PHYSICAL REVIEW D VOLUME 26, NUMBER 8 15 OCTOBER 1982

Equivalence of the four-point interaction and the Yukawa interaction. I. A boson model

C. C. Chiang,* C. B. Chiu, E. C. G. Sudarshan,
and
Xerxes Tata
Center for Particle Theory, Department of Physics,
The University of Texas,
Austin, Texas 78712
(Received 11 August 1981)

It is shown that the compositeness conditions by themselves do not ensure the complete equivalence of the Yukawa-type interaction with the corresponding four-point interaction. In our study of the corresponding situation in the context of the Lee model versus the separable-potential model, we find in the limit in which the bare Yukawa coupling constant becomes infinite, the finite-energy wave functions calculated from the two theories coincide. It is shown, however, that although the finite-energy spectra of the two theories are the same, the Yukawa theory has additional spectral contributions at infinity. It is further demonstrated that it is only by removing all contributions from the spectrum at infinity that the two theories can be made essentially identical. It is therefore suggested that the proofs of the equivalence of the four-fermion interaction and the Yukawa-type interaction need to be reexamined.

I. INTRODUCTION

The possibility of a close connection between a renormalizable Yukawa-type interaction and the nonrenormalizable four-spinor interaction was first suggested by Houard and Jouvet[1] in the context of the soluble Lee model and the separable-potential model. In their pioneering paper, Nambu and Jona-Lasinio[2] demonstrated that it was possible, starting from a four-fermion interaction, to obtain a collective bosonic state that effectively coupled to the fermions via a Yukawa interaction. Since then, a number of authors have considered the possibility of constructing Yukawa-type theories starting from a single quartically self-coupled fermion field, thereby reducing the number of fundamental fields in interaction. The interesting possibility that the resulting Yukawa theory admits a gauge type of interaction has been considered[3] in the context of quantum electrodynamics since the early sixties. Also, in recent years, there have been several attempts[4] to derive non-abelian gauge theories starting from an appropriate four-fermion theory. The possibility that the Higgs fields are composites of fermion fields has also been considered.[5]

The condition for the equivalence of the renormalized four-fermion and Yukawa-type Lagrangians, it is widely believed, is that the effect of the bare boson field in the Yukawa theory vanish. This can be realized by the vanishing of certain renormalization constant Z_i, of the corresponding Yukawa theory.[6] We note that the sense in which the equivalence is demonstrated is that the renormalized Green's functions of the two theories are shown to coincide to all orders in perturbation theory.

It has been suspected that direct comparison of the forms of the two Lagrangians and a one-to-one correspondence of the Feynman diagrams of the two theories may not be entirely satisfactory demonstration of their equivalence. Recently Rajaraman[7] has questioned the validity of some of these "proofs" basing his arguments on the consideration of two systems each with a finite number of degrees of freedom. He concluded that the compositeness condition $Z=0$ led to a renormalized coupling $g=\sqrt{Z}\, g_0=0$, i.e., the equivalence may, almost, be realized for free field theories.

The nonrenormalizable four-fermion interaction and the renormalizable Yukawa-type interaction intuitively appear to be of intrinsically different character. The purpose of this paper is to study the sense in which these two types of theories have been stated to be equivalent, and to point out the limitations of such equivalence proofs. For this purpose we consider the corresponding situation for two soluble models, viz, the Lee model (Yukawa-type theory) and the attractive separable-potential model (four-point interaction). The two theories are clearly distinct since the matrix elements of the Hamiltonian in the basis of bare states are clearly different irrespective of the existence of any untraviolet cutoff and of the strength of the coupling. In this paper, we study the extent to which these two nonidentical theories can be considered equivalent and also the mechanism by which the Lee-model interaction can be transmuted into the four-point interaction.

We proceed by first showing that it is possible to realize the compositeness conditions $Z=0$ for an interacting theory[8] in the strong-coupling (SC) limit, defined by introducing an ultraviolet cutoff on the integrals and taking the bare coupling to infinity. The cutoff may then be taken to infinity. In the cutoff theory, however, the spectrum of the Lee model develops additional contributions beyond the cutoff; these move to infinity in the SC limit. These are absent in the separable-potential model, cut off in a similar manner. We have explicitly shown that in the SC limit, these additional spectral contributions do not contribute to the finite-energy S-matrix elements or scattering amplitudes, i.e., at finite energies, the scattering amplitudes and the S matrix calculated from either the Lee model or the separable-potential model coincide in the SC limit. We emphasize, though that the Hamiltonians of the two theories are different, even in the SC limit.

To elucidate the mechanism by which the Lee model may be transmuted into the separable-potential model, we made a detailed study of the Hamiltonian matrix elements of the two theories in terms of the "base basis." We find that the Lee-model Hamiltonian can be written as a sum of two parts, one arising solely from contributions of those eigenstates whose energy remains finite in the SC limit, and the other from those whose eigenvalues move to infinity along with the coupling. Moreover, in the SC limit, we find that when the latter contributions are explicitly omitted, the Lee-model Hamiltonian reduces essentially to that of the separable-potential model. Thus, although the finite-energy scattering amplitudes and S-matrix elements for the two theories are numerically the same in the SC limit, the Hamiltonians of the two theories are distinct and the two theories are clearly distinguishable. Only when the spectral contributions at infinity are removed does the Lee model get transmuted into the separable-potential model. Some of our preliminary results were reported in an earlier paper.[9]

The plan of this paper is as follows. In Sec. II we discuss the solutions and Hamiltonian matrix elements for the separable-potential model. Section III is devoted to the corresponding study for the Lee model. It is shown that in the SC limit the scattering amplitudes and S matrices for the two theories coincide. The explicit removal pf spectral contributions to the Hamiltonian that arise due to the presence of the cutoff is carried out in Sec. IV. It is demonstrated that the Lee model then transmutes into the attractive separable-potential model. We conclude in Sec. V with some general remarks. Some useful mathematical formulas are listed in Appendix A. The explicit solutions to the Lee model with an ultraviolet cutoff are discussed in Appendix B. A demonstration of the completeness of the physical states is outlined in Appendix C. In Appendix D, we show that the equations of motion for the Källén-Pauli amplitudes can be obtained from those for the overcomplete amplitudes defined in Sec. III provided the latter satisfy the constraint relationships discussed in the same section.

II. THE SEPARABLE-POTENTIAL MODEL

The system described by the attractive separable-potential model consists of two fields associated with the N and θ particles, interacting via a four-point contact interaction. The Hamiltonian for the system is given by

$$H=\int d^3k\, k a^\dagger(k)a(k) - \int d^3k\, h(k)N^\dagger a^\dagger(k)\int d^3l\, h(l)Na(l)\ . \tag{2.1}$$

All fields are quantized as bosons. In Eq. (2.1), and in the rest of this paper, it is assumed as a matter of notation that all momentum integrals are cut off at an upper value of momentum $|\vec{k}|=L$. We consider, for simplicity, the θ particle to be massless and also ignore any recoil energy of the N particle.

The theory, as is well known, decomposes into a countable number of disconnected sectors labeled by the eigenvalues of the operators $\mathcal{N}_\theta$ and $\mathcal{N}_N$ defined by

$$\mathcal{N}_\theta = \int d^3k\, a^\dagger(k) a(k)$$

and

$$\mathcal{N}_N = N^\dagger N \ . \qquad (2.2)$$

The lowest nontrivial sector is characterized by $\mathcal{N}_N = 1$ and $\mathcal{N}_\theta = 1$.

In what follows, we use $|\ \rangle$ to denote the eigenstates of the free Hamiltonian (defined as the bilinear part) and $|\ \rangle\rangle_\lambda$ to denote those of the total Hamiltonian. The entry within the ket denotes the state under consideration and also the eigenvalue in the case of the free Hamiltonian eigenstates while λ denotes the eigenvalue of the corresponding states for the total Hamiltonian.

It can be easily checked that the Hamiltonian matrix elements for the $N\theta$ and $N\theta\theta$ sectors can be written as

$$\langle N\theta_k | H | N\theta_l \rangle = k\delta(\vec{k} - \vec{l}) - h(k)h(l) \qquad (2.3)$$

and

$$\langle N\theta_k\theta_l | H | N\theta_{k'}\theta_{l'} \rangle = (k+l)[\delta(\vec{k}-\vec{k}')\delta(\vec{l}-\vec{l}') + \delta(\vec{k}-\vec{l}')\delta(\vec{l}-\vec{k}')] - [h(k)h(k')\delta(\vec{l}-\vec{l}') + (\vec{k}\leftrightarrow\vec{l}) + (\vec{k}'\leftrightarrow\vec{l}') + (\vec{k}\leftrightarrow\vec{l})(\vec{k}'\leftrightarrow\vec{l}')] \ . \qquad (2.4)$$

These equations will be used later for the comparison of the "transmuted" Lee model with the separable-potential model.

In terms of the function $\rho_\lambda(k)$ defined as

$$\rho_\lambda(k) = \langle N\theta_k | N\theta \rangle\rangle_\lambda \ , \qquad (2.5)$$

the equation of motion in the $N\theta$ sector can easily be seen to be

$$(\lambda - k)\rho_\lambda(k) = -\int h(k)h(l)\rho_\lambda(l) d^3l \ . \qquad (2.6)$$

The solution for the scattering states can be readily obtained from Eq. (2.6) to be

$$\rho_\lambda(k) = \delta(\vec{\lambda} - \vec{k}) - \frac{h(k)h(\lambda)}{\lambda - k + i\epsilon} \frac{1}{D^+(\lambda)} \ , \qquad (2.7)$$

with

$$D(z) = 1 + \int d^3k \frac{h^2(k)}{z-k} \qquad (2.8)$$

and

$$D^\pm(\lambda) = D(\lambda \pm i\epsilon) \ .$$

By writing the equation of motion as

$$D(\lambda) \int d^3k\, h(k)\rho_\lambda(k) = 0 \ ,$$

we see that the discrete eigenvalues are given by the roots of

$$D(z) = 0 \ . \qquad (2.9)$$

It is clear that $D(z \to -\infty) \to 1$ and that D is continuous for real $z \in (-\infty, 0)$. Noting that $D'(z) < 0$ in the same domain, it is evident that a necessary and sufficient condition for a bound state below the $N\theta$ threshold is

$$\lim_{z \to 0^-} D(z) < 0 \ . \qquad (2.10)$$

The monotonicity of $D(z)$ ensures that the bound state, if it exists, is unique. Also, the fact that

$$\lim_{z \to +\infty} D(z) = 1$$

together with the continuous and monotonically decreasing nature of $D(z)$ for real z ensures there is no real zero of $D(z)$ for $z > L$, i.e., there is no bound state beyond the cutoff. It is well known that $D(z)$ has no complex zeros. In our considerations, we will assume that there is a bound state B of mass $M < 0$. Choosing this state to have unit norm, we write the solution as

$$\rho_M(k) = [-D'(M)]^{1/2} \frac{h(k)}{M-k} \ . \qquad (2.11)$$

Finally, from Eqs. (2.7) and (2.11), the eigenstates for the $N\theta$ sector of the separable-potential model can be written as

$$|N\theta\rangle\rangle_\lambda = \int d^3k\, \rho_\lambda(k) |N\theta_k\rangle \qquad (2.12a)$$

for the scattering states and

$$|B\rangle\rangle_M = \int d^3k\, \rho_M(k) |N\theta_k\rangle \qquad (2.12b)$$

for the bound state. In obtaining Eqs. (2.12) we have made use of the completeness of the bare eigenstates.

