):$ which can be shown to be a local field. [A. S. Wightman, *Cours de la Faculte des Sciences de l'Université de Paris*, 1957-8; p. 57 (unpublished)]. It may be seen to be irreducible over all odd (even) particle states of the free field $\varphi(x)$. But in the set of all states of $\varphi(x)$ with an odd number of quanta, there is no invariant state! In spite of this, $\phi(x)$ is a local field defined over these states and undergoing local automorphisms implemented by a unitary family.

[3] T. F. Jordan and E. C. G. Sudarshan, J. Math. Phys. 3, 587 (1962); D. Ruelle, Helv. Phys. Acta 35, 147 (1962); H. J. Borchers, "On Structure of the Algebra of Field Operators" (Institute for Advanced Study, Princeton, New Jersey, preprint).

field operator, but discuss a class of field theories in which this is not so.

The important question to be investigated is whether every relativistic field theory admitting the local automorphism has a unitary family $U(a, \Lambda)$ implementing it. There has recently been considerable interest in the nonstandard representations in field theory and in the possibility of breakdown of formal symmetry properties of the theory in the actual realization. It is then natural to ask if the unitary family $U(a, \Lambda)$ exists for the nonstandard representations of any theory. In the following sections, we investigate this question in detail for a free neutral scalar field: the result is somewhat unexpected. Within the framework of an *irreducible* representation of the operator algebra, only the standard Fock–Cook representation admits such a unitary family.[4] This result, obtained within the axiomatic framework outlined, is most simply stated as follows: "Of all representations of the relativistic (neutral, scalar) free field, only the standard representation obtains a manifestly covariant local unitary transformation." The role of the irreducibilty assumption is seen from the results of Shale and Segal[5] discussed in detail in Sec. 5; that, if it is relaxed, there exists a one-parameter infinity of theories in which a unitary family $U(a, \Lambda)$ implementing the local automorphism exist, but the spectrum conditions are violated in that the Hamiltonian is not positive definite (except for the Fock–Cook representation).

II. AUTOMORPHISMS AND REPRESENTATIONS OF THE FIELD RING

Let $\mathfrak{L}$ be the (one-particle) Hilbert space of square integrable functions $\psi(\mathbf{r})$ of the three-vector variable $\mathbf{r}$. Then a unitary representation of the (proper, orthochronous) inhomogeneous Lorentz group is furnished by the choice[6]

$$(h\psi)(\mathbf{r}) = \int \omega(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\, d^3r'$$

$$(\mathbf{p}\psi)(\mathbf{r}) = -i\nabla\psi(\mathbf{r}) \tag{2.1}$$

$$(\mathbf{j}\psi)(\mathbf{r}) = -i(\mathbf{r}\times\nabla)\psi(\mathbf{r})$$

$$(\mathbf{k}\psi)(\mathbf{r}) = \frac{1}{2}\int (\mathbf{r}+\mathbf{r}')\omega(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\, d^3r'.$$

for the ten generators h, $\mathbf{p}$, $\mathbf{j}$. $\mathbf{k}$ of the group. We have used the kernel $\omega(\mathbf{r}, \mathbf{r}')$ defined by

$$\omega(\mathbf{r}, \mathbf{r}') = (2\pi)^{-3}\int \exp\,[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')](\mathbf{q}^2+m^2)^{\frac{1}{2}}\, d^3q.$$

This representation is irreducible[7] and corresponds to a particle of spin zero and mass m. We shall denote the finite unitary transformation corresponding to a Lorentz transformation (a, Λ) obtained from these ten generators by the unitary operator $R(a, \Lambda)$ in the space $\mathfrak{L}$.

Let $u_\alpha(\mathbf{r})$ be an orthonormal basis in $\mathfrak{L}$ so that

$$\int u_\alpha^*(\mathbf{r})u_\beta(\mathbf{r})\, d^3r = \delta_{\alpha\beta}. \tag{2.2}$$

We define the kernel $\mathcal{C}(\mathbf{r}, \mathbf{r}')$ by

$$\mathcal{C}(\mathbf{r}, \mathbf{r}') = 2^{-\frac{1}{2}}(2\pi)^{-3} \times \int \exp\,[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')](\mathbf{q}^2+m^2)^{-\frac{1}{4}}\, d^3q.$$

Let a_α be a sequence of (unbounded) operators and let a_α^+ be their adjoints which satisfy the commutation relations

$$[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}; \qquad [a_\alpha, a_\beta] = 0. \tag{2.3}$$

We shall call the ring generated by the operators the field ring. Then the (relativistic, neutral scalar) field operator is given by the construction[8]

$$\phi(\mathbf{r}, t) = \sum_\alpha \int \mathcal{C}(\mathbf{r}, \mathbf{r}')\exp\,(iht) \times [a_\alpha u_\alpha(\mathbf{r}') + a_\alpha^+ u_\alpha^*(\mathbf{r}')]\, d^3r'. \tag{2.4}$$

