Generalized uncertainty relations and characteristic invariants for the multimode states

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(Received 9 November 1994; revised manuscript received 17 March 1995)

The close relationship between the zero-point energy, the uncertainty relation, coherent states, squeezed states, and correlated states for one mode is investigated. This group theoretic perspective of the problem enables the parametrization and identification of their multimode generalization. A simple and efficient method of determining the canonical structure of the generalized correlated states is presented. Implication of canonical commutation relations for correlations are not exhausted by the Heisenberg uncertainty relation, not even by the Schrödinger-Robertson uncertainty inequality, but there are relations in the multimode case that are the generalization of the Schrödinger-Robertson relation.

PACS number(s): 03.65.Bz

I. INTRODUCTION

In a previous contribution one of us [1] has demonstrated the close relationship between the zero-point energy, the uncertainty relations, the coherent states, the squeezed states, and the correlated states for a single mode. The group-theoretic significance of the states that have minimum Schrödinger uncertainty under canonical transformations was discussed and the application of the same approach to multimode correlated states was also indicated. In this paper we give a detailed demonstration of the group-theoretic approach to the problem, which enables the parametrization and identification of the multimode generalization. We develop an efficient and simple method of determining the canonical structure of the generalized correlated states. We also show that the implication of the canonical commutation relation for correlated states leads to inequalities that are generalizations of the Schrödinger-Robertson inequality.

Much work has been done on the Gaussian wave functions and the minimum-uncertainty states. Several papers dealing with the Gaussian wave function and their relationship with minimum-uncertainty relations, which appeared about a decade ago, include those of Sudarshan and co-workers [2,3], Milburn [4], and Schumaker [5]. The present work goes considerably beyond these papers. In particular we deal with general (mixed) states and the full $\mathrm{Sp}(2n,R)$ transformation. Naturally, instead of the Heisenberg uncertainty relation, we generalize the Schrödinger-Robertson uncertainty relations.

The plan of the paper is as follows. In this section we bring out some of the earlier results on the relationship between the zero-point energy, the Heisenberg uncertainty relation (HUR), the Schrödinger-Robertson inequality (SRI), the coherent, squeezed, and correlated states, and the effects of canonical transformations. In Sec. II we illustrate the application of the group-theoretic methods with some examples and prove a theorem on the minimum-uncertainty state. In Sec. III we develop in detail the group-theoretic approach to handle the two-mode case in such a manner as to enable us to generalize it to the multimode case. In Sec. IV we demonstrate a simple method to relate the general correlation matrix with the standard form of the correlation matrix of the minimum-uncertainty state, through canonical transformations and group-theoretic methods, for the two-mode case. We find that the method developed is "efficient" in the sense that it uses the minimum number of parameters. In Sec. V we extend the above considerations to the multimode correlated states. Once again we find that the route followed is an efficient one. Finally, we discuss the generalized uncertainty relations or inequalities that result in the multimode case.

A. Planck, Heisenberg, and Schrödinger-Robertson inequalities and canonical commutation relations

Let p and q be two canonical operators satisfying the commutation relations

$$[q,p]=i \quad (\hbar=1) \ . \tag{1.1}$$

For every ω, $0<\omega<\infty$, non-negativity of the square of Hermitian operator implies that

$$E(\omega)=(\omega q-ip)(\omega q+ip)\geq 0 \ . \tag{1.2}$$

So for any state $|\Psi\rangle$, the corresponding expectation value satisfies the relation

$$\omega^2\langle q^2\rangle+\langle p^2\rangle+i\omega\langle qp-pq\rangle=\omega^2\langle q^2\rangle-\omega+\langle p^2\rangle\geq 0 \ . \tag{1.3}$$

Hence the discriminant of this quadratic form should be negative or zero, that is,

$$4\langle q^2\rangle\langle p^2\rangle\geq 1 \ . \tag{1.4}$$

Noting that the deviations from the mean $Q=q-\langle q\rangle$, $P=p-\langle p\rangle$ also satisfy the canonical commutation relations, the inequality of (1.2) should continue to hold with the replacement of q by Q and p by P. We derive therefore

$$\langle(\Delta q)^2\rangle\langle(\Delta p)^2\rangle\equiv\langle Q^2\rangle\langle P^2\rangle\geq\tfrac{1}{4} \ , \tag{1.5}$$

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which is the Heisenberg uncertainty relation [6].

Next we can obtain the Planck [7] inequality and the zero-point energy by starting from the inequality (1.2). Since energy is given by $\frac{1}{2}(p^2+\omega^2 q^2)$, we have

$$E=\frac{1}{2}(p^2+\omega^2q^2)=\omega\left\{\frac{\omega q-ip}{\sqrt{2\omega}}\frac{\omega q+ip}{\sqrt{2\omega}}\right\}+\frac{\omega}{2}\geq\frac{\omega}{2}\,, \tag{1.6}$$

$$E\geq\frac{\omega}{2}\,. \tag{1.6'}$$

Since the first term is non-negative in (1.6), the energy has a nonzero minimum value, the so-called zero-point energy $\omega/2$ for the ground state $|\Psi_0\rangle$, which is annihilated by the operator a, given by

$$a=(\omega q+ip)/\sqrt{2\omega},\quad a|\Psi_0\rangle=0\,. \tag{1.7}$$

Therefore we may say that while the Planck energy relation (1.6) is not invariant under the linear canonical transformations

$$q\to Q=q-\langle q\rangle,\quad p\to P=p-\langle p\rangle\,, \tag{1.8}$$

or under

$$q\to\sqrt{\omega}q,\quad p\to p/\sqrt{\omega}\,, \tag{1.9}$$

these canonical transformations on the Planck energy inequality lead to the generic form of the HUR (1.5). However, there are other canonical transformations that leave the Planck energy relation invariant but not the HUR. One such transformation is given by

$$q\to q\cos\theta-\omega^{-1}p\sin\theta\,, \tag{1.10}$$

$$p\to\omega q\sin\theta+p\cos\theta\,. \tag{1.11}$$

Using (1.10) and (1.11) the relation (1.4) becomes

$$\{\omega^2\langle q^2\rangle+\langle p^2\rangle\}^2-\{(\omega^2\langle q^2\rangle-\omega\langle p^2\rangle)\cos2\theta-\omega\langle qp+pq\rangle\sin2\theta\}^2\geq\omega^2\,. \tag{1.12}$$

The minimum of the left-hand side of (1.12) can be shown to be

$$\langle q^2\rangle\langle p^2\rangle-\frac{\langle qp+pq\rangle^2}{4}\geq\frac{1}{4}\,, \tag{1.13}$$

which occurs at

$$\tan2\theta=-\omega\langle qp+pq\rangle/\{\omega^2\langle q^2\rangle-\langle p^2\rangle\}\,. \tag{1.14}$$

The inequality (1.13) is the generic form of the Schrödinger-Robertson inequality [8]. If we replace q and p by Q and P we obtain the relation first obtained by Schrödinger [8] and by Robertson [8]. It is stronger than the HUR and reduces to it in the special case of "uncorrelated states," for which

$$\langle(q-\langle q\rangle)(p-\langle p\rangle)+(p-\langle p\rangle)(q-\langle q\rangle)\rangle=0 \tag{1.15}$$

or, equivalently,

$$\langle qp+pq\rangle=\langle q\rangle\langle p\rangle+\langle p\rangle\langle q\rangle\,. \tag{1.16}$$

Even for a harmonic oscillator the condition (1.16) does not obtain in general and a Heisenberg "minimum-uncertainty state" is not canonically invariant. This has been known for the harmonic oscillator for decades and the general systematics of such a derivation has been given by Dodunov and Man'ko [9].