We now turn our attention to the solutions of the separable-potential model in the $N\theta\theta$ sector. In anticipation of the discussion in the next section, we introduce the amplitudes

$$\Omega_\lambda(k,l) = {}_k\langle\langle N\theta | a(l) | \ \rangle\rangle_\lambda \qquad (2.13a)$$

and

$$\Omega_\lambda(l)={}_M\langle\langle B\,|\,a(l)\,|\,\rangle\rangle_\lambda\ . \tag{2.13b}$$

The equations of motion for these amplitudes can be readily obtained by considering the action of the Hamiltonian on the states $a^\dagger(l)\,|\,N\theta\rangle\rangle_k$ and $a^\dagger(l)\,|\,B\rangle\rangle_M$. We find

$$(\lambda-k-l)\Omega_\lambda(k,l) = -h(l)\int d^3p\, h(p)\Omega_\lambda(k,p) \tag{2.14a}$$

and

$$(\lambda-M-l)\Omega_\lambda(l)=-h(l)\int d^3p\, h(p)\Omega_\lambda(p)\ . \tag{2.14b}$$

These equations are of the form of the equation of motion (2.6) in the lower sector. The solution to these can be readily obtained. The spectrum admits $N\theta\theta$ scattering states for $0\le\lambda\le 2L$ and $B\theta$ scattering states for $M\le\lambda\le M+L$. The corresponding solutions are written below.

$N\theta\theta$ **scattering states**

With $\zeta_1+\zeta_2=\lambda$, $0\le\zeta_1,\ \zeta_2\le L$,

$$\Omega_\lambda(k,l)=\delta(\vec{\zeta}_1-\vec{k})\delta(\vec{\zeta}_2-\vec{l})+\delta(\vec{\zeta}_2-\vec{k})\delta(\vec{\zeta}_1-\vec{l}) -\frac{h(l)h(\zeta_2)}{\lambda-k-l+i\epsilon}\frac{1}{D^+(\zeta_2)}\delta(\vec{\zeta}_1-\vec{k})-\frac{h(l)h(\zeta_1)}{\lambda-k-l+i\epsilon}\frac{1}{D^+(\zeta_1)}\delta(\vec{\zeta}_2-\vec{k}) \tag{2.15a}$$

and

$$\Omega_\lambda(l)=0\ . \tag{2.15b}$$

$B\theta$ **scattering states**

Let $\lambda=M+\xi$. Then

$$\Omega_\lambda(l)=\delta(\vec{\xi}-\vec{l})-\frac{h(l)h(\xi)}{\xi-l+i\epsilon}\frac{1}{D^+(\xi)} \tag{2.16a}$$

and

$$\Omega_\lambda(k,l)=\frac{1}{[-D'(M)]^{1/2}}\frac{h(l)}{\lambda-k-l}\delta(\vec{\xi}-\vec{k})\ . \tag{2.16b}$$

We see from Eq. (2.16a) that there is a bound state in the $B\theta$ channel at $\lambda=2M$, since at that point a nontrivial solution is possible in the absence of any driving term. The corresponding solution can be readily obtained to be

$$\Omega_{2M}(k)=\frac{\sqrt{2}}{[-D'(M)]^{1/2}}\frac{h(k)}{M-k} \tag{2.17a}$$

and

$$\Omega_{2M}(k,l)=0\ . \tag{2.17b}$$

The normalization is chosen so as to ensure unit norm for the bound state.

The eigenstates of the separable potential can be written in terms of the bare states $|\,N\theta_k\theta_l\rangle$ as

$$|\,\rangle\rangle_\lambda=\tfrac{1}{2}\int d^3k\, d^3l\,\Xi_\lambda(k,l)\,|\,N\theta_k\theta_l\rangle \tag{2.18a}$$

with

$$\Xi_\lambda(k,l)=\int \rho_p(k)\Omega_\lambda(p,l)d^3p +\rho_M(k)\Omega_\lambda(l)\ , \tag{2.18b}$$

where in obtaining the last equation we have made use of the completeness of physical states in the $N\theta$ sector.

The bare basis components for the various eigenstates can be readily worked out from Eqs. (2.18) and our solutions presented earlier. We find

$$\Xi_\lambda(k,l)=\delta(\vec{\zeta}_1-\vec{k})\delta(\vec{\zeta}_2-\vec{l})-\frac{h(k)h(\zeta_1)}{(\zeta_1-k)D^+(\zeta_1)}\delta(\vec{\zeta}_2-\vec{l})-\frac{h(k)h(\zeta_2)\delta(\zeta_1-l)}{(\zeta_2-k)D^+(\zeta_2)} +\frac{h(k)h(l)h(\zeta_1)h(\zeta_2)}{(\zeta_2-k)(\zeta_1-l)D^+(\zeta_1)D^+(\zeta^2)}+(k\leftrightarrow l) \tag{2.19a}$$

for the $|N\theta\theta\rangle\rangle_\lambda$ states,

$$\Xi_\lambda(k,l)=\frac{1}{[-D'(M)]^{1/2}}\frac{h(l)}{M-l}\delta(\vec{\xi}-\vec{k})-\frac{1}{[-D'(M)]^{1/2}}\frac{h(\xi)h(k)h(l)}{D^{+}(\xi)}\frac{1}{(\xi-k)}\frac{1}{(M-l)}+(k\leftrightarrow l) \quad (2.19b)$$

for the $|B\theta\rangle\rangle_\lambda$ states, and

$$\Xi_{2M}(k,l)=\frac{\sqrt{2}}{[-D'(M)]^{1/2}}\frac{h(k)h(l)}{(M-k)(M-l)} \quad (2.19c)$$

for the discrete state at $\lambda=2M$.

We will show in subsequent sections that the Lee model, with suitable spectral contributions removed, becomes effectively equivalent to the separable-potential model discussed in this section.

III. THE LEE MODEL

In addition to the N and θ fields, the Lee model[10] involves a third field, V, interacting with the N and θ fields via a Yukawa type of interaction. $N\theta$ scattering thus occurs through an exchange of a V particle. The Hamiltonian for the system is given by

$$H=m_0V^\dagger V+\int d^3k\,ka^\dagger(k)a(k) + \int d^3k\,g_0f(k)[V^\dagger Na(k)+VN^\dagger a^\dagger(k)]\ . \quad (3.1)$$

Again, all the fields are quantized as bosons. The Lee model also breaks up into a countable number of disconnected sectors labeled by the eigenvalues of the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ defined by

$$\mathcal{N}_1=\mathcal{N}_V+\mathcal{N}_N\ ,\qquad \mathcal{N}_2=\mathcal{N}_V+\mathcal{N}_\theta \quad (3.2)$$

with

$$\mathcal{N}_V=V^\dagger V\ .$$

The lowest nontrivial sector, the $N\theta$ or the V sector, is characterized by $\mathcal{N}_1=\mathcal{N}_2=1$, while the next higher sector referred to as the $N\theta\theta$ or $V\theta$ sector is labeled by $\mathcal{N}_1=1$, $\mathcal{N}_2=2$. In this section we present the detailed solutions for these two sectors.

The $N\theta$ sector

The eigenstates of the Hamiltonian in the $N\theta$ sector of the Lee model can be expanded in terms of the bare states, $|N\theta_k\rangle$ and $|V\rangle$, as

$$|\ \rangle\rangle_\lambda=\int\rho_\lambda(k)|N\theta_k\rangle d^3k+\sigma_\lambda|V\rangle \quad (3.3a)$$

with

$$\rho_\lambda(k)=\langle N\theta_k|\ \rangle\rangle_\lambda \quad (3.3b)$$

and

$$\sigma_\lambda=\langle V|\ \rangle\rangle_\lambda\ . \quad (3.3c)$$

The equations of motion for the Källén-Pauli components $\rho_\lambda(k)$ and σ_λ can be written as

$$(\lambda-m_0)\sigma_\lambda=g_0\int d^3k\,f(k)\rho_\lambda(k) \quad (3.4a)$$

and

$$(\lambda-k)\rho_\lambda(k)=g_0f(k)\sigma_\lambda\ . \quad (3.4b)$$

From Eqs. (3.4a) and (3.4b) we see there is a scattering-state continuum from $\lambda=0$ to $\lambda=L$. The corresponding solutions to these equations are

$$\sigma_\lambda=\frac{g_0f(\lambda)}{\alpha^{+}(\lambda)}\equiv g_\lambda^{*}\ , \quad (3.5a)$$

$$\rho_\lambda(k)=\delta(\vec{\lambda}-\vec{k})+\frac{g_0f(k)g_\lambda^{*}}{\lambda-k+i\epsilon} \equiv G_{\lambda k}\ , \quad (3.5b)$$

with

$$\alpha(z)=z-m_0-g_0{}^2\int d^3k\frac{f^2(k)}{z-k} \quad (3.6)$$

and $\alpha^{\pm}(\lambda)=\alpha(\lambda\pm i\epsilon)$.

It is well known that the discrete states are given by the roots of the equation

$$\alpha(z)=0\ . \quad (3.7)$$

By noting that $\lim_{x\to-\infty}\alpha(x)=-\infty$, $\alpha'(x)>0$, and that $\alpha(x)$ is continuous for $x\in[-\infty,0]$, it can be concluded that a necessary and sufficient condition for the existence of a discrete state at $\lambda=M_1<0$ is

$$\lim_{x\to 0^-}\alpha(x)>0\ . \quad (3.8)$$

As in the case of the separable potential, we assume that g_0 and m_0 are such that a discrete state $|V_1\rangle\rangle$ with an eigenvalue M_1 does indeed exist. To avoid the complications of discrete states in the continuum,[11] we also assume that $f(k)$ has no zeros.

Also, from the facts that $\alpha(x)$ is continuous for $x > L$ and that $\lim_{x\to+\infty}\alpha(x) = +\infty$, we see that a necessary and sufficient condition for a discrete state to exist at $\lambda = M_2 > L$ is

$$\lim_{x\to L^+} \alpha(x) < 0 \ .$$

Since $f(k)$ is assumed to have no zeros, the above-mentioned condition is indeed satisfied since the last term in Eq. (3.6) diverges logarithmically to $-\infty$ as $x \to L^+$. Hence, unlike in the separable-potential model, there is an additional discrete state beyond the cutoff.[12] We denote it by $|V_2\rangle\rangle$. We will subsequently see that in the SC limit, it is the presence of this state that is responsible for the difference between the Lee model and the separable-potential model.

The solution to Eq. (3.4) for the case of bound states is

$$\sigma_{M_i} = \sqrt{Z_i} \ , \qquad (3.9a)$$

$$\rho_{M_i}(k) = \frac{g_0 f(k)}{M_i - k}\sqrt{Z_i} \equiv F^i(k) \ . \qquad (3.9b)$$

Here, the normalization factor Z_i is defined by

$$Z_i = \left[1 + g_0{}^2 \int \frac{f^2(k) d^3k}{(M_i - k)^2}\right]^{-1} = \frac{1}{\alpha'(M_i)} \ . \qquad (3.10)$$

Z_i plays the role of the renormalization constant that relates the bare state $|V\rangle$ to the physical state $|V_i\rangle\rangle$ in the sense that $|\langle V | V_i \rangle\rangle|^2 = Z_i$.