The field operator $\phi(x) = \phi(\mathbf{x}, x_0)$ then satisfies the commutation relation

$$[\phi(x), \phi(x')] = i\Delta(x-x')$$

$$= -i(2\pi)^{-3}\int (\mathbf{q}^2+m^2)^{\frac{1}{2}}\exp\,(i\mathbf{k}\cdot\mathbf{y}) \times \sin\,\{y_0(\mathbf{q}^2+m^2)^{\frac{1}{2}}\}\, d^3q.$$

Under the transformation

$$u_\alpha(\mathbf{r}) \to R(a, \Lambda)u_\alpha(\mathbf{r}),$$

the field operator $\phi(x)$ transforms locally:

$$\phi(x) \to \phi(\Lambda x + a).$$

[4] This statement refers only to the free field. The essential point is not the existence of the vacuum (compare reference 2), but the existence of the energy operator; see Sec. 3.

[5] I. E. Segal, Illinois J. Math. 6, 500 (1962).

[6] See, for example, L. L. Foldy, Phys. Rev. 102, 568 (1956).

[7] This representation is not equivalent to the reducible representation obtained from a (local) relativistic wave equation with a (manifestly) covariant amplitude. To see this explicitly, we note that a local relativistic wave equation is invariant not only under the real Lorentz transformations, but also under complex Lorentz transformations. It is, in particular, invariant under the antichronous proper transformation $\mathbf{r}\to -\mathbf{r}$, $t\to -t$. But under this operation, frequencies change sign; hence the "energy" must also change sign for the one-particle amplitude. This is of course true in the familiar spin-0, spin-1/2 and spin-1 covariant wave equations.

[8] For a detailed discussion see, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961) Sec. 7.

By virtue of the invariance of the commutation relations under the change $x \to \Lambda x + a$, $x' \to \Lambda x' + a$, it follows that the local transformation $\phi(x) \to \phi(\Lambda x + a)$ is an automorphism.

We may now consider this automorphism of the field operator as being generated by a linear automorphism of the field ring [rather than by a linear transformation on the $u_\alpha(\mathbf{r})$]. For this purpose let us write

$$R_{\alpha\beta}(a, \Lambda) = \int u_\alpha(\mathbf{r}) R(a, \Lambda) u_\beta^*(\mathbf{r})\, d^3r. \qquad (2.5)$$

Then the local transformation of the field operators is equivalent to the linear automorphism

$$a_\alpha \to \sum_\beta R_{\alpha\beta} a_\beta; \qquad a_\alpha^+ \to \sum_\beta R_{\alpha\beta}^* a_\beta^+ \qquad (2.6)$$

of the field ring. The question of the representation of the field operator is the same as that of the representation of the field ring.

The interesting question to be discussed now is whether these automorphisms of the field ring can be generated as inner automorphisms, i.e., whether there exists an operator family $U(a, \Lambda)$ such that

$$U(a, \Lambda) a_\alpha U^{-1}(a, \Lambda) = \sum_\beta R_{\alpha\beta}(a, \Lambda) a_\beta. \qquad (2.8)$$

Since Hermiticity relations are preserved by these automorphisms, if the representation of the field ring is irreducible, the $U(a, \Lambda)$ would be a unitary family (apart from an unessential scalar). Even in the more general reduction into factors,[9] if $U(a, \Lambda)$ belongs to the field ring, the same property holds.

III. LORENTZ COVARIANCE OF IRREDUCIBLE REPRESENTATIONS OF THE FIELD RING

In this section we wish to investigate the automorphism of the field operators more closely. We shall be particularly interested in the necessary and sufficient conditions under which there may exist an inner automorphism of the operator algebra for every Lorentz transformation (a, Λ). Let e be any one (or linear combination) of the ten generators $\mathbf{h}$, $\mathbf{p}$, $\mathbf{j}$, $\mathbf{k}$. Define

$$e_{\alpha\beta} = \int u_\alpha^*(\mathbf{r}) e u_\beta(\mathbf{r})\, d^3r \qquad (3.1)$$

in terms of the basic set of one-particle wavefunctions $u_\alpha(\mathbf{r})$. Let us also assume that we have an irreducible representation (or, more generally, a factor representation) of the field ring.