B. SRI and canonical transformations

The clue to Schrödinger-Robertson generalization of the HUR is the requirement of invariance under the group of linear canonical transformations. The SRI is the most general relation that is canonically invariant. States that satisfy the SRI are called correlated states and the states that satisfy the HUR are squeezed states. A subclass of the squeezed states are coherent states, which satisfy $\omega^2(\Delta q^2)=\omega^2\langle q^2-\langle q\rangle^2\rangle=\omega^2(\Delta q^2)=\langle p^2-\langle p\rangle^2\rangle$ with energy minimum.

The state of minimum energy for the harmonic oscillator with Hamiltonian ($\omega=1$)

$$H=\tfrac{1}{2}(p^2+q^2)=(a^\dagger a+\tfrac{1}{2}) \tag{1.17}$$

is the vacuum state $|\Psi_0\rangle$ satisfying

$$a|\Psi_0\rangle=0\,, \tag{1.18}$$

with the associated wave function

$$\psi_0(x)=(\pi)^{-1/4}\exp(-x^2/2)\,. \tag{1.19}$$

This is a state of minimum uncertainty

$$\langle\Delta p^2\rangle=\langle\Delta q^2\rangle=\tfrac{1}{2}\,. \tag{1.20}$$

But the minimum-uncertainty class is wider, and among these are the states with

$$a|z\rangle=z|z\rangle\,, \tag{1.21}$$

where z is a complex number, with the wave function

$$\psi(x,z)=(\pi)^{-1/4}\exp\{-(x-\sqrt{2}z)^2/2\}\,. \tag{1.22}$$

These are the coherent states introduced by Schrödinger [10] and rediscovered decades later by Glauber [11] and by Sudarshan [12], in the context of quantum optics. They constitute an overcomplete family of states in terms of which every state can be expressed in infinitely many ways; further, in terms of them every density matrix can be exhibited as a sum of projectors $|z\rangle\langle z|$ to the coherent states with distribution valued weight [12,13]. But coherent states are not canonically invariant. For example, the scale transformation ("squeezing") $q\to\sqrt{\omega}q$ and $p\to p/\sqrt{\omega}$ takes coherent states into a class of states [14] that are now called squeezed states. In terms of the operators a and $a^\dagger$ these are Bogoliubov-Valatin transformations [15]. The unitary transformation

$$V=\exp\{-i\omega^{1/2}(qp+pq)/2\} \tag{1.23}$$

accomplishes the squeezing and thus obtained are the one-parameter family of overcomplete sets of squeezed coherent states with the wave functions

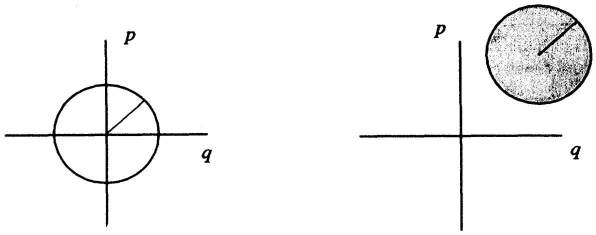

FIG. 1. Planck diagrams of the minimum-energy state and the coherent states. The coherent states are centered at the point $[(z+z^*)/\sqrt{2}, (z-z^*)/i\sqrt{2}]$.

$$\psi(x)=(\pi)^{-1/4}\exp\{-\omega(x-\sqrt{2}z)^2/2\} \ . \tag{1.24}$$

These are labeled by three parameters ω, Rez, and Imz. For each ω we have an overcomplete family of states.

This is still not general enough. There are more canonical transformations that will leave the state no longer a minimum-uncertainty state in the Heisenberg sense but that would be a minimum-uncertainty state in the Schrödinger sense. These are the correlated states whose wave functions have been obtained by Dodunov, Kurmyshev, and Man'ko [16]. A simpler form of this is a complex Gaussian

$$\psi(x)=(\pi)^{-1/4}\exp[-\tfrac{1}{2}(\alpha x^2-2\beta x+\gamma)] \ , \tag{1.25}$$

where α,β,γ are complex parameters satisfying $(\beta+\beta^*)^2/(\alpha+\alpha^*)=\gamma+\gamma^*$.

The imaginary part of γ is arbitrary. Therefore these contain two complex parameters $\alpha_1+i\alpha_2$ and $\beta_1+i\beta_2$ with

$$\begin{aligned}(\Delta q)^2&=\langle q^2\rangle-\langle q\rangle^2=\frac{1}{2\alpha_1} \ ,\\ (\Delta p)^2&=\langle p^2\rangle-\langle p\rangle^2=\frac{\alpha_1}{2}+\frac{\alpha_2^2}{2\alpha_1} \ ,\\ \langle qp+pq\rangle-\langle q\rangle\langle p\rangle-\langle p\rangle\langle q\rangle&=-\frac{\alpha_2}{\alpha_1} \ .\end{aligned} \tag{1.26}$$

It can now be shown that the inhomogeneous canonical transformations leave the SRI invariant but not the HUR. If we set $\alpha_1=1$, $\alpha_2=0$, and β_1 and β_2 arbitrary, the coherent states are obtained, which are labeled by $\tilde{\beta}$, where $\tilde{\beta}=\beta\sqrt{2}$. Then $a|\tilde{\beta}\rangle=\tilde{\beta}|\tilde{\beta}\rangle$, $\langle x\rangle=\sqrt{2}\,\mathrm{Re}\tilde{\beta}$, and $\langle p\rangle=\sqrt{2}\,\mathrm{Im}\tilde{\beta}$. If $\alpha_1\neq 1$ but $\alpha_2=0$ and β arbitrary, we obtain the squeezed states. The ground state is a special case of the coherent state, i.e., with $\beta=0$. A state with α and β arbitrary is called the correlated state.

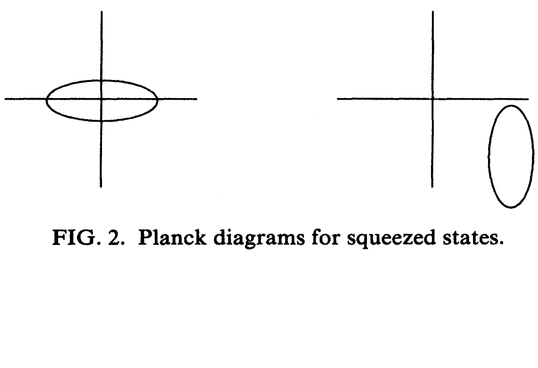

FIG. 2. Planck diagrams for squeezed states.

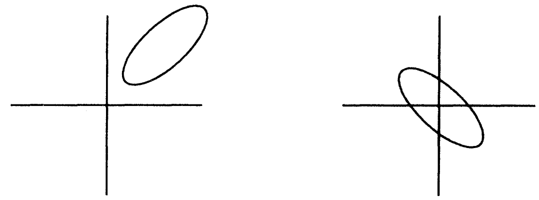

FIG. 3. Planck diagrams for correlated states.