Before proceeding with the calculation of the matrix elements of the Hamiltonian, we consider the behavior of certain relevant quantities in the SC limit. This limit is defined by first introducing a momentum cutoff on the integrals, and then taking the bare coupling to infinity. The cutoff may then be taken to infinity if required. Our results, however, do not depend on whether the cutoff is taken to infinity or not. We also require that the mass of the discrete state $|V_1\rangle\rangle$ is held at a finite, preassigned value $M_1 < 0$. Then, from

$$0 = \alpha(M_1) = M_1 - m_0 - g_0{}^2 \int \frac{f^2(k) d^3k}{M_1 - k} \ ,$$

we see that in the SC limit, in order to keep M_1 fixed, the bare mass m_0 has to be adjusted according to

$$m_0 \sim g_0{}^2 \ . \qquad (3.11a)$$

Noting that $\alpha(\lambda)$ defined in Eq. (3.6) can also be written as

$$\alpha(\lambda) = (\lambda - M_1) \times \left[1 - g_0{}^2 \int d^3k \frac{f^2(k)}{(k - M_1)(\lambda - k)}\right] ,$$

we get

$$0 = \alpha(M_2) = (M_2 - M_1)\left[1 - g_0{}^2 \int d^3k \frac{f^2(k)}{(k - M_1)(M_2 - k)}\right] ,$$

which implies

$$M_2 \sim g_0{}^2 \ \text{ in the SC limit} \ . \qquad (3.11b)$$

Also, from Eq. (3.10) we can easily see that in the SC limit,

$$Z_1 \sim \frac{1}{g_0{}^2} \qquad (3.11c)$$

and

$$Z_2 \sim (g_0)^0 \ . \qquad (3.11d)$$

We note also that for finite λ,

$$g_\lambda \sim \frac{1}{g_0} \ , \qquad (3.11e)$$

$$\alpha(\lambda) \sim g_0{}^2 \ , \qquad (3.11f)$$

and

$$G_{\lambda k} \sim (g_0)^0 \ , \qquad (3.11g)$$

also in the SC limit. We point out that since $\alpha(\lambda)$ has a zero at $\lambda = M_1$, and $\alpha'(M_1) = 1/Z_1$, $\alpha(\lambda) \sim (g_0)^0$ in the neighborhood $O(1/g_0{}^2)$ of M_1. This observation will play an essential role in the discussion of the strong-coupling behavior of the discrete state below the $V_1\theta$ threshold in the $V\theta$ sector of the Lee model. We note also that although M_2 and m_0 both move to infinity in the SC limit,

$$M_2 - m_0 = g_0{}^2 \int \frac{f^2(k)}{M_2 - k} d^3k \sim (g_0)^0 \ . \qquad (3.11h)$$

Lastly, from Eq. (3.9),

$$F^1(k) \sim (g_0)^0 \qquad (3.11i)$$

and

$$F^2(k) \sim \frac{1}{g_0} \ . \qquad (3.11j)$$

For the purpose of comparing our solutions (3.5) and (3.9) with those of the separable-potential model, we write

$$| V_i \rangle\rangle = \sqrt{Z_i}\, | V \rangle + \int d^3k\, F^i(k)\, | N\theta_k \rangle \tag{3.12a}$$

and

$$| N\theta \rangle\rangle_\lambda = g_\lambda^* \, | V \rangle + \int d^3k\, G_{\lambda k}\, | N\theta_k \rangle \,. \tag{3.12b}$$

In the SC limit, Eqs. (3.12) reduce to

$$| V_i \rangle\rangle \overset{s}{=} \int d^3k\, F^1(k)\, | N\theta_k \rangle \,, \tag{3.13a}$$

$$| V_2 \rangle\rangle \overset{s}{=} | V \rangle \,, \tag{3.13b}$$

and

$$| N\theta \rangle\rangle_\lambda \overset{s}{=} \int d^3k\, G_{\lambda k}\, | N\theta_k \rangle \,. \tag{3.13c}$$

Here, $\overset{s}{=}$ denotes equality in the SC limit. Thus, we see that in the SC limit, the finite-energy eigenstates $| V_1 \rangle\rangle$ and $| N\theta \rangle\rangle_\lambda$ have no contributions from the bare state $| V \rangle$ whereas the state $| V_2 \rangle\rangle$ is, in fact, coincident with the state $| V \rangle$. This leads us to interpret $| V_2 \rangle\rangle$ as the renormalized V particle with its mass shifted from its original value m_0 by a finite amount to M_2. The state $| V_1 \rangle\rangle$ may then be regarded, consistent with the compositeness condition $Z_1 = 0$ being satisfied, as being composed of an N and a θ particle. We now show that in the SC limit, Eqs. (3.13a) and (3.13c) reduce to Eqs. (3.12b) and (3.12a), respectively. To see this, we first note that for finite λ,

$$\begin{aligned} \alpha(\lambda) &= \lambda - m_0 - {g_0}^2 \int \frac{f^2(k) d^3k}{\lambda - k} \\ &\overset{s}{=} M_2 \left[-\frac{m_0}{M_2} - \frac{{g_0}^2}{M_2} \int \frac{f^2(k)}{\lambda - k} d^3k \right] \\ &\overset{s}{=} -M_2 \left[1 + \frac{{g_0}^2}{M_2} \int \frac{f^2(k)}{\lambda - k} d^3k \right] \,, \end{aligned} \tag{3.14}$$

where the last equality follows because $m_0/M_2 \overset{s}{=} 1$ as may be easily seen from Eq. (3.11h). With the identification of $h(k)$, the form factor of the separable-potential model with $(g_0/\sqrt{M_2})f(k)$, Eq. (3.14) can be written as

$$\alpha(\lambda) \overset{s}{=} -M_2 D(\lambda) \,, \tag{3.15}$$

where $D(\lambda)$ was defined in Eq. (2.8). Then, from Eq. (3.13a) it follows that

$$\begin{aligned} | V_1 \rangle\rangle &\overset{s}{=} \int d^3k \frac{1}{[\alpha'(M_1)]^{1/2}} \frac{g_0 f(k)}{M_1 - k} | N\theta_k \rangle \\ &\overset{s}{=} \int d^3k \frac{1}{[-D'(M_1)]^{1/2}} \frac{h(k)}{M_1 - k} | N\theta_k \rangle \,; \end{aligned} \tag{3.16a}$$

whereas from Eq. (3.13c) we obtain

$$\begin{aligned} | N\theta \rangle\rangle_\lambda &\overset{s}{=} \int d^3k \left[\delta(\vec{\lambda} - \vec{k}) + \frac{{g_0}^2 f(\lambda)}{(\lambda - k + i\epsilon)} \frac{f(k)}{\alpha^+(\lambda)} \right] | N\theta_k \rangle \\ &\overset{s}{=} \int d^3k \left[\delta(\vec{\lambda} - \vec{k}) - \frac{1}{D^+(\lambda)} \frac{h(\lambda) h(k)}{(\lambda - k + i\epsilon)} \right] | N\theta_k \rangle \,. \end{aligned} \tag{3.16b}$$

A comparison of Eqs. (3.16) and (2.12) shows that in the $N\theta$ sector the finite-energy eigenstates of the Lee model coincide with those of the separable-potential model in the SC limit provided the form factors are suitably chosen. As a result, the finite-energy scattering amplitudes and the S matrix as predicted by the two theories coincide in the SC limit. This follows since these depend only on the form of the eigenstates. Explicitly, for the Lee model, the S matrix takes the form

$$\begin{aligned} S_{\lambda\lambda'} &= \langle\langle N\theta, \text{out} \,|\, N\theta, \text{in} \rangle\rangle_{\lambda'} \\ &= \frac{\alpha^+(\lambda)}{a^-(\lambda)} \delta(\lambda - \lambda') \,. \end{aligned} \tag{3.17}$$

We point out, however, that in spite of the fact that the finite-energy state vectors and S matrix coincide in the SC limit, the spectra of the two theories are clearly different since the state $| V_2 \rangle\rangle$ is present only in the Lee model.

Moreover, the Lee-model Hamiltonian matrix elements

$$\begin{aligned} &\langle V | H | V \rangle = m_0 \,, \\ &\langle V | H | N\theta_k \rangle = g_0 f(k) \,, \\ &\langle N\theta_k | H | N\theta_l \rangle = k\delta(\vec{k} - \vec{l}) \,, \end{aligned} \tag{3.18}$$

are different from those of the separable-potential model [see Eq. (2.3)]. This is a direct consequence of the difference in the spectra of the two theories. In fact, we will show that the difference in the Hamiltonian matrix elements of the two theories in the bare states basis comes precisely from that part of the spectrum that is present in the Lee model but not in the separable-potential model. To reiterate, in the $N\theta$ sector, although we have shown that the finite-energy scattering amplitudes and S matrices for the two theories are the same in the SC limit, the theories in question are clearly nonidentical since the respective Hamiltonians differ even in this limit. We shall see in the next section that to make them identical, all spectral contributions to the Lee-model Hamiltonian that arise due to the extra state $|V_2\rangle\rangle$ have to be removed. (We recall that these did not contribute to the finite-energy scattering amplitudes or the S matrix in the SC limit.) Then, the Lee-model Hamiltonian is transmuted into that of the separable potential, and the two theories become truly equivalent. This, as we shall see, holds for the $N\theta\theta$ sector of the model as well.

The $V\theta$ ($N\theta\theta$) sector

As in the case of the lower sector, we denote the eigenstates of the complete Hamiltonian by $|\ \rangle\rangle_\lambda$, with λ labeling the eigenvalue and the entry inside the ket denoting the state under consideration. We also denote the discrete states with eigenvalues Λ_j by $|\Lambda_j\rangle\rangle$.

The eigenstates of the Hamiltonian can be expanded in terms of the bare states $|V\theta_k\rangle$ and $|N\theta_k\theta_l\rangle$ as

$$|\ \rangle\rangle_\lambda=\int d^3k\,\psi_\lambda(k)\,|V\theta_k\rangle+\tfrac{1}{2}\int d^3k\,d^3l\,\psi_\lambda(k,l)\,|N\theta_k\theta_l\rangle\ . \qquad (3.19)$$

The equations of motion for the Källén-Pauli components

$$\psi_\lambda(k)=\langle V\theta_k\,|\ \rangle\rangle_\lambda$$

and

$$\psi_\lambda(k,l)=\langle N\theta_k\theta_l\,|\ \rangle\rangle_\lambda$$

are

$$(\lambda-m_0-k)\psi_\lambda(k)=g_0\int d^3p\,f(p)\psi_\lambda(k,p) \qquad (3.20a)$$

and

$$(\lambda-k-l)\psi_\lambda(k,l)=g_0f(l)\psi_\lambda(k)+g_0f(k)\psi_\lambda(l)\ . \qquad (3.20b)$$

As has been suggested by Bolsterli[13] who originally solved this sector of the Lee model but without any cutoff, this system of integral equations is most conveniently solved by introducing an overcomplete set of basis vectors, $a^\dagger(k)\,|V_i\rangle\rangle$, $a^\dagger(k)\,|N\theta\rangle\rangle_l$, $V^\dagger N\,|V_i\rangle\rangle$, and $V^\dagger N\,|N\theta\rangle\rangle_p$. Accordingly, we introduce the amplitudes

$$\langle\langle V_i\,|a(l)|\ \rangle\rangle_\lambda=\phi_\lambda{}^i(l)\ , \qquad (3.21a)$$

$$_k\langle\langle N\theta\,|a(l)|\ \rangle\rangle_\lambda=\phi_\lambda(k,l)\ , \qquad (3.21b)$$

$$\langle\langle V_i\,|N^\dagger V|\ \rangle\rangle_\lambda=\chi_\lambda{}^i\ , \qquad (3.21c)$$

and

$$_k\langle\langle N\theta\,|N^\dagger V|\ \rangle\rangle_\lambda=\chi_\lambda(k)\ . \qquad (3.21d)$$