We shall now explicitly assume that $U(a, \Lambda)$ belong to this ring.[9] Let $U(\tau)$ be any one-parameter family belonging to $U(a, \Lambda)$. Since $U(\tau)$ constitute a one-parameter unitary family on the Hilbert space $\mathcal{H}$ on which the field operators are represented, by Stone's theorem,[10] there exists a Hermitian generator E for this family in $\mathcal{H}$ which satisfies the relations

$$U(\tau) = 1 + i\tau E + O(\tau^2),$$
$$U(\tau) a_\beta U^{-1}(\tau) = a_\beta + i\tau[E, a_\beta] + O(\tau^2).$$

If $U(\tau)$ is the unitary family corresponding to the generator e in the one-particle Hilbert space $\mathcal{L}$, it follows then that

$$a_\alpha \to \sum_\beta R_{\alpha\beta}(\tau) a_\beta = a_\alpha + i\tau \sum_\beta e_{\alpha\beta} a_\beta + O(\tau^2).$$

We have, on comparing the two transformations to first order in τ,

$$[E, a_\alpha] = \sum_\beta e_{\alpha\beta} a_\beta. \qquad (3.2)$$

Consider the operator

$$F = \sum_{\alpha,\beta} e_{\alpha\beta} a_\alpha^+ a_\beta; \qquad [F, a_\alpha] = \sum_\beta e_{\alpha\beta} a_\beta,$$

if it exists. Then it follows that (if F exists),

$$[E - F, a_\alpha] = 0,$$

so that in every irreducible (or factor) realization of the oscillator ring, $E - F$ must be a scalar, which we denote by C. Hence if the operator F exists, the generator E has the form

$$E = \sum_{\alpha,\beta} e_{\alpha\beta}(a_\alpha^+ a_\beta + \mathfrak{C}_{\alpha\beta}), \qquad (3.3)$$

where $\mathfrak{C}_{\alpha\beta}$ may be chosen arbitrarily except insofar as to require

$$C = \sum_{\alpha,\beta} e_{\alpha\beta} \mathfrak{C}_{\alpha\beta}.$$

On the other hand, we may show that if this structure (3.3) for E does not exist, there can be no operator associated with the oscillator ring which satisfies (3.2). To see this, let us consider the index α to be restricted to take on a finite set of values. Then the corresponding condition (3.2) states that

$$[E, a_\alpha] = \sum_\beta e_{\alpha\beta} a_\beta; \qquad \alpha \in \{\alpha_1, \cdots, \alpha_N\}.$$

The right-hand side exists by definition of the automorphism. This is satisfied only if

$$E = \sum_\alpha a_\alpha^+ (\sum_\beta e_{\alpha\beta} a_\beta) + \mathcal{E}, \qquad (3.4)$$

[9] Compare T. F. Jordan and E. C. G. Sudarshan, reference 3.

[10] M. H. Stone, Ann. Math. 33, 643 (1932); J. von Neumann, Ann. Math. 33, 567 (1932); F. Riesz and B. Sz-Nagy, "Functional Analysis," translated from French by L. F. Boron, (Frederick Ungar Publishing Company, New York, 1955), p. 383.

where

$$[\mathcal{E}, a_\alpha] = 0,$$

i.e., $\mathcal{E}$ is associated with the oscillator ring a_γ, a_γ^+ where γ does not assume any of the N values $\alpha_1, \cdots, \alpha_N$. Since the subset $\alpha_1, \cdots, \alpha_N$ is arbitrary, this can be true if and only if E has the form (3.4). Hence we have proved that the necessary and sufficient condition for the existence of the operator E associated with the field ring is that, for a suitable set of constants $\mathcal{C}_{\alpha\beta}$, there exists a nontrivial operator E of the form (3.3).

Let now E, E' be the operators corresponding to the one-particle operators e, e' belonging to some group of continuous automorphisms. This implies that E, E' have the structures

$$E = \sum_{\alpha,\beta} e_{\alpha\beta}(a_\alpha^+ a_\beta + \mathcal{C}_{\alpha\beta}),$$

$$E' = \sum_{\alpha,\beta} e'_{\alpha\beta}(a_\alpha^+ a_\beta + \mathcal{C}'_{\alpha\beta}),$$

with suitable constants $\mathcal{C}_{\alpha\beta}$, $\mathcal{C}'_{\alpha\beta}$. Then we have

$$[E, E'] = \sum_{\alpha,\beta} \sum_{\alpha',\beta'} e_{\alpha\beta} e'_{\alpha'\beta'} [a_\alpha^+ a_\beta, a_{\alpha'}^+ a_{\beta'}]$$

$$= \sum_{\alpha,\beta} [e, e']_{\alpha\beta} a_\alpha^+ a_\beta. \qquad (3.5)$$

It is important to note that on the right-hand side of (3.5) there are no constant terms. Hence the correspondence between e and E expressed by (3.5) may now be further restricted by stating that if $e = [e', e'']$, then $\mathcal{C}_{\alpha\beta} = 0$ in the expression (3.5) for E. Hence for such E we have