Making use of the appealing phase space picture introduced by Planck [7,18] for the quantum oscillator, the ground state with the zero-point energy (for $\omega=1$) has a phase-space patch that is a circle with unit radius and an area π that is 2π times the zero-point energy. The mean value of $\frac{1}{2}(p^2+q^2)$ within this circular disk is $\frac{1}{2}$, which enabled Planck to derive the zero-point energy; his picture of the ground state is a circle of unit radius centered at the origin. By displacing the origin to $\sqrt{2}z$ we get the two-parameter (one complex parameter) family of coherent states (Fig. 1).

Squeezed states are obtained by area-preserving deformations of the circles into ellipses with major (minor) axis along the coordinate directions (Fig. 2). When the ellipse is tilted we get the more general family of correlated states discussed by Dodunov, Kurmyshev, and Man'ko [16,19]. Of course this tilting alters things only for the squeezed states but not for the coherent states (Fig. 3).

II. GROUP-THEORETIC APPROACH TO SCHRÖDINGER MINIMAL UNCERTAINTY STATES

A. An example: Simple harmonic oscillator

We illustrate our discussion of the preceding section with the example of simple harmonic oscillator. Defining $q=(a+a^\dagger)/\sqrt{2\omega}$ and $p=(a-a^\dagger)\sqrt{\omega}/i\sqrt{2}$, the Hamiltonian is given by

$$H=\tfrac{1}{2}(p^2+q^2)=(aa^\dagger+\tfrac{1}{2})\omega \quad \text{where } [a,a^\dagger]=1 \ . \tag{2.1}$$

The Heisenberg equation of motion for a and $a^\dagger$ leads to the standard time dependence of $\exp(+i\omega t)$ and $\exp(-i\omega t)$, respectively. This in turn leads to

$$\begin{aligned} q&=q_0\cos\omega t-\frac{p_0}{\omega}\sin\omega t \ ,\\ p&=\omega q_0\sin\omega t+p_0\cos\omega t \ ,\end{aligned} \tag{2.2}$$

where $q=q_0$ and $p=p_0$ at $t=0$. This is identical to the transformations (1.10) and (1.11), when $\theta=\omega t$.

Using (2.2), it can be shown that

$$\begin{aligned}&\langle q^2\rangle\langle p^2\rangle-\left\langle\frac{qp+pq}{2}\right\rangle^2\\ &\quad=\langle q_0^2\rangle\langle p_0^2\rangle-\left\langle\frac{q_0p_0+p_0q_0}{2}\right\rangle^2\equiv k^2\geq\frac{1}{4} \ . \end{aligned}\tag{2.3}$$

Notice that for the present example, one may think of the transformation (2.2) as a variable rotation conjugated by a fixed scaling transformation, i.e., $T(t) = S(\omega)\mathcal{R}(\theta=-\omega t)S^{-1}(\omega)$. That is the initial state with q_0 and p_0 undergoing the transformation

$$\begin{bmatrix} q_0 \\ p_0 \end{bmatrix} \to \begin{bmatrix} \sqrt{\omega} q_0 \\ \frac{p_0}{\sqrt{\omega}} \end{bmatrix} \to \begin{bmatrix} \sqrt{\omega} q \\ \frac{p}{\sqrt{\omega}} \end{bmatrix} = \begin{bmatrix} \sqrt{\omega} q_0 \cos\omega t + \frac{p^0}{\sqrt{\omega}} \sin\omega t \\ -\sqrt{\omega} q_0 \sin\omega t + \frac{p_0}{\sqrt{\omega}} \cos\omega t \end{bmatrix} . \tag{2.4}$$

Equation (2.3) reflects the fact that k^2 is invariant under the canonical transformations. A little algebra shows that the Heisenberg inequality is not invariant under this canonical transformation and as t increases, the uncertainty $\langle q^2\rangle\langle p^2\rangle$ oscillates according to

$$\langle q^2\rangle\langle p^2\rangle = v_0^2 - (v_1 \cos 2\omega t + v_2 \sin 2\omega t)^2 , \tag{2.5}$$

where

$$v_0 = \frac{1}{2}\left[\omega\langle q_0^2\rangle + \frac{1}{\omega}\langle p_0^2\rangle\right] ,$$
$$v_1 = \frac{1}{2}\left[\omega\langle q_0^2\rangle - \frac{1}{\omega}\langle q_0^2\rangle\right] , \tag{2.6}$$
$$v_2 = \frac{1}{2}\langle q_0 p_0 + p_0 q_0\rangle .$$

We see that $\langle q^2\rangle\langle p^2\rangle$ oscillates with twice the frequency ω and is bounded by the maximum $v_0^2+v_1^2+v_2^2$ and the minimum $v_0^2-v_1^2-v_2^2 \geq \frac{1}{4}$.

B. A theorem on the minimal uncertainty state

Theorem. A state with absolute minimum uncertainty is a pure state. Given

$$\langle q^2\rangle = a, \quad \langle p^2\rangle = b, \quad \left\langle \frac{qp+pq}{2}\right\rangle = c \quad \text{with } ab - c^2 = \tfrac{1}{4} , \tag{2.7}$$

there exists one and only one pure state ψ, which satisfies the relation (2.7).

Proof. Let Q and P be arbitrary operators that are related to q and p through a canonical transformation

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} . \tag{2.8}$$

For a real parameter μ, Hermiticity of $Q+i\mu P$ implies

$$(Q-i\mu P)(Q+i\mu P) = Q^2 + \mu^2 P^2 - \mu \geq 0 . \tag{2.9}$$

Evaluating the expectation value of (2.9) by using (2.7) and (2.8), we obtain

$$F(\theta,\mu) \equiv \langle (Q-i\mu P)(Q+i\mu P)\rangle = (1+\mu^2)\frac{a+b}{2} + \frac{1-\mu^2}{2}(a-b)\cos 2\theta + (\mu^2-1)c \sin 2\theta - \mu \geq 0 . \tag{2.10}$$

The minimum of $F(\theta,\mu)$ is obtained by requiring both

$$\frac{\partial F}{\partial \theta}, \quad \frac{\partial F}{\partial \mu} = 0 \tag{2.11}$$

to be satisfied simultaneously. This occurs at $\theta=\theta_1$ and $\mu=\mu_1$, with

$$\tan 2\theta_1 = -\frac{2c}{a-b} , \quad \mu_1 = \frac{1}{a+b-r} , \quad r = \sqrt{(a-b)^2+4c^2} . \tag{2.12}$$

At this point F vanishes, in other words,

$$\langle (Q_1 - i\mu P_1)(Q_1 + i\mu P_1)\rangle = F(\theta_1,\mu_1) = 0 \tag{2.13}$$

with

$$Q_1 = Q(\theta_1,\mu_1;q,p), \quad P_1 = P(\theta_1,\mu_1;q,p) . \tag{2.14}$$

Consider the state (mixed) to be represented by a density matrix ρ with the canonical decomposition

$$\rho = \sum_j c_j \psi_j \psi_j^\dagger \tag{2.15}$$

with all $c_j \geq 0$ and the ψ_j's forming an orthonormal set. Thus $\mathrm{Tr}\rho = \mathbb{1}$. Now, we have shown that

$$\langle (Q_1 - i\mu_1 P_1)(Q_1 + i\mu_1 P_1)\rangle = \mathrm{Tr}(\rho a^\dagger a) = \sum_j c_j a \psi_j \psi_j^\dagger a^\dagger = \sum_j c_j |a\psi_j|^2 = 0 . \tag{2.16}$$

Since all $c_j \geq 0$, this implies that every term of the sum must be equal to zero. If $c_1 \neq 0$ then $|a\psi_1|^2=0$, or ψ_1 is the vacuum state. Since all other ψ_j's (i.e., for $j>1$) are orthogonal to ψ_1 we have

$$\sum_{j>1} c_j |a\psi_j|^2 > 0 \tag{2.17}$$

unless all $c_j=0$ for $j>1$. Thus $\mathrm{Tr}(\rho a^\dagger a)=0$ leads to $\rho=\psi_1\psi_1^\dagger$ or that the original state, which has the zero expectation value, i.e., $F(\mu,\theta_1)=0$, is a pure state and not a mixed state.