For brevity, we have suppressed the degeneracy index for the $N\theta\theta$ scattering states. By considering the action of the Hamiltonian on each element of the aforementioned overcomplete basis, the equations of motion can be written in the form

$$(\lambda-M_i-l)\phi_\lambda{}^i(l)=g_0f(l)\chi_\lambda{}^i\ , \qquad (3.22a)$$

$$(\lambda-k-l)\phi_\lambda(k,l)=g_0f(l)\chi_\lambda(k)\ , \qquad (3.22b)$$

$$(\lambda-M_i-m_0)\chi_\lambda{}^i=g_0\int d^3l\,f(l)\phi_\lambda{}^i(l)-2\sqrt{Z_i}g_0\int d^3k\,f(k)\sum_j\sqrt{Z_j}\phi_\lambda{}^i(k)-2\sqrt{Z_i}g_0\int d^3k\,d^3l\,f(k)g_l^*\phi_\lambda(l,k) \qquad (3.22c)$$

and

$$(\lambda-k-m_0)\chi_\lambda(k)=g_0\int d^3l\,f(l)\phi_\lambda(k,l)-2g_0g_k\int d^3p\,f(p)\sum_j\sqrt{Z_j}\phi_\lambda{}^j(p)-2g_0g_k\int d^3p\,d^3q\,f(p)g_q^*\phi_\lambda(q,p)\ . \qquad (3.22d)$$

It is important to note that the amplitudes introduced in Eqs. (3.21) are not linearly independent. This is a reflection of the fact that the vectors $a^\dagger(l)\,|\,V_i\rangle\rangle$ and $a^\dagger(l)\,|\,N\theta\rangle\rangle_k$, though oblique, form a basis. By writing the vectors $V^\dagger N\,|\,V_i\rangle\rangle$ and $V^\dagger N\,|\,N\theta\rangle\rangle_k$ in terms of the above-mentioned oblique basis, the functions $\chi_\lambda{}^i$ and $\chi_\lambda(k)$ can be expressed in terms of the functions $\phi_\lambda{}^i(l)$ and $\phi_\lambda(k,l)$ as

$$\chi_\lambda{}^i=\int d^3p\sum_j\sqrt{Z_j}\phi_\lambda{}^j(p)F^i(p)+\int d^3p\,d^3q\,F^i(p)g_q^*\phi_\lambda(q,p) \tag{3.23a}$$

and

$$\chi_\lambda(k)=\int d^3p\,G_{kp}^*\sum_j\sqrt{Z_j}\phi_\lambda{}^j(p)+\int d^3p\,d^3q\,G_{kp}^*g_q^*\phi_\lambda(q,p)\ . \tag{3.23b}$$

The full content of the Lee model is realized by solving the equations of motion (3.22) subject to the constraints, Eqs. (3.23).

We proceed by first obtaining all possible solutions to the system of equations (3.22) and then ruling out those that do not satisfy Eqs. (3.23). The equations of motion are solved by first eliminating the functions $\phi_\lambda{}^i(l)$ and $\phi_\lambda(k,l)$ from Eqs. (3.22c) and (3.22d) using Eqs. (3.22a) and (3.22b) and then solving these for the functions $\chi_\lambda{}^i$ and $\chi_\lambda(k)$. The details are presented in Appendix B.

We may now turn to the analysis of the condition for the existence of discrete states. For a discrete state at $\lambda=\Lambda$, we have from Eqs. (3.22a) and (3.22b) that

$$\phi_\Lambda{}^i(l)=\frac{g_0f(l)}{\Lambda-M_i-l}\chi_\Lambda{}^i \tag{3.24a}$$

and

$$\phi_\Lambda(k,l)=\frac{g_0f(l)\chi_\Lambda(k)}{\Lambda-k-l}\ . \tag{3.24b}$$

Using these, the last two equations of motion can be written in the form

$$\alpha(\Lambda-M_i)\chi_\Lambda{}^i=-2\sqrt{Z_i}\,Y_\Lambda \tag{3.24c}$$

and

$$\alpha(\Lambda-k)\chi_\Lambda(k)=-2g_kY_\Lambda\ , \tag{3.24d}$$

with

$$Y_\Lambda=g_0\int d^3p\,f(p)\sum_j\sqrt{Z_j}\phi_\Lambda{}^j(p)+g_0\int d^3p\,d^3q\,f(p)g_q^*\phi_\Lambda(q,p)\ . \tag{3.24e}$$

If $\Lambda\neq 2M_1$, $2M_2$, or M_1+M_2, Eqs. (3.24c) and (3.24d) can be solved for $\chi_\Lambda{}^i$ and $\chi_\Lambda(k)$. Using these solutions (we will consider the exceptions later), together with Eqs. (3.24a) and (3.24b) in Eq. (3.24e), we obtain

$$\left[1+2\sum_j Z_j\frac{A_{\Lambda-M_j}}{\alpha(\Lambda-M_j)}+I_\Lambda\right]Y_\Lambda=0\ ,$$

with

$$A_\lambda^\pm=g_0{}^2\int d^3l\frac{f^2(l)}{\lambda-l\pm i\epsilon}$$

and

$$I_\lambda^\pm=2\int d^3k\,|\,g_k\,|^2A_{\lambda-k}^\pm/\alpha^\pm(\lambda-k)\ .$$

There are, therefore, two possibilities for obtaining discrete eigenvalue solutions to the system of Eqs. (3.22), viz.,

$$1+2\sum_j Z_j\frac{A_{\Lambda-M_j}}{\alpha(\Lambda-M_j)}+I_\Lambda=0 \tag{3.25a}$$

or

$$Y_\Lambda=0\ . \tag{3.25b}$$

As can be seen from Eq. (3.24c), the second is a possibility only if $\Lambda=2M_1$, $2M_2$ or M_1+M_2. As mentioned previously, we will consider these cases later.

The condition, Eq. (3.25a), can be rewritten in terms of the function K_λ defined by

$$K_\lambda^\pm=-\int d^3k\frac{|\,g_k\,|^2}{\alpha^\pm(\lambda-k)} \tag{3.26}$$

as

$$(\Lambda-2m_0)\left[\frac{Z_1}{\alpha(\Lambda-M_1)}+\frac{Z_2}{\alpha(\Lambda-M_2)}-K_\Lambda\right]=0\ . \tag{3.27}$$

[See Appendix A, Eq. (A15).]

We now analyze the possibility $Y_\Lambda=0$. As can be seen from Eq. (3.24c) this is possible only if $\Lambda=2M_1$, $2M_2$, or M_1+M_2. Also, from Eqs. (3.24d) and (3.24b), we have $\chi_\Lambda(k)=\phi_\Lambda(k,l)=0$. Finally, if $\Lambda=2M_1$ or $2M_2$ (let us say $2M_1$ for definiteness), we see from Eqs. (3.24c) and (3.24a) that $\chi_\Lambda{}^2=\phi_\Lambda{}^2=0$. Then Eqs. (3.24e) and (3.24a) lead to the null solution. For $\Lambda=M_1+M_2$, it is simple to check that the relation

$$\chi_\Lambda{}^2=-\left[\frac{Z_1}{Z_2}\right]^{1/2}\frac{M_2-m_0}{M_1-m_0}\chi_\Lambda{}^1$$

implied by Eq. (3.25b) together with $\chi_\Lambda(k)=0$ and Eqs. (3.24a) and (3.24b) constitute a nontrivial solution to Eqs. (3.22). This solution, however, does not satisfy the constraint equations (3.23) and hence is unacceptable.

The set of admissible eigenvalues is further reduced by noting that the solution [see Eqs. (3.31)] corresponding to $\Lambda=2m_0$ again does not satisfy the constraints, Eqs. (3.23). The set of possible discrete eigenvalues of the Lee-model Hamiltonian in the $V\theta$ sector is, therefore, given by the roots of the equation

$$\eta(\Lambda)\equiv\frac{Z_1}{\alpha(\Lambda-M_1)}+\frac{Z_2}{\alpha(\Lambda-M_2)}-K_\Lambda=0\ . \tag{3.28}$$

Before proceeding to locate the roots of Eq. (3.28), we present the solutions to the system of equations (3.22) corresponding to different possible states in the $V\theta$ sector of the Lee model.

$V\theta$ scattering states

In terms of the quantity $\xi_i=\lambda-M_i$, we have

$$\chi_\lambda{}^i=\tilde{g}^*_{\xi_i}-\frac{2\sqrt{Z_i}}{\alpha^\dagger(\xi_i)}\frac{\sqrt{Z_1}\tilde{g}^*_{\xi_1}+\sqrt{Z_2}\tilde{g}^*_{\xi_2}}{\eta^+(\lambda)} \tag{3.29a}$$

with

$$\tilde{g}^*_{\xi_i}=g^*_{\xi_i}\theta(L-\xi_i)\theta(\xi_i)\ ,$$

$$\phi_\lambda{}^i(l)=\delta(\vec{\xi}_i-\vec{l}\,)+\frac{g_0f(l)\chi_\lambda{}^i}{\xi_i-l+i\epsilon}\ , \tag{3.29b}$$

$$\chi_\lambda(k)=\sqrt{Z_1}\delta(\vec{\xi}_1-\vec{k})+\sqrt{Z_2}\delta(\vec{\xi}_2-\vec{k})+\frac{g_k}{\alpha^+(\lambda-k)}\frac{[\alpha^+(\xi_i)\chi_\lambda{}^i-g_0\tilde{f}(\xi_i)]}{\sqrt{Z_i}} \tag{3.29c}$$

with

$$\tilde{f}(\xi_i)=f(\xi_i)\theta(\xi_i)\theta(L-\xi_i)$$

and

$$\phi_\lambda(k,l)=\frac{g_0f(l)\chi_\lambda(k)}{\lambda-k-l+i\epsilon}\ . \tag{3.29d}$$

$N\theta\theta$ scattering states

Write $\lambda=\zeta_1+\zeta_2$, $0\le\zeta_1,\ \zeta_2\le L$. Then,

$$\chi_\lambda{}^i=-\frac{2\sqrt{Z_i}}{\alpha^+(\lambda-M_i)}\frac{g^*_{\zeta_1}g^*_{\zeta_2}}{\eta^+(\lambda)}\ , \tag{3.30a}$$

$$\chi_\lambda(k)=g^*_{\zeta_1}\delta(\vec{\zeta}_2-\vec{k})+g^*_{\zeta_2}\delta(\vec{\zeta}_1-\vec{k})+\frac{g_k}{\alpha^+(\lambda-k)}\frac{\alpha^+(\lambda-M_i)\chi_\lambda{}^i}{\sqrt{Z_i}}\ , \tag{3.30b}$$

$$\phi_\lambda{}^i(l)\frac{g_0f(l)\chi_\lambda{}^i}{\lambda-M_i-l}\ , \tag{3.30c}$$

and

$$\phi_\lambda(k,l)=\delta(\vec{\zeta}_1-\vec{k})\delta(\vec{\zeta}_2-\vec{l}\,)+\delta(\vec{\zeta}_1-\vec{l}\,)\delta(\vec{\zeta}_2-\vec{k})+\frac{g_0f(l)\chi_\lambda(k)}{\lambda-k-l+i\epsilon}\ . \tag{3.30d}$$