$$E = \sum_{\alpha,\beta} e_{\alpha\beta} a_\alpha^+ a_\beta. \qquad (3.6)$$

For the case of the Lorentz group, the ten generators h, $\mathbf{p}$, $\mathbf{j}$, $\mathbf{k}$ have this property; it follows that for any of these operators, the corresponding generator associated with the oscillator ring is given by expressions of the form (3.6). Hence the necessary and sufficient conditions for the Lorentz invariance of the theory, i.e., for the existence of a unitary family $U(a, \Lambda)$ associated with the oscillator ring, is that the ten quantities H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ defined by the equations

$$H = \sum_{\alpha,\beta} h_{\alpha\beta} a_\alpha^+ a_\beta,$$

$$\mathbf{P} = \sum_{\alpha,\beta} \mathbf{p}_{\alpha\beta} a_\alpha^+ a_\beta,$$

$$\mathbf{J} = \sum_{\alpha,\beta} \mathbf{j}_{\alpha\beta} a_\alpha^+ a_\beta,$$

$$\mathbf{K} = \sum_{\alpha,\beta} \mathbf{k}_{\alpha\beta} a_\alpha^+ a_\beta, \qquad (3.7)$$

all exist.

We now observe that only in the standard representation of the (finite-mass m, spin-0) field can the Hamiltonian exist, since

$$H \geq \sum_{\alpha,\beta} m\delta_{\alpha\beta} a_\alpha^+ a_\beta = m \sum a_\alpha^+ a_\alpha; \qquad (3.8)$$

but the right-hand side is infinite (i.e. does not exist) for any representation except the standard one.[11] It thus follows that, in the case of the relativistic (finite-mass, spin-0) field, none except the standard (Fock) representation is Lorentz-covariant. With unessential technical modifications, the proof can be adapted to any finite-mass free field.

IV. SHALE–SEGAL STATES AND REDUCIBLE COVARIANT REPRESENTATIONS

The Wightman formulation[12] of the free field is well known. It suffices here to say that in this formulation the existence of the ten generators of the Lorentz group and the existence of an invariant state (vacuum) are postulated. The analysis in the previous sections asserts that the nonstandard representations of the free field do not fall within the Wightman framework.

The relevant point here is that, by virtue of a familiar construction,[12,13] if we can define a linear functional over the field ring which is left invariant under the automorphism of the oscillator ring, the theory furnishes a unitary representation of the group of automorphisms. We outline the proof of this assertion.

Let Ω be a linear functional over the oscillator ring which is invariant under the Lorentz automorphisms $A \to \Omega(A)$, and the collection of linear functionals Ω_B defined by $A \to \Omega_B(A) = \Omega(BA)$, for any operators A, B. Then we can define a Hilbert space with a standard state ω and operators $\mathcal{O}_B$ associated with the elements B of the algebra defined by

$$\Omega \to \omega; \qquad B \to \mathcal{O}_B; \qquad \Omega_B \to \omega_B,$$

with the representation

$$\mathcal{O}_B \omega_A = \omega_{BA}$$

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[11] For readable accounts, see A. S. Wightman and S. S. Schweber, Phys. Rev. 98, 812 (1955); R. Haag, *Lectures on Theoretical Physics* edited by W. E. Brittin, B. W. Downs and J. Downs (Interscience Publishers, Inc., New York, 1961), Vol. III.

[12] A. S. Wightman, Phys. Rev. 101, 860 (1956).

[13] I. E. Segal, Bull. Am. Math. Soc. 53, 73 (1947); I. Gelfand and M. Naimark, Mat. Sbornik 54, 197 (1943) (in Russian); M. A. Naimark, *Normed Rings*, translated by L. F. Boron, (P. Noordhoff Ltd., Groningen, The Netherlands, 1959). See also R. Haag and B. Schroer, J. Math. Phys. 3, 248 (1962).

for the operators $\mathcal{O}_B$. The scalar product is defined as

$$(\omega_B, \omega_A) = \Omega(B^+A).$$

Let us now consider the automorphism of the oscillator ring associated with $R(a, \Lambda)$. Let $A \to A'$, etc. under this automorphism. Then $\omega_A \to \omega_{A'}$ etc.; but the scalar product becomes

$$(\omega_B, \omega_A) \to (\omega_{B'}, \omega_{A'}) = \Omega(B'^+A')$$
$$= \Omega(B^+A) = (\omega_B, \omega_A),$$

so that there exists the true *unitary* family $U(a, \Lambda)$ in the Hilbert space which yields

$$U(a, \Lambda)\omega_A = \omega_{A'}.$$

It satisfies, in particular, the property of leaving the standard state invariant:

$$U(a, \Lambda)\omega = \omega.$$

The representation is a true representation, and by Stone's theorem[10] there exists generators for every one-parameter family. In particular, the ten generators of the Lorentz group all exist. However, *there is no assurance that $U(a, \Lambda)$ belongs to the oscillator ring.* Nor is it guaranteed that the oscillator ring has an irreducible (a factor) representation. But it is true that the "vacuum state" ω is cyclic with respect to the oscillator ring.