C. Further studies

Group-theoretic significance of states with minimum Schrödinger uncertainty

The linear canonical transformations on a pair of canonical variables form a group $SL(2,R) \ltimes T(2)$, the semidirect product of the special linear group with translations. (The symbol $\ltimes$ denotes semidirect product.) The minimum-uncertainty states of Planck are invariant under the harmonic SO(2) subgroup of this group; this is its stability group. So the quotient of the canonical group and the harmonic stability group, the correlated states, is in one-to-one correspondence with the elements of the coset of dimension $5-1=4$. These states are realized by single-mode lasers and states with substantial squeezing and/or correlation have been generated and identified [17].

Given any state, represented by any density matrix ρ with the quadratic moments being given by C, there exists a whole orbit of density matrices $U(g)\rho U^\dagger(g)$ with $U(g)$ any linear canonical transformation for which the matrix $C\beta$ undergoes a similarity transformation

$$C\beta \to SC\beta S^{-1} .$$

Hence the only characteristic invariant of the antisymmetric matrix $C\beta$ is its determinant, which is the determinant of C (multiplied by $\det\beta=1$). So if ρ is any matrix the family $U\rho U^\dagger$ also has the same invariant. But the converse is not necessarily true; given C the family $U\rho U^\dagger$ is not unique. There are, in general, an infinity of mixed states for the same value of C. The only exception is the case when $\det C=\frac{1}{4}$, which corresponds to the family $U\rho_0 U^\dagger$ with $\rho_0=|0\rangle\langle 0|$ the vacuum density matrix. The perceptive reader will see that this situation obtains for the multimode correlation matrices and their invariants.

It is a natural question to ask whether these notions and correspondences can be generalized to n degrees of freedom and multimode laser beams. Group theory can be invoked to get a general answer to the problem.

D. Extension to multimode correlated states

Consider a system of N canonical pairs $\{q_r,p_r\}$, $1\le r, s\le N$. The homogeneous linear transformations are $\mathrm{Sp}(2N,R)$ and the translations are $T(2N)$. So the linear canonical group is the semidirect product $\mathrm{Sp}(2N,R)\boxminus T(2N)$ with $N(2N+1)+2N=N(2N+3)$ parameters. We seek canonical invariants bilinear in the $2N$ canonical variables and look for the appropriate conditions to get the minimal generalized Schrödinger uncertainty. We expect this to come from the ground state $|\Omega\rangle$ annihilated by all annihilation operators $(q_r+ip_r)/\sqrt{2}$ and states obtained from $|\Omega\rangle$ by the action of the linear canonical group. Since these involve individual harmonic SO(2) elements for each degree of freedom and any $O(N)$ rotation between the various degrees of freedom the stability group of $|\Omega\rangle$ has $N+[N(N-1)]/2=\frac{1}{2}N(N+1)$ parameters, we expect a family of correlated states with $\frac{1}{2}N(3N+5)$ parameters corresponding to the dimension of the coset space.

Even for small values of N this dimension grows rapidly; we adopt a more elementary method to obtain the generalized correlated states. The multimode coherent states are $2N$ parameter states obtained by $T(2N)$ acting on $|\Omega\rangle$. We describe the situation in detail for $N=2$ in Secs. III and IV. The generalization of the same approach to arbitrary N will be given in Sec. V. For the $N=2$ case, we consider the group $\mathrm{Sp}(4,R)$, which is a double covering of SO(3,2) and has the same Lie algebra of dimension ten. This algebra can be obtained by the three (p_rp_s), the three (q_rq_s), and the four $\frac{1}{2}(q_rp_s+p_sq_r)$, which close under commutation. The generic SO(3,2) algebra has two invariants, one of second order and one of fourth order. If we consider the expectation values of the ten quantities (p_rp_s), (q_r,q_s), and $\frac{1}{2}(q_rp_s+p_sq_r)$ they furnish a 4×4 symmetric non-negative matrix that is bounded below by the zero-point energy.

III. GENERALIZATIONS OF CANONICAL TRANSFORMATION: A STUDY ON THE TWO-MODE CASE

A. Generators of canonical transformation

The canonical transformation A is defined as a transformation that leaves the commutation relations invariant. Introduce the spinor

$$\xi=(q_1,p_1,q_2,p_2), \quad [\xi_a,\xi_b]=i\beta_{ab} . \tag{3.1}$$

Invariance of the commutator under A implies that

$$\begin{aligned}[\xi_a,\xi_b]&=A_{ai}A_{bj}[\xi_i,\xi_j]\\ &=iA_{ai}\beta_{ij}A^T_{jb}\\ &=i\beta_{ab} .\end{aligned} \tag{3.2}$$

So the canonical transformation must satisfy the relation

$$A\beta A^T=\beta . \tag{3.3}$$

It is convenient to establish the corresponding criteria for the generators of the transformations. Denote the metric matrix by β. As we shall discuss below, there are ten independent generators G_i, where βG_i are symmetric real matrices. They are given in Table I.

The finite canonical transformation can be represented by the exponential form

$$A=\exp\sum_{i,j} d_{ij}\sigma_i\rho_j . \tag{3.4}$$

The corresponding canonical transformation condition of (3.3) in the infinitesimal form becomes

$$A\beta A^T \to \left[1+\sum\epsilon_{ij}\sigma_i\rho_j\right]\beta\left[1+\sum\epsilon_{ij}\sigma_i\rho_j\right]^T=\beta . \tag{3.5}$$

Using $\beta=i\sigma_2 I$, the verification of (3.3) is equivalent to checking the following relations to first order in ϵ_{ij} which are valid for all relevant i's and j's:

$$E_{ij}\equiv\sigma_i\rho_j\sigma_2+\sigma_2(\sigma_i\rho_j)^T=\sigma_i\rho_j\sigma_2+\sigma_2\rho_j^T\sigma_i^T=0 . \tag{3.6}$$

TABLE I. Generators for two modes.

i/j	ρ_0	ρ_1	$i\rho_2$	ρ_3
σ_0			$i\rho_2$	
σ_1	σ_1	$\sigma_1\rho_1$		$\sigma_1\rho_3$
$i\sigma_2$	$i\sigma_2$	$i\sigma_2\rho_1$		$i\sigma_2\rho_3$
σ_3	σ_3	$\sigma_3\rho_1$		$\sigma_3\rho_3$