Discrete states

We first present the solutions for the discrete states. The conditions for their existence will be subsequently discussed. For a discrete state with eigenvalue Λ [note that it follows from the previous discussion that $\Lambda\neq2M_1$, $2M_2$ or M_1+M_2; also, since $f(l)$ never vanishes for $0\le l\le L$, it follows that Λ lies outside the continuum between 0 and $2L$],

$$\chi_\Lambda{}^1=c\ , \tag{3.31a}$$

$$\chi_\Lambda{}^2=\left[\frac{Z_2}{Z_1}\right]^{1/2}\frac{\alpha(\Lambda-M_1)}{\alpha(\Lambda-M_2)}c\ , \tag{3.31b}$$

$$\chi_\Lambda(k)=\frac{g_k}{\alpha(\Lambda-k)}\frac{\alpha(\Lambda-M_1)}{\sqrt{Z_1}}c\ , \tag{3.31c}$$

$$\phi_\Lambda{}^i(l)=\frac{g_0f(l)}{\Lambda-M_i-l}\chi_\Lambda{}^i\ , \tag{3.31d}$$

and

$$\phi_\Lambda(k,l)=\frac{g_0f(l)\chi_\Lambda(k)}{\Lambda-k-l}\ . \tag{3.31e}$$

The normalization c is determined by the condition

$$\langle\langle\Lambda\mid\Lambda\rangle\rangle=1\ .$$

We find (see Appendix B)

$$|c|^2=-\frac{2K_\Lambda}{\eta'(\Lambda)\alpha(\Lambda-M_1)}\times\frac{1}{1+(Z_2/Z_1)\alpha(\Lambda-M_1)/\alpha(\Lambda-M_2)}\ . \tag{3.31f}$$

By writing the Källén-Pauli components in terms of the overcomplete-basis components as

$$\psi_\lambda(l)=\sqrt{Z_1}\phi_\lambda{}^1(l)+\sqrt{Z_2}\phi_\lambda{}^2(l) + \int d^3p\, g_p^* \phi_\lambda(p,l) \quad (3.32a)$$

and

$$\psi_\lambda(k,l)=\phi_\lambda{}^1(k)F^1(l)+\phi_\lambda{}^2(k)F^2(l) + \int d^3p\, G_{pk}\phi_\lambda(p,l) , \quad (3.32b)$$

it can be verified that our solutions satisfy Eqs. (3.20) discussed earlier.

Having presented the solutions that are possible in the $V\theta$ sector of the Lee model, we now turn our attention to investigating the existence of discrete eigenvalues. As already discussed, these are given by solutions of

$$\eta(x)=0 .$$

The search for real zeros of $\eta(x)$ rather naturally breaks up the domain of possibilities into three parts, viz.,

(a) $x<M_1$,

(b) $x>M_2+L$,

and

(c) $2L<x<M_2$.

The last is possible only if $M_2>2L$. Since we are interested in the SC limit, in which event $2L << M_2$, this case is clearly of interest. We examine each of these, in turn.

(a) If $x<2M_1$, $\alpha(x-k)<0$, and hence, from Eq. (3.26) $K_x>0$. Also, $\alpha(x-M_1)<0$ and $\alpha(x-M_2)<0$. As a result, each term on the right-hand side of Eq. (3.28) is negative, and hence, there are no eigenvalues for $x<2M_1$. A necessary and sufficient condition for there to be a discrete state at $x=\Lambda_1$, with $2M_1\leq\Lambda_1\leq M_1$, is

$$\left| -K_{M_1}+\frac{Z_2}{\alpha(M_1-M_2)} \right| > \frac{Z_1}{\alpha(0^-)} . \quad (3.33)$$

For the case of the Lee model without a cutoff, the condition for the existence of a discrete state below the $V\theta$ threshold is the inequality (3.33), but with $Z_2=0$.[13] It is amusing to note that if such a state exists in the absence of any cutoff, (3.33) mandates its presence in a cutoff theory. The eigenvalue, however, shifts to an algebraically lower value due to the cutoff. There are no other discrete states for $x<0$.

It is interesting to note that in the SC limit, $\Lambda_1\to 2M_1$. To see this, we first note that in the SC limit,

$$\alpha(\Lambda_1-M_2)\sim {g_0}^2 .$$

The bound-state condition $\eta(\Lambda_1)=0$ then readily yields

$$\frac{\alpha(\Lambda_1-M_2)}{\alpha(\Lambda_1-M_1)}\sim {g_0}^2 ,$$

i.e., $\alpha(\Lambda_1-M_1)\sim(g_0)^0$ and not as ${g_0}^2$ as might be expected from naively counting powers of g_0 as in Eq. (3.11f). It then follows from the equation

$$\alpha(\Lambda_1-M_1)=(\Lambda_1-2M_1)\left[1+{g_0}^2\int d^3k\frac{f^2(k)}{(\Lambda_1-M_1-k)(M_1-k)}\right]$$

that $\Lambda_1\to 2M_1+O(1/{g_0}^2)$ in the SC limit.

(b) We now turn to the examination of discrete states beyond the cutoff. It can be easily seen that Eq. (3.28) has not roots for $x>2M_2$. We now consider the possibility of a root at x such that $M_2+L<x<2M_2$. To this end, we rewrite Eq. (3.28) as

$$\eta(x)=\frac{Z_1}{\alpha(x-M_1)}+\frac{Z_2}{\alpha(x-M_2)}+\int d^3k\frac{|g_k|^2}{\alpha(x-k)}=0 . \quad (3.34)$$

We first note that $\eta(x)\to-\infty$ as $x\to 2M_2$ from below. Moreover, recalling that $\alpha(L^+)$ diverges and noting that $\alpha(M_2+L-M_1)$ and $\alpha(M_2+L-k)$ are both non-negative, we have from Eq. (3.34) that $\eta(M_2+L)>0$. Finally, by writing $\eta(x)$ in the form

$$\eta(x)=\frac{{Z_1}^2}{x-2M_1}+\frac{2Z_1Z_2}{x-M_1-M_2}+\frac{{Z_2}^2}{x-2M_2}+\int d^2l\,|g_l|^2\left[\frac{2Z_1}{x-l-M_1}+\frac{2Z_2}{x-l-M_2}\right] + \int d^3k\, d^3l\frac{|g_k|^2|g_l|^2}{x-k-l} , \quad (3.35)$$

we see that $\eta(x)$ is continuous and monotonically decreasing in the domain under consideration and hence passes through a single zero. There is, therefore, one discrete state at Λ_2, with $\Lambda_2 \in [M_2+L, 2M_2]$. It is easy to see that the difference, $\Lambda_2-(M_2+L)$, remains finite in the SC limit, since in this limit, Eq. (3.32) reduces to

$$\eta(\Lambda_2)=\frac{2Z_1}{\Lambda_2-M_1-M_2}+\frac{1}{\Lambda_2-2M_2}+2\int d^3l\frac{|g_l|^2}{\Lambda_2-M_2-l} .$$

It follows then that $\eta(\Lambda_2)=0$ can be satisfied only if Λ_2 is finitely removed from M_2. (Recall $\Lambda_2<2M_2$.)

(c) Finally, since M_2 always exceeds $2L$ in the SC limit, the domain $2L<x<M_2$ has also to be examined for possible roots of Eq. (3.28). For $2L<x<M_1+M_2$, $\alpha(x-M_1)$ and $\alpha(x-M_2)$ are both negative, whereas $K_x>0$. As a result, $\eta(x)<0$, leading us to conclude the absence of a discrete state in this domain. We now turn to the possibility of discrete states in the domain $M_1+M_2<x<M_2$. We note from Eq. (3.35) that $\eta'(x)<0$ in this domain and that $\eta(x)$ diverges to $+\infty$ as $x\rightarrow(M_1+M_2)^+$. We now show that $\eta(M_2)>0$, so that $\eta(x)$ remains positive in the whole domain.

From Eq. (3.35), we have

$$\eta(M_2)=\frac{Z_1^2}{M_2-2M_1}+\frac{2Z_1Z_2}{-M_1}+\frac{Z_2^2}{-M_2}+\int d^3l\,|g_l|^2\left[\frac{2Z_1}{M_2-l-M_1}+\frac{2Z_2}{-l}\right]+d^3k\,d^3l\frac{|g_k|^2|g_l|^2}{M_2-k-l} . \tag{3.36a}$$

Also, we have

$$\frac{2Z_1Z_2}{-M_1}+\frac{Z_2^2}{-M_2}+2Z_2\int d^3l\frac{|g_l|^2}{-l} =\frac{Z_2^2}{M_2}+\frac{2Z_2}{\alpha(0^-)}>0 \tag{3.36b}$$

since Z_2, M_2, and $\alpha(0^-)$ are all positive. Finally, since $M_2>2L$ in our present considerations, we note that those terms in Eq. (3.36b) that have not been included on the left-hand side of Eq. (3.36c) are all positive, i.e., $\eta(M_2)>0$. We are thus led to conclude that there is no discrete state for $M_1+M_2\leq x\leq M_2$.

To summarize, the spectrum of the Lee model in the $V\theta$ sector consists of (i) an $N\theta\theta$ continuum from $\lambda=0$ to $\lambda=2L$, (ii) a $V_1\theta$ continuum from $\lambda=M_1$ to $\lambda=M_1+L$, (iii) a $V_2\theta$ continuum from $\lambda=M_2$ to $\lambda=M_2+L$, and (iv) discrete states at Λ_1 and Λ_2 with $2M_1<\Lambda_1<M_1$ and $M_2+L<\Lambda_2<2M_2$. In the SC limit, $\Lambda_1\rightarrow 2M_1$ whereas Λ_2 remains finitely removed from M_2+L.

This spectrum is illustrated in Fig. 1.

The eigenstates can be written in terms of the functions $\phi_\lambda{}^i(l)$, $\phi_\lambda(k,l)$ without any mention of $\chi_\lambda{}^i$ or $\chi_\lambda(k)$ because the vectors $a^\dagger(l)\,|\,V_i\rangle\rangle$ and $a^\dagger(l)\,|\,N\theta\rangle\rangle_k$ form a complete set. It can then be directly verified using our solutions together with the SC behavior of the various quantities discussed earlier in this section, that the states $|\,V_1\theta\rangle\rangle$, $N\theta\theta\rangle\rangle$ and the bound state below the threshold (these would be the only eigenstates in the absence of any cutoff) do not have any contribution from the bare V particle in the SC limit, i.e., $\psi_\lambda(l)$ vanishes for these states. Moreover, for these states, $\psi_\lambda(k,l)$ has no contribution from the $\phi_\lambda{}^2(l)$ term in Eq. (3.32b). (Note that this term is the contribution of the state $|\,V_2\rangle\rangle$.) In fact, the solutions [Eqs. (3.29), (3.30), and (3.31)] for the $|\,V_1\theta\rangle\rangle$, $|\,N\theta\theta\rangle\rangle$, and $|\,\Lambda_2\rangle\rangle$ states in the SC limit take the forms

$$\phi_\lambda{}^1(l)\overset{s}{=}\delta(\vec{\xi}_1-\vec{l})+\frac{g_0f(l)f(\lambda-M_1)}{(\xi_1-l)\alpha(\lambda-M_1)}$$

and

$$\phi_\lambda(k,l)\overset{s}{=}\frac{g_0f(l)}{\lambda-k-l}\frac{1}{\alpha'(M_1)}\delta(\vec{\xi}_1-\vec{k})\quad\text{for the }|\,V_1\theta\rangle\rangle\text{state} ,$$

$$\phi_\lambda{}^1(l)\overset{s}{=}0 ,$$

$$\phi_\lambda(k,l)\overset{s}{=}\delta(\vec{\xi}_1-\vec{k})\delta(\vec{\xi}_2-\vec{l})+\delta(\vec{\xi}_2-\vec{k})\delta(\vec{\xi}_1-\vec{l})$$
$$+\frac{g_0f(l)}{\lambda-k-l+i\epsilon}\left[\frac{f(\xi_1)}{\alpha^+(\xi_1)}\delta(\vec{\xi}_2-\vec{k})+\frac{f(\xi_2)}{\alpha^+(\xi_2)}\delta(\vec{\xi}_1-\vec{k})\right]\quad\text{for the }|\,N\theta\theta\rangle\rangle_\lambda\text{ state},$$