If $U(a, \Lambda)$ did belong to the oscillator ring, and if the vacuum is cyclic, then all irreducible (factor) representations into which the given representation may be decomposed have an invariant vacuum state, and the unitary family $U(a, \Lambda)$ simultaneously decomposes.[9] Then, by virtue of the results above, it would follow that all the irreducible (factor) representations must be the standard (Fock) representation of the oscillator ring.

Two remarkable results concerning linear functionals invariant under automorphisms have been presented by Segal[5]; we state the results here without proof (and suitably paraphrased):

Theorem (Shale): There exists an infinite one-parameter family of invariant linear functionals on the oscillator ring and associated inequivalent representations of the oscillator ring.

Theorem (Segal): Any universally invariant linear functional is a convex integral of these fundamental linear functionals. In every one of these inequivalent representations, except the standard Fock representation, the generator, associated with a one-parameter automorphism corresponding to a positive-definite one-particle generator, has a partially negative spectrum. In particular, the Hamiltonian is not positive definite.

In the Shale theorem, the universal invariance refers to an *arbitrary* linear automorphism corresponding to an arbitrary unitary transformation in the space of one-particle wavefunctions. If we restrict ourselves to the linear automorphisms corresponding to the Lorentz group, it may be necessary to weaken the theorem by omitting the second part of the theorem. Shale fundamental linear functionals are defined as follows: Let A be any operator associated with a finite subset of oscillator variables

$$\{a_{\beta_1}, \cdots, a_{\beta_N}; a_{\beta_1}^+, \cdots, a_{\beta_N}^+\}.$$

Let $D(n; \beta_1, \cdots, \beta_N)$ be the projection operator associated with the operator $\sum_{i=1}^N a_{\beta_i}^+ a_{\beta_i}$ corresponding to the eigenvalue n. Then consider the linear functional that assigns the numerical value

$$E_C(A) = (1 - C)^N \sum_{n=0}^{\infty} C^n \operatorname{tr} \{AD(n; \beta_1, \cdots, \beta_N)\}, \tag{4.1}$$

where tr corresponds to the trace relative to the finite subset of oscillators and $0 \leq C < 1$. The universal invariance of the linear functional is apparent since the projection operator $D(n; \beta_1, \cdots, \beta_N)$, as well as the operation of relative trace, are invariant under arbitrary linear automorphisms of the finite set of oscillator variables, corresponding to a finite-dimensional unitary transformation.

Observing that the Shale linear functionals are invariant under the Lorentz automorphisms, making use of the linear functional construction, we have a unitary representation of the Lorentz group on a Hilbert space and an invariant state. If the unitary family $U(a, \Lambda)$ belonged to the oscillator ring, the known result on the reduction of representations of fields with an invariant vacuum state then assert that the representation is reducible,[3] and further, that in each of the reduced representations there exists an invariant state.[9] But since $E_C(1) = 1$,

$$E_C(\sum_\alpha a_\alpha^+ a_\alpha) = \lim_{N\to\infty} (1 - C)^N \times \sum_{n=0}^{\infty} C^n \cdot n \cdot \operatorname{tr} \{D(n; \beta_1, \cdots, \beta_N)\}$$
$$= \lim_{N\to\infty} (1 - C)^N \sum_{n=0}^{\infty} \frac{nC^n(N + n - 1)!}{(N - 1)!n!} = \infty, \tag{4.2}$$

it follows that not all of them can be standard representations. *Hence the unitary family $U(a, \Lambda)$ does not belong to the oscillator ring.*

These results point out that the Shale states furnish a new class of representations of an operator algebra by linear operators, and its automorphisms by unitary operators, in terms of direct integrals of representations of the algebra which do not in general, furnish a representation of the automorphisms. The unitary operators representing the automorphisms do not leave the component representations of the operator algebra invariant, but in fact intertwine these component representations. We are then led to conjecture that the Shale states are not pure states even though they are extremal elements of the convex set of universally invariant states. To verify this conjecture, let us restrict ourselves to a finite subset of oscillator variables; then the density matrix representing the state is

$$\rho_N = (1 - C)^N \sum_{n=0}^{\infty} C^n D(n; \beta_1, \cdots, \beta_N), \tag{4.3}$$

so that $\operatorname{tr}\{\rho_N\} = 1$, but

$$\begin{aligned}\operatorname{tr}\{\rho_N^2\} &= (1 - C)^{2N} \sum_{n=0}^{\infty} C^{2n} \operatorname{tr}\{D(n; \beta_1, \cdots, \beta_N)\} \\ &= (1 - C)^{2N}(1 - C^2)^{-N} \\ &= [(1 - C)/(1 + C)]^N \neq 1 = \operatorname{tr}\{\rho_N\},\end{aligned}$$

for any value of N.