Consider first $j\neq 2$, where $\rho_j^T=\rho_j$. Here

$$E_{ij}=(\sigma_i\sigma_2+\sigma_2\sigma_i^T)\rho_j \ . \tag{3.7}$$

For $i\neq 0,2$, $\sigma_i^T=\sigma_i$, so

$$E_{ij}=(\sigma_i\sigma_2+\sigma_2\sigma_i)\rho_j=0 \ . \tag{3.8}$$

For $i=2$, $\sigma_2^T=-\sigma_2$, so

$$E_{2j}=(\sigma_2\sigma_2+\sigma_2\sigma_2^T)\rho_j=0 \ . \tag{3.9}$$

Notice that in Table I, for the column index $j\neq 2$ there is no entry for $i=0$. So for $j\neq 2$, the condition of (3.3) is satisfied. Now we turn to the case where $j=2$, where $\rho_2^T=-\rho_2$. According to Table I, we need to consider the $i=0$ case only. For this case,

$$E_{02}=(\sigma_0\sigma_2-\sigma_2\sigma_0)\rho_2=0 \ . \tag{3.10}$$

Again for $j=2$ the condition of (3.3) is satisfied. Therefore the ten operators listed in Table I are indeed the generators of the canonical transformations. One may also make a special choice of the 4×4 γ matrices to represent these generators. They are

$$\gamma_\mu=(i\sigma_2,\sigma_3\rho_3,-\rho_1,-\rho_1\sigma_3) \ . \tag{3.11}$$

The remaining generators in Table I are then given by

$$\tfrac{1}{2}[\gamma_0,\gamma_i]=(-\sigma_1\rho_3,-\sigma_3,\sigma_1\rho_1),\quad \tfrac{1}{2}[\gamma_1,\gamma_2]=-i\sigma_2\rho_3 \ ,$$
$$\tfrac{1}{2}[\gamma_2,\gamma_3]=-i\sigma_2\rho_2,\quad \tfrac{1}{2}[\gamma_3,\gamma_1]=-i\rho_2 \ . \tag{3.12}$$

One may also construct the ten independent symmetric bilinear expressions in ξ with

$$V_\mu=-\tfrac{1}{4}\xi^T\beta\gamma_\mu\xi,\quad S_{\mu\nu}=\tfrac{1}{8}\xi^T\beta[\gamma_\mu,\gamma_\nu]\xi \ . \tag{3.13}$$

One can verify that the vector and the tensor matrices $\beta\gamma_\mu$ and $\beta[\gamma_\mu,\gamma_\nu]$ are the ten independent symmetric real matrices. They reproduce the commutation relations of SO(3,2) [20]

$$\begin{aligned}&[S_{\mu\nu},S_{\rho\sigma}]=i(g_{\mu\rho}S_{\nu\sigma}-g_{\nu\rho}S_{\mu\sigma}+g_{\mu\sigma}S_{\rho\nu}-g_{\nu\sigma}S_{\rho\mu}) \ ,\\ &[S_{\mu\nu},V_\rho]=i(g_{\mu\rho}V_\nu-g_{\nu\rho}V_\mu) \ ,\\ &[V_\mu,V_\nu]=-iS_{\mu\nu} \ .\end{aligned} \tag{3.14}$$

B. Invariant subgroups

Let C be the canonical correlation matrix and C_0 the corresponding minimal vacuum correlation matrix given by

$$C=k\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix},\quad C_0=\frac{1}{2}\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} \ . \tag{3.15}$$

It can be shown easily that the generators that commute with $C\beta$ are the U(1) generator $i\sigma_2$ and the SU(2) generators $i\sigma_2\rho_1$, $i\sigma_2\rho_3$, and $i\rho_2$. These four generators together form the U(2) algebra. When the matrix C is specified by two parameters k_1 and k_2 as

$$C=\begin{bmatrix} k_1 & 0\\ 0 & k_2\end{bmatrix}=\frac{k_1+k_2}{2}i\sigma_2+\frac{k_1-k_2}{2}i\sigma_2\rho_3 \ ,$$

the generators that leave $C\beta$ invariant are only $i\sigma_2$ and $i\sigma_2\rho_3$, which generate only the algebra U(1)×U(1).

We now demonstrate how to construct the invariant subgroup for an arbitrary symmetric form for the correlation matrix denoted by M. First bring the matrix M to its canonical form C by

$$M=gCg^T \ . \tag{3.16}$$

Let $\mathcal{L}$ be the little group element of the C matrix defined by

$$\mathcal{L}C\mathcal{L}^T=C \ . \tag{3.17}$$

It follows that

$$\begin{aligned}M&=gCg^T\\ &=g\mathcal{L}C\mathcal{L}^Tg^T\\ &=g\mathcal{L}g^{-1}gCg^T(\mathcal{L}g^{-1})^Tg^T\\ &=(g\mathcal{L}g^{-1})M(g\mathcal{L}g^{-1})^T \ .\end{aligned} \tag{3.18}$$

Thus the invariant subgroup of M may be represented by

$$\mathcal{L}_g\equiv g\mathcal{L}g^{-1} \ . \tag{3.19}$$

For the two-mode case, with two distinct parameters k_1 and k_2 for the C matrix, the invariant subgroup element is the 4×4 matrix given by

$$\mathcal{R}(\theta_{11},\theta_{22})=\begin{bmatrix}\mathcal{R}(\theta_{11}) & \\ & \mathcal{R}(\theta_{22})\end{bmatrix} \ , \tag{3.20}$$

which belongs to the group U(1)×U(1), with separate rotations among mode-1 variables and among the mode-2 variables. On the other hand, if the C matrix is proportional to the identity matrix, $\mathcal{L}$, as seen earlier, is enlarged to the U(2) group, where there is rotation between mode 1 and mode 2 variables also.

IV. TWO-MODE CASE: REDUCTION OF THE 4×4 GENERAL MATRIX M TO THE STANDARD C MATRIX

Let the bilinear expectation values of the two-mode case be represented by the 4×4 matrix

$$M=\begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\quad\text{with } M^T=M \ . \tag{4.1}$$

The basis vector is chosen to be $\xi=(q_1,p_1,q_2,p_2)$, so that M_{11} contains mode-1 variables M_{22}, mode-2 variables, and M_{12} and M_{21}, the cross terms, which mix mode-1 and mode-2 variables. We proceed to diagonalize M through canonical transformations.

Step 1: Diagonalizing the 2×2 diagonal blocks. For a general 2×2 matrix

$$\mathcal{R}(\theta_{11},\theta_{22})=\begin{bmatrix}\mathcal{R}(\theta_{11}) & 0\\ 0 & \mathcal{R}(\theta_{22})\end{bmatrix} \tag{4.2}$$

each 2×2 diagonal block M_{11}, M_{22} can be diagonalized by the transformation

$$\mathcal{R}(\theta)\begin{bmatrix} a & b \\ b & c \end{bmatrix}\mathcal{R}^T(\theta)=\begin{bmatrix} a' & 0 \\ 0 & a'' \end{bmatrix} \tag{4.3}$$

with

$$\mathcal{R}(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \tag{4.4}$$

Then θ, a', and a'' are given by

$$\tan 2\theta=\frac{-2b}{a-d} \tag{4.5}$$

with

$$\begin{aligned} a' &= \frac{a+d}{2}+r , \\ a'' &= \frac{a+d}{2}-r , \\ r &= \left[b^2+\left[\frac{a-d}{2}\right]^2\right]^{1/2} . \end{aligned} \tag{4.6}$$

Thus matrix M of (4.1) now becomes

$$M'=\mathcal{R}(\theta_{11},\theta_{22})M\mathcal{R}^T(\theta_{11},\theta_{22})=\begin{bmatrix} \begin{matrix} a' & 0 \\ 0 & a'' \end{matrix} & M_{12} \\ M_{21} & \begin{matrix} b' & 0 \\ 0 & b'' \end{matrix} \end{bmatrix} . \tag{4.7}$$

Note that since M_{12} and M_{21} in (4.1) were not specified explicitly we have dropped the primes in (4.7) with the caveat that M_{12} and M_{21} in (4.7) differ from those in (4.1). The same caveat will also be used hereafter on all the unspecified 2×2 submatrix elements.