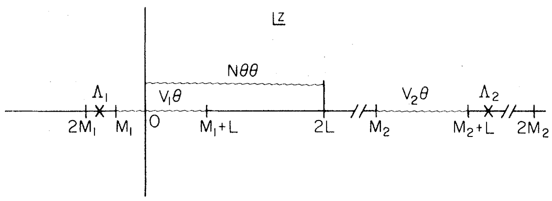

FIG. 1. The spectrun of the Lee model in the $N\theta\theta$ sector.

and finally,

$$\phi_{\Lambda_1}(l) \stackrel{s}{=} \frac{g_0 f(l)}{M_1 - l} \frac{\sqrt{2}}{\alpha'(M_1)}$$

and

$$\phi_{\Lambda_1}(k,l) \stackrel{s}{=} 0 \quad \text{for the} \mid \Lambda_1 \rangle\rangle \text{ state.}$$

Once again, we see using Eq. (3.15) and identifying the form factors $h(k)$ and $g_0 f(k)/\sqrt{M_2}$ that in the SC limit the solutions for these states reduce to the corresponding solutions Eqs. (2.16), (2.15), and (2.17) for the separable-potential model, i.e., $\phi_\lambda{}^1(l) \stackrel{s}{=} \Omega_\lambda(l)$, $\phi_\lambda(k,l) \stackrel{s}{=} \Omega_\lambda(k,l)$, and $\phi_\lambda{}^2(l) = 0$. At this point we remind the reader that $\Lambda_1 \stackrel{s}{=} 2M_1$. The state $\mid \Lambda_1 \rangle\rangle$ is identified with the bound state of the B and θ that occurs in the separable-potential model. We are thus led to interpret it as a bound state of the V_1 and the θ particles in the Lee model.

As with the case of the lower sector, since the wave functions for the finite-energy eigenstates of the Lee model and for the separable-potential model become identical, we are led to conclude that the scattering amplitudes and the S matrix for the two models are the same at finite energies, i.e., the agreement is not due to some fortuitous accident in the lower sector.

We now turn our attention to the matrix elements of the Hamiltonian of the Lee model. Direct computation of these from Eq. (3.1) yields

$$\langle V\theta_k \mid H \mid V\theta_l \rangle = (m_0 + l)\delta(\vec{k} - \vec{l}) , \tag{3.37a}$$

$$\langle V\theta_k \mid H \mid N\theta_p\theta_q \rangle = g_0 f(p)\delta(\vec{k} - \vec{q}) + g_0 f(q)\delta(\vec{k} - \vec{p}) , \tag{3.37b}$$

$$\langle N\theta_k\theta_l \mid H \mid N\theta_p\theta_q \rangle = (k + l)[\delta(\vec{k} - \vec{p})\delta(\vec{l} - \vec{q}) + \delta(\vec{k} - \vec{q})\delta(\vec{l} - \vec{p})] . \tag{3.37c}$$

Once again, we see that although the finite-energy wave functions, scattering amplitudes, and S matrices calculated from the Lee model and the separable-potential model coincide in the SC limit, the Hamiltonian matrix elements definitely differ. As with the case of the lower sector, the transmutation of these to the Hamiltonian matrix elements of the separable-potential model so as to make the two theories essentially identical, requires that the spectral contributions to the Hamiltonian, from that part of the spectrum which arises due to the cutoff, be explicitly removed. This transmutation mechanism forms the subject of the next section.

IV. THE TRANSMUTATION OF THE LEE MODEL INTO THE SEPARABLE-POTENTIAL MODEL

In the previous sections, we have concluded that although the finite-energy scattering amplitudes and S matrices for the Lee model and the separable-potential model are identical in the SC limit, the spectra of the two theories are clearly distinct since the Lee model develops spectral contributions at infinity which do not arise in the separable potential. Thus, the compositeness condition $Z_1 = 0$ does not ensure complete identity of the two theories. In this section, we demonstrate that if the spectral contributions of these extra states are explicitly removed, the bare V particle decouples from the theory and the Lee model becomes essentially equivalent to the separable-potential model in the SC limit. (This limit is required to ensure $Z_1 = 0$.) We have carried out our calculations for the $N\theta$ and the $N\theta\theta$ sectors so as to guard against the possibility that the "transmutation property" is accidental in the lower sector.

Dynamical rearrangement of the Hamiltonian

We denote by $\mid x \rangle$ the eigenstates of the free Hamiltonian and by $\mid z \rangle\rangle$ those of the total Hamiltonian. Then, using the closure relations we write

$$\langle x' \mid H \mid x \rangle = \sum_{\mid z \rangle\rangle} z \langle x' \mid z \rangle\rangle \langle\langle z \mid x \rangle , \tag{4.1}$$

where $H \mid z \rangle\rangle = z \mid z \rangle\rangle$. Again, for brevity, the degeneracy index is suppressed. The wave functions $\langle x \mid z \rangle\rangle$ for the $N\theta$ sector are given by Eqs. (3.12) while those for the $N\theta\theta$ sector can be calculated using our solutions together with Eqs. (3.30).

We have performed a rather lengthy calculation to verify that the dynamically rearranged right-hand side of Eq. (4.1) indeed reproduces the matrix elements listed in Eqs. (3.18) and (3.37) for the $N\theta$ and $V\theta$ sectors, respectively. Extensive use has been made of the compendium of integrals in Appendix A.

The transmutation of the Lee model

As discussed above, the dynamical rearrangement cannot, and does not, alter the content of the Lee model. We now show that if we take the SC limit, and explicitly exclude from the sum in Eq. (4.1) all spectral contributions that have their roots in the presence of a cutoff, the Lee model is essentially reduced to that for the separable potential. In the $N\theta$ sector the excluded part of the spectrum consists of the isolated point at M_2 whereas for the $V\theta$ sector the excluded parts are the discrete state beyond the cutoff together with the $|V_2\theta\rangle\rangle$ continuum. We remind the reader that in both sectors, the excluded contributions move to infinity in the SC limit. The Hamiltonian thus separates into two parts, each of which consists exclusively of contributions from the parts of the spectrum that remain finite in the SC limit or from those parts that move to infinity in the same limit.

We now consider the lower-sector case in some detail. Let us first concentrate, for definiteness, on the matrix element

$$\langle V|H|V\rangle = M_1\langle V|V_1\rangle\rangle\langle\langle V_1|V\rangle + \int d^3p\,p\langle V|N\theta\rangle\rangle_{p\,p}\langle\langle N\theta|V\rangle + [M_2\langle V|V_2\rangle\rangle\langle\langle V_2|V\rangle]\ . \tag{4.2}$$

In Eq. (4.2), and in what follows, that part of the matrix element which arises from the spectral contributions at infinity is enclosed in the square brackets. Using Eqs. (3.5) and (3.9) these can be readily evaluated to yield

$$\begin{aligned} M_1Z_1 + \int d^3p\,|g_p|^2 p + [M_2Z_2] &= M_1Z_1 + \frac{1}{2i\pi}\int_0^L dp\,p\left[\frac{1}{\alpha^-(p)} - \frac{1}{\alpha^+(p)}\right] + [M_2Z_2] \\ &= M_1Z_1 - \frac{1}{2i\pi}\int_\Gamma dp\frac{p}{\alpha(p)} + [M_2Z_2] \\ &= M_1Z_1 - M_1Z_1 - M_2Z_2 + m_0 + [M_2Z_2] \stackrel{s}{=} m_0 - M_2 + [M_2]\ , \end{aligned} \tag{4.3a}$$

where Γ is the contour illustrated in Fig. 2. We remind the reader that the symbol, $\stackrel{s}{=}$, denotes equality in the SC limit.

In a similar manner, we readily obtain

$$\langle V|H|N\theta_k\rangle = 0 + [g_0 f(k)] \tag{4.3b}$$

and

$$\langle N\theta_k|H|N\theta_l\rangle = k\delta(\vec{k}-\vec{l}) - \frac{g_0{}^2 f(k)f(l)}{M_2} + \frac{[g_0{}^2 f(k)f(l)]}{M_2}\ . \tag{4.3c}$$

We thus see from Eqs. (4.3) and (2.11) that, at least in the lowest nontrivial sector, in the SC limit, when the spectral contributions due to the single state at infinity are removed (these are the ones in the square brackets), the V particle decouples from the N and θ particles. Then the Lee model and the separable-potential model, with the identification of the form factors $g_0 f(k)/\sqrt{M_2}$ and $h(k)$, become effectively identical. We emphasize that the removal of the additional spectral contributions is an essential step in this demonstration since, as is obvious from Eqs. (4.3), if this is not done the

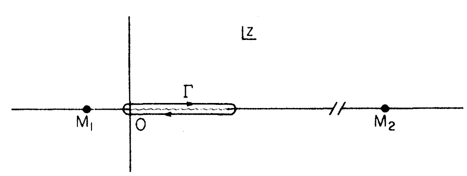

FIG. 2. The contour Γ that occurs in Eq. (4.5a) of the text and Eq. (A16) of Appendix A.

Lee-model matrix elements are reproduced as should be the case.

We now show that this "strong-coupling" transmutation can also be done in the $V\theta$ sector of the Lee model. To see this, we have once again to isolate all the spectral contributions that arise due to the cutoff. Once again, these move to infinity in the SC limit. The separation of these contributions is most simply achieved by noting that the sum in Eq. (4.1) consists of quantities of the generic form

$$\sum_{|z\rangle\rangle} z A_z B_z^* ,$$

where A_z and B_z are any one of $\phi_z{}^j(k)$ or $\phi_z(k,l)$. Using the equations of motion (3.22) the above sums can be reduced to linear combinations of the form

$$\sum_{|z\rangle\rangle} C_z D_z^* , \tag{4.4}$$

where C_z and D_z are any one of $\phi_z{}^j(k)$, $\phi_z(k,l)$, $\chi_z{}^j$, or $\chi_z(k)$. Before evaluating these sums we use the relations in Eqs. (3.23) to eliminate the functions $\chi_z{}^i$ and $\chi_z(k)$ from the sum in (4.4). The isolation of the spectral contributions that arise due to cutoff is simply achieved by separating from the sum in (4.4) all terms with eigenvalues z that move to infinity in the SC limit, i.e., those due to the $|V_2\theta\rangle\rangle$ scattering states and all the discrete states beyond the cutoff. We have explicitly verified that the contributions of the $a^\dagger(k)\,|V_1\rangle\rangle$ and $a^\dagger(k)\,|N\theta\rangle\rangle_l$ components of the states "at infinity" to the sum (4.4) vanish in the SC limit. Those from the $a^\dagger(k)\,|V_2\rangle\rangle$ component of the finite-energy states $|V_1\theta\rangle\rangle_\lambda$, $|N\theta\theta\rangle\rangle_\lambda$, and $|\Lambda_1\rangle\rangle$ also vanish in this limit. That this occurs can be readily seen by merely counting the leading powers of g_0 in the solutions presented in Sec. III and using these to compute the g_0 behavior of the sum (4.4). In doing so, particular care needs to be taken in dealing with the discrete state at Λ_1 below the $V_1\theta$ threshold, since as we saw in the last section, $\alpha(\Lambda_1-M_1)\sim(g_0)^0$ rather than $g_0{}^2$ as might be expected by simple power counting. The reason for this seemingly anomalous behavior, of course, is that $\Lambda_1\to 2M_1$ in the SC limit.