V. DISCUSSION

We thus find that the various aspects of relativistic invariance of quantized field theories imply different things, and to a large extent, these requirements are independent. We may have a local automorphism $\phi(x) \to \phi(\Lambda x + a)$ but no unitary operator $U(a, \Lambda)$ in a particular representation of the field operators; it then means that it is meaningless to talk about an energy-momentum operator and the spectral conditions. This comes about since in an irreducible representation of a set of operators, it is not automatic that groups of (linear) automorphisms of the operators get represented; in general such an automorphism generates an inequivalent representation of the operator algebra. Since automorphisms of a Hamiltonian dynamical system are called canonical transformations, we see that not all canonical transformations are unitary transformations. It is curious to observe that in the logical structure of dynamics, the primitive dynamical attributes of energy, momentum, and angular momentum are associated with automorphisms (canonical transformations) of the dynamical variables, rather than directly with functions of the dynamical variables themselves.

On the other hand, the existence of a unitary family $U(a, \Lambda)$ representing Lorentz transformations does not imply the *local* manifestly covariant transformation:

$$\phi(x) \to U(a, \Lambda)\phi(x)U^{-1}(a, \Lambda) \neq \phi(\Lambda x + a). \tag{5.1}$$

Such a theory may be constructed as follows. Choose the standard (Fock) representation of the free field. Then we can explicitly construct the projection operator to a two-particle state, following a construction of von Neumann.[14] We have

$$\mathcal{P}(\alpha, \beta) = \tfrac{1}{2} a_\alpha^+ a_\beta^+ \mathcal{P}(O) a_\alpha a_\beta. \tag{5.2}$$

Here $\mathcal{P}(O)$ is the vacuum state projection operator:

$$\begin{aligned}\mathcal{P}(O) = \prod_{\alpha=1}^{\infty} (2\pi)^{-1} \int_{-\infty}^{\infty} dx_\alpha \int_{-\infty}^{\infty} dy_\alpha \\ \times \exp[(x_\alpha + iy_\alpha)/\sqrt{2}a_\alpha] \\ \times \exp[(x_\alpha - iy_\alpha)/\sqrt{2}a_\alpha^+],\end{aligned} \tag{5.3}$$

which does not vanish by definition of the standard representation. Let $V(\alpha_1\,\beta_1;\,\alpha_2\beta_2)$ be the Möller matrix for a relativistic interacting two-particle system. Such unitary Möller matrices exist.[15] Now construct the field operator:

$$\begin{aligned}\psi(x) = \{1 - \sum_{\alpha_1,\beta_1} \mathcal{P}(\alpha_1, \beta_1)\}\phi(x)\{1 - \sum_{\alpha_2,\beta_2} \mathcal{P}(\alpha_2, \beta_2)\} \\ + \tfrac{1}{2} \sum_{\alpha_1,\beta_1} \sum_{\alpha_2,\beta_2} V^*(\alpha_1\beta_1;\,\alpha_2\beta_2)\mathcal{P}(\alpha_2, \beta_2)\phi(x) \\ + \tfrac{1}{2} \sum_{\alpha_1,\beta_1} \sum_{\alpha_2,\beta_2} V(\alpha_1\beta_1;\,\alpha_2\beta_2)\phi(x)\mathcal{P}(\alpha_2, \beta_2).\end{aligned} \tag{5.4}$$

This field operator is unitarily equivalent to the field operator $\phi(x)$, since the transformation $\phi(x) \to \psi(x)$ is equivalent to the unitary transformation in the Hilbert space of the field operator in which the "two-particle" states undergo the unitary transformation by the Möller matrix $V(\alpha_1\beta_1;\,\alpha_2\beta_2)$. The resulting theory leads to nontrivial scattering in the two-particle channel, and only in that channel; it is hence highly artificial. On the other hand, the transformation of $\psi(x)$ when $\phi(x)$ transforms by $U(a, \Lambda)$, is by the family

$$\begin{aligned}U'(a, \Lambda) = U(a, \Lambda)\{1 - \sum_{\alpha,\beta} \mathcal{P}(\alpha, \beta)\} \\ + \frac{1}{4} \sum_{\alpha_1,\beta_1} \sum_{\alpha_2,\beta_2} \sum_{\alpha_1',\beta_1'} \sum_{\alpha_2',\beta_2'} V(\alpha_1'\beta_1';\,\alpha_2'\beta_2') \\ \times \mathcal{P}(\alpha_2', \beta_2')U(a, \Lambda)\mathcal{P}(\alpha_1, \beta_1)V(\alpha_1\beta_1;\,\alpha_2\beta_2).\end{aligned} \tag{5.5}$$

These transformations are nonlocal, but are never-

[14] J. von Neumann, Math. Ann. 104, 570 (1931). See also A. S. Wightman and S. S. Schweber, reference 8.