Step 2: Diagonalizing M_{12} through little group transformations. The canonical transformation that keeps a 2×2 matrix invariant is referred to as the little group for that matrix. Note that the correlation matrix may be shown to be related to the standard form as

$$\begin{bmatrix} a' & 0 \\ 0 & a'' \end{bmatrix}=cB(\lambda)IB^T(\lambda)=c\begin{bmatrix} e^{2\lambda} & \\ & e^{-2\lambda} \end{bmatrix} . \tag{4.8}$$

Then

$$\frac{a'}{a''}=\frac{e^{2\lambda}}{e^{-2\lambda}} \quad \text{or } e^{2\lambda}=\left[\frac{a'}{a''}\right]^{1/2} , \quad c=\sqrt{a'a''} . \tag{4.9}$$

The little group that leaves the first diagonal block element M_{11} in (4.7) invariant may be obtained based on the following consideration:

$$\begin{aligned} B(\lambda)\mathcal{R}(\theta)B(-\lambda)&\begin{bmatrix} a' & \\ & a'' \end{bmatrix}B^T(-\lambda)\mathcal{R}^T(\theta)B^T(\lambda) \\ &=B(\lambda)\mathcal{R}(\theta)\sqrt{a'a''}\mathcal{R}^T(\theta)B^T(\lambda) \\ &=\sqrt{a'a''}\begin{bmatrix} e^{2\lambda} & 0 \\ 0 & e^{-2\lambda} \end{bmatrix}=\begin{bmatrix} a' & 0 \\ 0 & a'' \end{bmatrix} . \end{aligned} \tag{4.10}$$

The little group element is therefore given by

$$\mathcal{R}_x(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta e^{-2\lambda} \\ \sin\theta e^{2\lambda} & \cos\theta \end{bmatrix} \tag{4.11}$$

with $e^{-2\lambda}=\sqrt{a''/a'}$, where the subscript x serves as a reminder that the little group element leaves a certain 2×2 matrix invariant. For the 4×4 matrix of (4.7) the little group transformation matrix that leaves the diagonal 2×2 blocks invariant is given by

$$\mathcal{R}(\theta_{12},1;\varphi_{12},2)\equiv\begin{bmatrix} \mathcal{R}_1(\theta_{12}) & 0 \\ 0 & \mathcal{R}_2(\varphi_{12}) \end{bmatrix} , \tag{4.12}$$

where

$$\mathcal{R}_1(\theta_{12})=\begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12}\left[\dfrac{a''}{a'}\right]^{1/2} \\ \sin\theta_{12}\left[\dfrac{a'}{a''}\right]^{1/2} & \cos\theta_{12} \end{bmatrix} \tag{4.13}$$

is the little group element that leaves M_{11} of (4.7) invariant, while

$$\mathcal{R}_2(\varphi_{12})=\begin{bmatrix} \cos\varphi_{12} & -\sin\theta_{12}\left[\dfrac{b''}{b'}\right]^{1/2} \\ \sin\varphi_{12}\left[\dfrac{b'}{b''}\right]^{1/2} & \cos\varphi_{12} \end{bmatrix} \tag{4.14}$$

leaves M_{22} of (4.5) invariant. The diagonalization of M' of (4.7) may then be carried out through

$$\begin{aligned} M''&=\mathcal{R}(\theta_{12},1;\varphi_{12},2)M'\mathcal{R}^T(\theta_{12},1;\varphi_{12},2) \\ &=\begin{bmatrix} \begin{matrix} a' & \\ & a'' \end{matrix} & \mathcal{R}_1(\theta_{12})M_{12}\mathcal{R}_2^T(\varphi_{12}) \\ \mathcal{R}_2(\varphi_{12})M_{21}\mathcal{R}_1^T(\theta_{12}) & \begin{matrix} b' & \\ & b'' \end{matrix} \end{bmatrix} \end{aligned} \tag{4.15}$$

For ease of reference we give the general form of M''_{12} here, i.e.,

$$M''_{12}\equiv\mathcal{R}_1(\theta)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathcal{R}_2^T(\varphi)=\begin{bmatrix} -c\sin(\theta+\varphi)+d\cos(\theta-\varphi) & 0 \\ 0 & b\sin(\theta+\varphi)+d\cos(\theta-\varphi) \end{bmatrix} , \tag{4.16}$$

where

$$\tan(\theta+\varphi)=-\frac{b+c}{a-d},\quad \tan(\theta-\varphi)=\frac{b-c}{a+d}\ . \tag{4.17}$$

Thus the 4×4 matrix takes the form

$$M''=\begin{bmatrix} a' & 0 & r' & 0 \\ 0 & a'' & 0 & r'' \\ r' & 0 & b' & 0 \\ 0 & r'' & 0 & b'' \end{bmatrix}\ . \tag{4.18}$$

Step 3: Diagonalizing M''. We find it instructive to temporarily rearrange matrix elements in the order $\tilde{\xi}=(q_1,q_2,p_1,p_2)$. Correspondingly

$$\tilde{M}''=\begin{bmatrix} a' & r' & & \\ r' & b' & & 0 \\ & & a'' & r'' \\ 0 & & r'' & b'' \end{bmatrix}\ . \tag{4.19}$$

The diagonalization may then be achieved through

$$\tilde{M}'''=\tilde{L}(\lambda,\alpha)\tilde{M}''\tilde{L}^T(\lambda,\alpha)\ , \tag{4.20}$$

where

$$\tilde{L}(\lambda,\alpha)\equiv\tilde{S}(\alpha,\alpha)\tilde{B}(\lambda,-\lambda) \tag{4.21}$$

with

$$\tilde{B}(\lambda,-\lambda)=\begin{bmatrix} \tilde{B}(\lambda) & \\ & \tilde{B}(-\lambda) \end{bmatrix}$$

$$\text{where } \tilde{B}(\lambda)=\begin{bmatrix} e^{\lambda} & \\ & e^{-\lambda} \end{bmatrix}, \tag{4.22}$$

and with

$$\tilde{S}(\alpha,\alpha)=\begin{bmatrix} \tilde{S}(\alpha) & \\ & \tilde{S}(\alpha) \end{bmatrix}$$

$$\text{where } \tilde{S}(\alpha)=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\ . \tag{4.23}$$

The parameters λ and α may be determined through the two diagonalization conditions implied by (4.20) [see (4.26) below], which leads to

$$\lambda=\frac{1}{4}\ln\frac{a''r'+b'r''}{a'r''+b''r'} \tag{4.24}$$

and

$$\tan2\alpha=\frac{-2}{a'a''-b'b''}[(a'r''+b''r')(a''r'+b'r'')]^{1/2} \tag{4.25}$$

Reverting back to the original basis $\xi=(q_1,p_1,q_2,p_2)$. The corresponding transformation is given by

$$M'''=L(\alpha,\lambda)M''L^T(\alpha,\lambda)$$

$$=\begin{bmatrix} a'(12) & & & \\ & a''(12) & & \\ & & b'(12) & \\ & & & b''(12) \end{bmatrix}, \tag{4.26}$$

with

$$L(\alpha,\lambda)=S(\alpha,\alpha)B(\lambda,-\lambda)\ , \tag{4.27}$$

where

$$B(\lambda,-\lambda)=\begin{bmatrix} e^{\lambda} & & & \\ & e^{-\lambda} & & \\ & & e^{-\lambda} & \\ & & & e^{\lambda} \end{bmatrix},$$

$$S(\alpha,\alpha)=\begin{bmatrix} \cos\alpha & & -\sin\alpha & \\ & \cos\alpha & & -\sin\alpha \\ \sin\alpha & & \cos\alpha & \\ & \sin\alpha & & \cos\alpha \end{bmatrix}\ . \tag{4.28}$$

The parameters λ and α are those given in (4.24) and (4.25).