The fact that the above-mentioned contributions vanish facilitates the removal of the spectral contributions at infinity to the Hamiltonian matrix elements since this is equivalent to dropping all $\phi_\lambda{}^2(k)$ contributions from the sum (4.4) after $\chi_\lambda{}^i$ and $\chi_\lambda(k)$ have been eliminated. These reduced sums can then be evaluated[14] using Eqs. (A6)–(A14) of Appendix A. We thus obtain in the SC limit

$$\langle V\theta_k\,|H\,|\,V\theta_l\rangle=(m_0-M_2)\delta(\vec{k}-\vec{l}\,)+[M_2\delta(\vec{k}-\vec{l}\,)]\ , \tag{4.5a}$$

$$\langle V\theta_k\,|H\,|\,N\theta_{k'}\theta_{l'}\rangle=0+[g_0f(k')\delta(\vec{k}-\vec{l}\,')+g_0f(l')\delta(\vec{k}-\vec{k}')]\ , \tag{4.5b}$$

and

$$\begin{aligned}\langle N\theta_k\theta_l\,|H\,|\,N\theta_{k'}\theta_{l'}\rangle=&(k+l)[\delta(\vec{k}-\vec{k}')\delta(\vec{l}-\vec{l}')+\delta(\vec{k}'-\vec{l}\,)\delta(\vec{k}-\vec{l}')]\\&-\left\{\frac{g_0f(k)f(k')\delta(\vec{l}-\vec{l}')}{M_2}+(\vec{k}\leftrightarrow\vec{l}\,)+(\vec{k}'\leftrightarrow\vec{l}')+(\vec{k}\leftrightarrow\vec{l}\,)(\vec{k}'\leftrightarrow\vec{l}')\right\}\\&+\left[\left\{\frac{g_0f(k)f(k')\delta(\vec{l}-\vec{l}')}{M_2}+(\vec{k}\leftrightarrow\vec{l}\,)+(\vec{k}'\leftrightarrow\vec{l}')+(\vec{k}\leftrightarrow\vec{l}\,)(\vec{k}'\leftrightarrow\vec{l}')\right\}\right]\ .\end{aligned} \tag{4.5c}$$

We remind the reader that the terms in the square brackets are those that occur from the spectral contributions that arise due to the cutoff. We see once again that when, and only when, these are removed that the bare V particle decouples from the bare N and θ particles and the Lee model and the separable-potential [see Eqs. (2.11)] model become effectively equivalent.

At this point, it seems appropriate to delineate the role of the ultraviolet cutoff in the transmutation of the Lee model into the separable-potential model. Irrespective of the existence of any cutoff, the finite-energy S-matrix elements and scattering amplitudes calculated from the two theories coincide in the SC limit; the Hamiltonian matrix elements, however, are always different. (Note that the presence of a cutoff does not alter these.) The two theories are thus clearly distinct and since, in the absence of a cutoff, there are no additional states in the Lee model that are not present in the separable-potential model, we see no obvious way in which one can be transformed into the other by omitting suitable contributions. We also remark that the equivalence proofs existing in the literature confine themselves to demonstrating the equivalence of the Green's functions (or equivalently, scattering amplitudes) without any reference to the Hamiltonian. These proofs, it is therefore suggested, may not be regarded as complete.

To summarize, we have explicitly shown in the two lowest sectors of the Lee model that the condition $Z_1=0$ by itself does not ensure its equivalence with the model of the separable potential. To see the additional ingredient required to make the two theories identical, we introduce an ultraviolet cutoff on both theories. This causes additional states to appear in the Lee model but not in separable-potential model. It is only when the spectral contributions due to these additional states are explicitly removed that the two theories become effectively identical in the SC limit. The latter is, of course, needed to ensure the compositeness condition $Z_1=0$.

V. CONCLUDING REMARKS

In this paper, we have presented the solutions for the two lowest nontrivial sectors of the Lee model and the separable-potential model when a momentum cutoff L is imposed on all the integrals. We have shown that the finite-energy scattering amplitudes and the S matrices calculated from the two theories can be made numerically the same provided we proceed in a certain manner, i.e., we take the bare coupling to infinity first (this ensures the compositeness condition $Z_1=0$) and then let the ultraviolet cutoff become arbitrarily large. This demonstration is the analog of the demonstration of the equivalence of the two field theories via a one-to-one correspondence of their Feynman diagrams[6] (or their renormalized Green's functions). We emphasize, however, that even in the SC limit, the Lee model and the separable-potential model have manifestly different spectra. To ensure the identity of the two theories, the additional spectral contributions that occur in the Lee model due to the presence of the cutoff have to be explicitly removed; then the Lee model is transmuted into the separable-potential model. This amputation and the strong-coupling limit are both essential ingredients.

In the last few years, some authors[15] have suggested the renormalizability of the four-fermion interaction, basing their arguments on the equivalence of the four-fermion theory and the corresponding Yukawa theory. Our investigation of this alleged equivalence, at least for the soluble models, shows that the transmutation of the Yukawa type of interaction to the four-point interaction occurs only when certain contributions from the spectrum of the Yukawa theory are removed. We have also checked this transmutation property for the case of the Lee model wherein the N and θ fields are quantized as fermion fields to ensure that the transmutation process is not a characteristic of Bose fields. The results of this investigation will be reported elsewhere.[16] To conclude, we note that if a scenario similar to the one discussed here also prevails in the case of relativistic field theories, the present proofs of equivalence and the resulting conclusion that some four-fermion type of interactions may be renormalizable needs further examination.

ACKNOWLEDGMENT

This research was supported in part by the U.S. Department of Energy Contract No. DE-AS05-76ER03992.

APPENDIX A

In this appendix, we list some formulas which are useful in deriving many of the results presented in the text. The scattering states are normalized as

$$\langle N\theta_k \mid N\theta_l \rangle = \delta(\vec{k}-\vec{l}) \,, \tag{A1}$$

$$\langle V\theta_k \mid V\theta_l \rangle = \delta(\vec{k}-\vec{l}) \,, \tag{A2}$$

$$\langle N\theta_k\theta_l \mid N\theta_{k'}\theta_{l'} \rangle = \delta(\vec{k}-\vec{k}')\delta(\vec{l}-\vec{l}') + \delta(\vec{k}-\vec{l}')\delta(\vec{l}-\vec{k}') \,. \tag{A3}$$

The discrete states are normalized to have unit norm. From the above, the completeness relations take the form

$$|V\rangle\langle V| + \int d^3k \, |N\theta_k\rangle\langle N\theta_k| = I \tag{A4}$$

and

$$\int d^3k\, | V\theta_k \rangle \langle V\theta_k | + \tfrac{1}{2} \int d^3k\, d^3l\, | N\theta_k\theta_l \rangle \langle N\theta_k\theta_l | = I \quad \text{(A5)}$$

in the $N\theta$ and $V\theta$ sectors, respectively.

In what follows, we list certain integrals which have been extensively used in the text:

$$\int d^3k\, F^i(k)F^j(k) = \delta_{ij} - (Z_iZ_j)^{1/2} , \quad \text{(A6)}$$

$$\int d^3k\, | g_k |^2 = 1 - Z_1 - Z_2 , \quad \text{(A7)}$$

$$\int d^3k\, G_{pk}G^*_{qk} = \delta(\vec{p} - \vec{q}) - g_q g_p^* , \quad \text{(A8)}$$

$$\int d^3k\, F^i(k)G^*_{pk} = -\sqrt{Z_i}g_p , \quad \text{(A9)}$$

$$\int d^3k\, g_k G_{kp} = -\sqrt{Z_1}F^1(p) - \sqrt{Z_2}F^2(p) , \quad \text{(A10)}$$

$$\int d^3k\, G^*_{kp}G_{kq} = \delta(\vec{p} - \vec{q}) - F^1(p)F^1(q) - F^2(p)F^2(q) , \quad \text{(A11)}$$

$$\int d^3k\, g_k^* G^*_{kp}(k+l) = -\sum_i (M_i + l)\sqrt{Z_i}F^i(p) + g_0 f(p) , \quad \text{(A12)}$$

$$\int d^3k\, G_{kp}G^*_{kq}(k+l) = (p+l)\delta(\vec{p} - \vec{q}) - \sum_i (M_i + l)F^i(p)F^i(q) , \quad \text{(A13)}$$

$$\int d^3k\, | g_k |^2 (k+l) = -\sum_i [(M_i + l)Z_i] + m_0 + l . \quad \text{(A14)}$$

Also, the spectral functions I_λ and K_λ defined in the text are related by

$$1 + I_\lambda = -\sum_i \frac{Z_i}{\alpha(\lambda - M_i)}(\lambda - 2M_i) - (\lambda - 2m_0)K_\lambda + 2(Z_1 + Z_2) . \quad \text{(A15)}$$

We complete our compendium of useful results by pointing out that many momentum integrals occurring in the text can be written as contour integrals of functions, analytic in the cut k plane as

$$\int_0^L d^3k\, | g_k |^2 \Psi(k) = -\frac{1}{2\pi i} \int_\Gamma \frac{dk}{\alpha(k)} \Psi(k) , \quad \text{(A16)}$$

where Γ is the closed contour defined in Fig. 2. Finally, the discontinuity

$$\alpha^+(k) - \alpha^-(k) = 2\pi i \tilde{f}^2(k) ,$$

where

$$\int d^3k\, f^2(k) \cdots = \int dk\, \tilde{f}^2(k) \cdots . \quad \text{(A17)}$$

APPENDIX B

In this appendix we present, in some detail, the method of obtaining the solution to the system of equations (3.22) for the various states of the Lee model.