[15] T. F. Jordan, A. Macfarlane, and E. C. G. Sudarshan, "A Hamiltonian Model of Lorentz Invariant Particle Interactions," (to be published).

theless unitary, and are obtained by a unitary transformation on the family $U(a, \Lambda)$. Consequently, the family $U'(a, \Lambda)$ is a unitary representation of the Lorentz transformations and is in accord with the spectral conditions. However, since the transformations under Lorentz transformations are *nonlocal*, the elegant *analyticity* properties of the Wightman functions do not obtain for the "interacting" field.

We have already seen that the existence of local unitary automorphisms of the field operator does not imply the existence of a vacuum state (invariant linear functional) even if the spectrum conditions are satisfied. On the other hand, from the Segal theorem, we see that a unitary family of local automorphisms does not imply the spectrum conditions even if a vacuum state (invariant linear functional) exists.

The lack of Lorentz covariance of the nonstandard representations of the free relativistic field implies that the so-called "thermodynamic limit" of the free field (in which the particle density is finite over all space) is not Lorentz-covariant.

Instead of the relativistic field and covariance under the Lorentz group, we could consider other dynamical systems and other groups of automorphisms. One familiar example of this type is a spin assembly with a ferromagnetic Hamiltonian, i.e., an infinite number of localized "spins" (constituting a spatial lattice and with mutual interactions favoring parallel alignment of spins). There then exist states of infinite spin ("ferromagnetic states") with the resultant spin of the ferromagnet oriented along an arbitrary axis in space ("along the direction of the trace magnetic field"). On the other hand, since the (interacting) Hamiltonian of the spin assembly is rotationally invariant, no direction is preferred over any other; it is usually stated that the ground state must be infinitely degenerate since every one of these states with the "infinite" spin has the same energy. In the light of the results stated before, it is clear that to refer to this phenomenon as "degeneracy of the ground state" is misleading since each one of these states of infinite spin corresponds to a different representation; degeneracy refers to states in the same irreducible representation of the dynamical system. It also follows that while there is a rotation automorphism of the spin algebra (which leaves the ferromagnet Hamiltonian unchanged), the "infinite spin" states do not furnish a unitary representation of these automorphisms. In other words the "ferromagnet" is not rotation-covariant, and its angular momentum is undefined. We can, however, construct the Shale–Segal representations for the spin assembly in which ferromagnetic states are included; but in these representations the spin assembly is not irreducibly represented. In actual physical situations, one does not consider infinite spin assemblies, and it is clear that only a countable number of inequivalent representations exist (corresponding to all values of total spins up to a maximum finite spin, and these with suitable multiplicities). However, the restriction of the inequivalent ferromagnetic states to a finite number of spins forms a convenient starting point for a perturbation theory which may be useful below the Curie temperature. It is perhaps important to note that *the existence of the various representations is purely kinematic* (i.e., depending only on the operator structure of the dynamical system), and not on its dynamics (Hamiltonian); the dynamics merely help "stabilize" the states and make them occur in physically interesting applications.

It is tempting to believe that the considerations outlined here apply to the structure of the representations of interacting fields. For the trivial nonfree system of theories involving Wightman polynomials[9] (since normal ordering is still defined!), these considerations certainly apply. But no result of this kind is known for any genuine interacting theories. (Nor does one know if there are genuine interacting field theories!). For the time being, the relevance of these considerations to interacting fields must remain a hope.

APPENDIX A. REPRESENTATIONS OF THE FIELD RING

The question of the representation (see reference 9) of the field operator $\phi(x)$ is the same as the representation of the ring a_α, a_α^+. The most familiar representation of the field ring is the "Fock" representation furnished by all sequences of nonnegative integers $\{n_i\}$, with $\sum_{i=1}^{\infty} n_i < \infty$ considered as basis vectors of a Hilbert space so that

$$(\{n_i^{(1)}\}, \{n_i^{(2)}\}) = \prod_{i=1}^{\infty} \delta(n_i^{(1)}, n_i^{(2)}).$$

The oscillator variables a_α, a_α^+ have the representation

$$a_\alpha\{n_i\} = n_\alpha^{\frac{1}{2}}\{n_i - \delta_{\alpha i}\};$$

$$a_\alpha^+\{n_i\} = (n_\alpha + 1)^{\frac{1}{2}}\{n_i + \delta_{\alpha i}\},$$

so that the n_i can be thought of as occupation numbers, and a_α, a_α^+ as annihilation and creation operators. We shall refer to this representation as the standard representation (or the Fock representation).