Step 4: Transformation to the standard C matrix. The final step to obtain the standard form of the C matrix is obtained through rescaling the diagonal 2×2 matrices to bring each diagonal 2×2 matrix block to be proportional to the identity matrix

$$C=B(\lambda_1,\lambda_2)M'''B(\lambda_1,\lambda_2)$$

$$=\begin{bmatrix} k_1 & & & \\ & k_1 & & \\ & & k_2 & \\ & & & k_2 \end{bmatrix}\ . \tag{4.29}$$

V. MULTIMODE CORRELATED STATES

A. Reduction of the $2N\times2N$ matrix M to the standard C matrix

We can readily extend the treatment of Sec. IV to the arbitrary N-mode case. Here the corresponding matrix to be diagonalized is the $2N\times2N$ matrix, which can be conveniently written in the form

$$M=\begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots \\ M_{21} & M_{22} & M_{23} & \cdots \\ M_{31} & M_{32} & M_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\ . \tag{5.1}$$

Step 1: Diagonalize all the 2×2 diagonal blocks by rotations

$$M \xrightarrow{\mathcal{R}(\theta_{11},\theta_{22},\ldots,\theta_{NN})} \begin{bmatrix} \begin{matrix} a' & 0 \\ 0 & a'' \end{matrix} & M_{12} & \cdots & \cdots \\ M_{21} & \begin{matrix} b' & 0 \\ 0 & b'' \end{matrix} & \cdots & \cdots \\ \cdots & \cdots & \begin{matrix} c' & 0 \\ 0 & c'' \end{matrix} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \equiv M' . \tag{5.2}$$

Repeat steps 2 and 3 of Sec. IV, $N(N-1)/2$ times to systematically transform away symmetric pairs of off-diagonal 2×2 matrices. Generalizing the notation of the preceding section, we write

$$L_{ij}=L_{ij}(\theta_{ij},\varphi_{ij};\lambda_{ij},\alpha_{ij}) , \tag{5.3}$$

where the subscript ij serves as a reminder that through the transformation by the L matrix, with the set of parameters θ_{ij}, ϕ_{ij}, λ_{ij}, and α_{ij}, the element M_{ij} is to be eliminated. The successive transformations may be chosen following the steps indicated below

$$M' \xrightarrow{L_{12}} \begin{bmatrix} \begin{matrix} a'(12) & 0 \\ 0 & a''(12) \end{matrix} & 0 & M_{13} & \cdots \\ 0 & \begin{matrix} b'(12) & 0 \\ 0 & b''(12) \end{matrix} & \cdots & \cdots \\ M_{31} & \cdots & \begin{matrix} c' & 0 \\ 0 & c'' \end{matrix} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} , \tag{5.4}$$

$$\xrightarrow{L_{13}} \begin{bmatrix} \begin{matrix} a'(13) & \\ & a''(13) \end{matrix} & 0 & 0 & \cdots \\ 0 & \begin{matrix} b'(12) & 0 \\ 0 & b''(12) \end{matrix} & M_{23} & \cdots \\ 0 & M_{32} & \begin{matrix} c'(13) & 0 \\ 0 & c''(13) \end{matrix} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \xrightarrow{L_{23}} \cdots , \tag{5.5}$$

$$\xrightarrow{L_{N-1,N}} \begin{bmatrix} \begin{matrix} a'(1N) & 0 \\ 0 & a''(1N) \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} b'(2N) & \\ & b''(2N) \end{matrix} & 0\cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \begin{matrix} z'(N-1,N) & 0 \\ 0 & z''(N-1,N) \end{matrix} \end{bmatrix} \equiv C' . \tag{5.6}$$

Step 4: From the diagonal matrix C' to the standard C matrix

$$C=\begin{bmatrix} k_1 & 0\cdots & & 0 \\ & k_{\hat{1}} & & \\ 0 & & & \vdots \\ \vdots & & \ddots & 0 \\ & & k_N & \\ 0 & \cdots 0 & & k_N \end{bmatrix} . \tag{5.7}$$

The total number of parameters can now be tallied as follows: The number of parameters in original M matrix is $N_M=N(2N+1)$; the numbers of parameters of transformations N_T are

$$R(\theta_{11},\ldots,\theta_{nn})\rightarrow N\ ,$$
$$L_{12},\ldots,L_{1N};\ L_{23},\ldots,L_{2N}\ ;\ L_{33},\ldots,L_{3N};\ldots;\ldots L_{N-1N}\rightarrow 4\times\tfrac{1}{2}(N-1)N\ ,$$
$$B(\lambda_1,\ldots,\lambda_N)\rightarrow N\ ;$$

and the number of parameters in C matrix is $N_C=N$. For the present case, where all the k_i's are distinct, the invariant subgroup leaving the ground-state invariant has N parameters. Notice that $N_M=N_T+N_C=N(2N+1)$. This implies that the set of canonical transformations chosen to bring M to its standard C matrix follows an efficient route, in the sense that all the transformation parameters are independent parameters.

Dimension of the invariant subgroup

Consider a general matrix M with the corresponding C matrix that has the following structure. There is n_1-fold degeneracy for k_1, n_2-fold degeneracy for k_2, and up to n_N-fold degeneracy for k_N, with

$$n_1+n_2+\cdots+n_N=N\ . \tag{5.8}$$

The corresponding little group that leaves C invariant will be given by

$$L=U(n_1)\otimes U(n_2)\cdots\otimes U(n_N)\ . \tag{5.9}$$

Here we follow the convention that if $n_j=0$, $U(0)=I$. This element will be removed from the direct product expression. The dimension of the invariant subgroup (IS), which leaves M invariant [see Eq. (3.20)], is given by

$$d_L=d_{\rm IS}=n_1^2+n_2^2+\cdots+n_N^2\ . \tag{5.10}$$

We display the two limiting cases here. For case 1, where all k_i's are distinct, (5.10) implies that

$$d_L=N\ . \tag{5.11}$$

The dimension of the corresponding coset space is

$$d_{\rm coset}=N(2N+1)-N=2N^2\ . \tag{5.12}$$

In case 2, where all the k_i's are equal, $n_1=N$ and $n_i=0$ for $i>1$. So

$$d_L=N^2,\ \ d_{\rm coset}=N(2N+1)-N^2=N(N+1)\ . \tag{5.13}$$

B. Reduction of the $2N\times 2N$ matrix M to the standard C matrix

The displacements and squeezings introduce $2N+N=3N$ parameters. But the generalized correlated state is obtained by the full coset of the linear canonical group $Sp(2N,R)\boxminus T(2N)$ by the stability group of the N-mode vacuum state $|\Omega\rangle$.

These correlated states may be displayed explicitly, but are too cumbersome. The multimode correlated states have wave functions that are displaced Gaussians with phase factors. Depending upon the experimental requirements, we may obtain intensity correlations, photocount statistics, etc., directly. The number of parameters describing such correlated states are enormous and would be restricted by the method of generation of such states.

C. Generalized uncertainty principles

For N modes with the parameters $k_1,\ldots,k_N$, the C matrix in its standard form expressed in 2×2 block notation

$$C=\begin{bmatrix} k_1 I & & \\ & \ddots & \\ & & k_N I\end{bmatrix}\ , \tag{5.14}$$

$$\beta=\begin{bmatrix} i\sigma_2 & & \\ & \ddots & \\ & & i\sigma_2\end{bmatrix}\ , \tag{5.15}$$

$$C\beta=\begin{bmatrix} ik_1\sigma_2 & & \\ & \ddots & \\ & & ik_N\sigma_2\end{bmatrix}\ , \tag{5.16}$$

$$\mathrm{Tr}(C\beta)^2=2N(k_1^2+\cdots+k_N^2)\ , \tag{5.17}$$

$$\mathrm{Tr}(C\beta)^{2n}=2N(k_1^{2n}+\cdots+k_N^{2n})\ . \tag{5.18}$$

For the minimum-uncertainty system $k_1=k_2=\cdots=k_N=\frac{1}{2}$. So for the N-mode case, the minimum-uncertainty state with the corresponding arbitrary matrix M_0 and the standard matrix C_0 gives

$$\mathrm{Tr}(M_0\beta)^{2n}=\mathrm{Tr}(C_0\beta)^{2n}=\frac{N}{2^{2n-1}}\quad\text{for } n=1,\ldots,N\ . \tag{5.19}$$

This leads to the complete set of the generalized uncertainty relations

$$\mathrm{Tr}(M\beta)^{2n}\geq\frac{N}{2^{2n-1}}\quad\text{for } n=1,\ldots,N\ . \tag{5.20}$$

For $N=1$, i.e., the one-mode case,

$$\mathrm{Tr}(M\beta)^2=C^2=\langle q^2\rangle\langle p^2\rangle-\left\langle\frac{qp+pq}{2}\right\rangle^2\geq\frac{1}{4}\ , \tag{5.21}$$

which is the Schrödinger-Robertson inequality relation. One may say that for the N-mode case, the number N of inequalities given in (5.20) is the generalization of the Schrödinger-Robertson inequality.

It is instructive to look at the case of $N=2$ in detail. For $k_1\neq k_2$,

$$C_2 \equiv \mathrm{Tr}(M\beta)^2 = 2(k_1^2 + k_2^2) \geq 1 \; , \quad (5.22)$$

$$C_4 \equiv \mathrm{Tr}(M\beta)^4 = 2(k_1^4 + k_2^4) = (k_1^2 + k_2^2)^2 + (k_1^2 - k_2^2)^2 \geq \tfrac{1}{4} \; . \quad (5.23)$$

In general, there are two independent inequalities, which reduces to one when $k_1 = k_2$. Here Eqs. (5.22) and (5.23) are reduced to one independent inequality. The inequality expressed in terms of k_1 and k_2 contains more information than those given by the corresponding traces. For instance, from the expression in terms of the k's, one could arrive at

$$C_4 - \tfrac{1}{4} C_2^2 = (k_1^2 - k_2^2)^2 \geq 0 \; , \quad (5.24)$$

which is not implied by

$$C_2 \geq 1, \quad C_4 \geq \tfrac{1}{4} \; . \quad (5.25)$$

When the equality is satisfied, i.e., $k_1 = k_2 = \frac{1}{2}$, this corresponds to the case of the minimum-uncertainty pure state. Much of the statements here can be extended to N modes. We will not detail them here.

The inequalities of (5.20) are statements on the invariants of matrix M. We observe that the invariants of a matrix can also be specified through its characteristics. The characteristic equation of the matrix M is given by

$$\det|M\beta - \lambda I| = \det|C\beta - \lambda I| = \prod_{i=1}^{N} (k_i^2 + \lambda^2) = \sum_{i=0}^{N} f_i \lambda^{2i} \; , \quad (5.26)$$

with f_i, the coefficient of λ^{2i}, being the characteristics of order i. The characteristics are

$$\begin{aligned} f_0 &= k_1^2 k_2^2 \cdots k_N^2 \; , \\ f_1 &= \left[\frac{1}{k_1^2} + \frac{1}{k_2^2} + \cdots \frac{1}{k_N^2} \right] f_0 \; , \\ &\vdots \\ f_{N-1} &= k_1^2 + k_2^2 + \cdots + k_N^2 \; , \\ f_N &= 1 \; . \end{aligned} \quad (5.27)$$

They are all positive quantities. The set of the invariant C_i can be expressed in terms of the characteristics f_i. The uncertainty relations may also be stated in terms of the characteristics giving

$$f_0 \geq \frac{1}{4^{N-1}}, \quad f_1 \geq \frac{N}{4^{N-1}}, \ldots, f_{N-1} \geq \frac{N}{4} \; . \quad (5.28)$$

When all the k_i's are distinct, the number of independent inequalities is again N. Evidently the set of inequalities in (5.28) expressed in terms of the k_i^2's is equivalent to those in (5.20).

VI. DISCUSSION

Some remarks are in order about the correlated states in quantum field theory. As long as the number of excited modes is finite, however many, there exists a unitary transformation from the multimode vacuum state to the multimode correlated state. These unitary transformations are generated by a quantity that is bilinear in the canonical variables. These operators are unbounded but do generate unitary transformations. When the number of modes becomes infinite, the generic correlated state cannot be obtained from the vacuum state. They would be in a different Hilbert space from the Fock vacuum [21], though canonical unitary transformations can generate a restricted set of correlated states.

It was the purpose of this paper to demonstrate the close relation between the correlated states and the linear canonical group and to show that the correlated states that minimize the Schrödinger uncertainties are related to the canonical multimode vacuum that is invariant under linear unitary transformations of the modes. The generic wave functions are Gaussians with a determined number of independent parameters.

The one- and two-mode analysis is equally applicable to the propagation of the Gaussian Schell mode paraxial wave fronts through a system of thin lenses that are, respectively, isotropic and nonisotropic. This has been carried out elsewhere [22].

Correlated states are the generic family that include squeezed states and coherent states as special cases. For each value of the complex parameter α, we have an overcomplete family of states in the case of one degree of freedom. For the multimode case the parameter defining the generic form of the M matrix of (5.1) from the canonical form of the C matrix of (5.7) is such a labeling parameter.

The implications of canonical commutation relations for correlations are not exhausted by the Heisenberg uncertainty relation, nor even the Schrödinger uncertainties, but there are relations in multimode case. In the case of partially coherent two-mode excitation these furnish a further test of quantum theory and also gives a more detailed characterization of the correlation function.

ACKNOWLEDGMENTS

We wish to thank Professor Luis Boya for invaluable discussions on the group-theoretical aspect of the problem. Thanks are also due to Professor Larry Beidenharn and Professor Vladimir Man'ko for a careful reading of an early version of this paper. This work was supported by the U.S. Department of Energy under Contract No. DOE-FG05-85ER40-200.

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