Inequivalent representations of indescribable multiplicity exist. An uncountable number of such representations is obtained by the above construction, but by relaxing the requirement that $\sum_{i=1}^{\infty} n_i$ be finite. We note that if $\{n_i^{(1)}\}$ and $\{n_i^{(2)}\}$ belong to the same representation, then $\sum_{i=1}^{\infty} |n_i^{(1)} - n_i^{(2)}| < \infty$. We may define equivalence of two sequences of nonnegative integers by requiring that $\sum_{i=1}^{\infty} |n_i^{(1)} - n_i^{(2)}|$ be finite; this is then reflexive, symmetric, and transitive, and consequently defines uncountably many equivalence classes of sequences of nonnegative integers. Each sequence defines a representation of the oscillator ring, and there are uncountably many inequivalent representations. All these representations are called "discrete."

A multitude of inequivalent representations of the oscillator ring can be obtained as follows: Consider the transformation

$$a_\alpha \to b_\alpha = \sum_\beta \{V(\alpha, \beta) a_\beta + W(\alpha, \beta) a_\beta^+\},$$

with

$$\sum_\beta \{V(\alpha, \beta) V^*(\alpha', \beta) - W(\alpha, \beta) W^*(\alpha', \beta)\} = \delta_{(\alpha, \alpha')}.$$

Then b_α, b_α^+ are also oscillator variables:

$$[b_\alpha, b_\beta^+] = \delta_{\alpha\beta}; \qquad [b_\alpha, b_\beta] = 0.$$

We can now construct the (uncountably many, inequivalent) discrete representations with respect to the oscillator variables. For almost all transformations V, W, these representations are inequivalent among themselves and in relation to the discrete representations with respect to the primitive variables. The simplest class of such representations is obtained by considering an infinite set $\{\nu_k^r\}$ of infinite subsets of the indices j, so that r and k run over an infinite set of values, and taking

$$V(\nu_k^r, \nu_l^s) = \delta_{rs} \delta_{kl} \cosh \theta(r),$$

$$W(\nu_k^r, \nu_l^s) = \delta_{rs} \delta_{kl} \sinh \theta(r).$$

There are uncountably many such choices of parameters, and the discrete states with respect to these oscillator variables are inequivalent.

A third class of representations is discussed in connection with the results of Segal and Shale in Sec. 5.

APPENDIX B. REPRESENTATION OF GUAGE TRANSFORMATIONS

There are cases in which a one-parameter group of automorphisms admit a unitary representation for one of the nonstandard representations of the oscillator ring. Consider the case of a *charged* (scalar) field $\psi(x)$ and the one-parameter group of gauge transformations of the first kind:

$$\psi(x) \to e^{i\lambda} \psi(x).$$

This is an automorphism since it leaves the commutation relations

$$[\psi(x), \psi^*(x)] = i\Delta(x - x')$$

invariant. Since we have a charged field, $\psi(x)$ is no longer Hermitian, and the expansion in terms of one-particle function introduces two sets of oscillators a_α, a_α^+; b_α, b_α^+. The automorphism on the oscillator ring is

$$a_\alpha \to e^{i\lambda} a_\alpha, \qquad b_\alpha \to e^{-i\lambda} b_\alpha,$$

$$a_\alpha^+ \to e^{-i\lambda} a_\alpha^+, \qquad b_\alpha^+ \to e^{i\lambda} b_\alpha^+.$$

For α running over a finite index set, this automorphism is generated by the unitary operator

$$V(\lambda) = \exp\{i\lambda \sum_\alpha (b_\alpha^+ b_\alpha - a_\alpha^+ a_\alpha)\}.$$

If the (unbounded) hermitian operator

$$Q = \sum_{\alpha=1}^{\infty} (a_\alpha^+ a_\alpha - b_\alpha^+ b_\alpha)$$

exists, then the unitary operator

$$U(\lambda) = \exp(-i\lambda Q)$$

also exists, and generates the automorphism. In other words, we have a realization of the gauge transformations of the first kind provided Q defined by (3.10) exists. However this is not a *necessary* condition since it is sufficient if the operator

$$Q' = \sum_{\alpha=1}^{\infty} (a_\alpha^+ a_\alpha - b_\alpha^+ b_\alpha + \mathfrak{C}_\alpha)$$

exists for suitable choice of the constants $\mathfrak{C}_\alpha$. Consequently, every one of the *discrete* representations of the oscillator ring is gauge-covariant. The question of the gauge covariance of the continuous representations is more complicated; in general they are not gauge-covariant. It is not known whether there are any gauge-covariant continuous representations of the oscillator ring.

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