$V\theta$ scattering states

From Eqs. (3.22a) and (3.22b) we can write

$$\phi_\lambda{}^i(l) = \delta(\vec{\xi}_i - \vec{l}) + \frac{g_0 f(l)\chi_\lambda{}^i}{\xi_i - l + i\epsilon} \quad \text{(B1a)}$$

and

$$\phi_\lambda(k,l) = \frac{g_0 f(l)\chi_\lambda(k)}{\lambda - l - k + i\epsilon} , \quad \text{(B1b)}$$

with

$$\xi_i = \lambda - M_i .$$

Substituting these into Eqs. (3.22c) and (3.22d) one easily obtains

$$\alpha^+(\xi_i)\chi_\lambda{}^i = \tilde{f}(\xi_i) - 2\sqrt{Z_1}\,Y_\lambda \quad \text{(B2a)}$$

and

$$\alpha^+(\lambda - k)\chi_\lambda(k) = -2g_k Y_\lambda . \quad \text{(B2b)}$$

Here

$$Y_\lambda \equiv g_0 \int d^3k\, f(k)[\sqrt{Z_1}\phi_\lambda{}^1(k) + \sqrt{Z_2}\phi_\lambda{}^2(k)] + g_0 \int d^3k\, d^3l\, f(k) g_l^* \phi_\lambda(l,k)$$

and

$$\tilde{f}(\xi_i) = f(\xi_i)\theta(\xi_i)\theta(L - \xi_i) .$$

In terms of

$$\tilde{g}_{\xi_i}=g_{\xi_i}\theta(\xi_i)\theta(L-\xi_i)\,,$$

the solution to (B2b) can be easily written as

$$\chi_\lambda(k)=\sqrt{Z_1}\delta(\vec{\xi}_1-\vec{k})+\sqrt{Z_2}\delta(\vec{\xi}_2-\vec{k})+\frac{g_k}{\alpha^+(\lambda-k)}\frac{[\alpha^+(\xi_i)\chi_\lambda{}^i-\tilde{f}(\xi_i)]}{\sqrt{Z_i}}\,, \tag{B3}$$

where we have written Y_λ using Eq. (B2a). Using (B1) and (B3) and the definition of Y_λ, Eq. (B2a) reduces to the system of equations

$$\begin{bmatrix}\alpha^+(\xi_1)(1+I_\lambda^+)+2Z_1A_{\xi_1}^+ & 2(Z_1Z_2)^{1/2}A_{\xi_2}^+\\ 2(Z_1Z_2)^{1/2}A_{\xi_1}^+ & \alpha^+(\xi_2)(1+I_\lambda^+)+2Z_2A_{\xi_2}^+\end{bmatrix}\begin{bmatrix}\chi_\lambda{}^1\\ \chi_\lambda{}^2\end{bmatrix}=\begin{bmatrix}\tilde{f}(\xi_1)(1+I_\lambda^+)-2\sqrt{Z_1}\Gamma_V\\ \tilde{f}(\xi_2)(1+I_\lambda^+)-2\sqrt{Z_2}\Gamma_V\end{bmatrix},$$

with

$$\Gamma_V=\sum_j\sqrt{Z_j}(\tilde{f}(\xi_j)+\tilde{g}^*_{\xi_j}A_{M_j})\,. \tag{B4}$$

This can be readily inverted to obtain

$$\chi_\lambda{}^1=\frac{(\lambda-2m_0)(1+I_\lambda^+)[-Z_1\tilde{g}^*_{\xi_1}\alpha^+(\xi_2)+Z_2\tilde{f}(\xi_1)-2(Z_1Z_2)^{1/2}\tilde{f}(\xi_2)-\tilde{f}(\xi_1)\alpha^+(\xi_2)K_\lambda^+]}{(\lambda-2m_0)(1+I_\lambda^+)\eta^+(\lambda)\alpha^+(\xi_1)\alpha^+(\xi_2)}\,. \tag{B5}$$

$\chi_\lambda{}^2$ can be obtained in a similar manner. A straightforward manipulation of this leads to the solution stated in the text.

Nθθ scattering states

Once again, from Eqs. (3.22a) and (3.22b) we can write

$$\phi_\lambda{}^i(l)=\frac{g_0f(l)\chi_\lambda{}^i}{\lambda-M_i-l+i\epsilon} \tag{B6a}$$

and

$$\phi_\lambda(k,l)=\delta(\vec{\zeta}_1-\vec{l})\delta(\vec{\zeta}_2-\vec{k})+\delta(\vec{\zeta}_2-\vec{l})\delta(\vec{\zeta}_1-\vec{k})+\frac{g_0f(l)\chi_\lambda(k)}{\lambda-k-l+i\epsilon} \tag{B6b}$$

with $\lambda=\zeta_1+\zeta_2$. Again, in terms of the quantity Y_λ defined as before, Eqs. (3.22c) and (3.22d) take the form

$$\alpha^+(\lambda-M_i)\chi_\lambda{}^i=-2\sqrt{Z_i}Y_\lambda\,, \tag{B7a}$$

$$\alpha^+(\lambda-k)\chi_\lambda(k)=g_0f(\zeta_1)\delta(\vec{\zeta}_2-\vec{k})+g_0f(\zeta_2)\delta(\vec{\zeta}_1-\vec{k})-2g_kY_\lambda\,. \tag{B7b}$$

The solution to Eq. (B7b), using Y_λ obtained from Eq. (B7a), can be readily written as

$$\chi_\lambda=g^*_{\zeta_1}\delta(\vec{\zeta}_2-\vec{k})+g^*_{\zeta_2}\delta(\vec{\zeta}_1-\vec{k})+g_k\frac{\alpha^+(\lambda-M_i)}{\alpha^+(\lambda-k)}\frac{\chi_\lambda{}^i}{\sqrt{Z_i}}\,. \tag{B8}$$

Also, as in the case of the $|V\theta\rangle\rangle$ states Eqs. (B7a) lead to the system of equations

$$\begin{bmatrix}\alpha^+(\lambda-M_1)(1+I_\lambda^+)+2Z_1A^+_{\lambda-M_1} & 2(Z_1Z_2)^{1/2}A^+_{\lambda-M_2}\\ 2(Z_1Z_2)^{1/2}A^+_{\lambda-M_1} & \alpha^+(\lambda-M_2)(1+I_\lambda^+)+2Z_2A^+_{\lambda-M_2}\end{bmatrix}\begin{bmatrix}\chi_\lambda{}^1\\ \chi_\lambda{}^2\end{bmatrix}=-\begin{bmatrix}2\sqrt{Z_1}\Gamma_N\\ 2\sqrt{Z_2}\Gamma_N\end{bmatrix}$$

with

$$\Gamma_N = f(\zeta_1) g^*_{\zeta_2} + f(\zeta_2) g^*_{\zeta_1} + g^*_{\zeta_1} g^*_{\zeta_2} (A^+_{\zeta_1} + A^+_{\zeta_2}) \ . \tag{B9}$$

The solution to (B9) reduces to that stated in the text.

Discrete states

As discussed in the text the discrete states are characterized by the roots of the equation

$$\eta(\Lambda) = 0 \ .$$

The solution to the system of equations (3.22), for $\Lambda \neq M_1 + M_2$, can be readily obtained by first writing from Eqs. (3.22a) and (3.22b)

$$\phi_\Lambda{}^1(l) = \frac{g_0 f(l) \chi_\Lambda{}^i}{\Lambda - M_i - l} \tag{B10a}$$

and

$$\phi_\Lambda(k,l) = \frac{g_0 f(l) \chi_\Lambda(k)}{\Lambda - k - l} \ . \tag{B10b}$$

Also from (3.22c) and (3.22d) we readily obtain

$$\alpha(\Lambda - k) \chi_\Lambda(k) = g_k \left[\frac{\alpha(\Lambda - M_i) \chi_\Lambda{}^i}{\sqrt{Z_i}} \right] \ . \tag{B11}$$

Since the left-hand side of (B11) is independent of i, we see that

$$\chi_\Lambda{}^2 = \left[\frac{Z_2}{Z_1} \right]^{1/2} \frac{\alpha(\Lambda - M_1)}{\alpha(\Lambda - M_2)} \chi_\Lambda{}^1 \ . \tag{B12}$$

Also, from Eq. (B11),

$$\chi_\Lambda(k) = \frac{g_k}{\alpha(\Lambda - k)} \frac{\alpha(\Lambda - M_1) \chi_\Lambda{}^1}{\sqrt{Z_1}} \ . \tag{B13}$$

The normalization constant $\chi_\Lambda{}^1$ is, of course, fixed by the requirement

$$\langle\langle \Lambda \mid \Lambda \rangle\rangle = 1 \ .$$

It can be easily shown that this reduces to

$$\sum_i \chi^{i*}_\Lambda \chi_\Lambda{}^i + \int d^3p\, \chi^*_\Lambda(p) \chi_\Lambda(p) + \sum_i \int d^3k\, \phi^{i*}_\Lambda(k) \phi_\Lambda{}^i(k) + \int d^3k\, d^3l\, \phi^*_\Lambda(l,k) \phi_\Lambda(l,k) = 2 \ . \tag{B14}$$

For $\Lambda \neq M_1 + M_2$, Eq. (B14) leads to the normalization, Eq. (3.28f), stated in the text.

APPENDIX C

In this appendix, we show that it is possible, using the explicit solutions presented in the text to reproduce the results one would expect simply on the grounds that the physical states form a complete set. This would serve as an independent check of these solutions. As an illustrative example, we consider

$$\sum_\lambda \langle\langle V_2 \mid a(k) \mid \rangle\rangle_{\lambda\,\lambda}\langle\langle \mid a^\dagger(l) \mid V_2 \rangle\rangle = \langle\langle V_2 \mid a(k) a^\dagger(l) \mid V_2 \rangle\rangle = \delta(\vec{k} - \vec{l}) + F^2(k) F^2(l) \tag{C1}$$

with λ running over the entire spectrum of the Hamiltonian. The left-hand side of Eq. (C1) can be written as

$$\sum_{\lambda_1} \phi_{\lambda_1}{}^2(k) \phi_{\lambda_1}{}^{2*}(l) + \sum_{\lambda_2} \phi_{\lambda_2}{}^2(k) \phi_{\lambda_2}{}^{2*}(l) \ , \tag{C2}$$

where λ_1 runs over the part of the spectrum that remains finite in the SC limit, and λ_2 over the remainder, i.e., the sum over λ_2 contains contributions from the $V_2\theta$ continuum eigenstates and the discrete state $\mid \Lambda_2 \rangle\rangle$. Here, we verify Eq. (C1) restricting ourselves to the SC limit for the sake of simplicity.

We first note that from Eqs. (3.29), (3.30), and (3.31) it follows that

$$\phi_{\lambda_1}(k) \sim \frac{1}{g_0{}^3}$$

in the SC limit when the state $\mid \ \rangle\rangle_{\lambda_1}$ is any of $\mid V_1\theta \rangle\rangle_{\lambda_1}$, $\mid N\theta\theta \rangle\rangle_{\lambda_1}$, or $\mid \Lambda_1 \rangle\rangle$. Also, since the right-hand side of Eq. (C1) $\sim 1/g_0{}^2$ in the SC limit we restrict our demonstration to the same order. This is not to say Eq. (C1) is not exactly valid, but only that we restrict our demonstration to terms

$\sim 1/g_0^2$ for the sake of simplicity. We thus evaluate the left-hand side of Eq. (C1) by retaining only the second term of the expression (C2).

We first define the function $\tilde{\eta}(z)$ in terms of the function $\eta(z)$, defined in Eq. (3.28) as

$$\tilde{\eta}(z)=\frac{\eta(z)\alpha(z-M_2)}{2Z_2} . \tag{C3}$$

We note that the numerator of Eq. (C3) is nonsingular since $\lambda_2 \geq M_2$. The function $\tilde{\eta}(\lambda)$ can be written in the SC limit, using the representation Eq. (3.35) for η as

$$\tilde{\eta}(z)\overset{s}{=}1-Z_2\frac{\alpha^+(z-M_2)}{2(z-2M_2)} . \tag{C4}$$

We thus get

$$\tilde{\eta}^+(x)-\tilde{\eta}^-(x)\overset{s}{=}-\frac{Z_2}{2(x-2M_2)}[\alpha^+(x-M_2)-\alpha^-(x-M_2)] . \tag{C5}$$

In terms of this newly defined function $\tilde{\eta}$ the solution $\phi_\lambda^{(2)}(k)$ for the $V_2\theta$ continuum can be written from Eqs. (3.29) as

$$\phi_\lambda^{(2)}=\delta(\vec{\xi}_2-\vec{1})-\frac{Z_2 g_0 f(l) f(\xi_2)}{2(\xi_2-l+i\epsilon)(\xi_2-M_2)}\frac{1}{\tilde{\eta}^+(\lambda)} . \tag{C6}$$

The contribution of the $V_2\theta$ continuum to the sum in (C2) can be written as

$$\begin{aligned}\int_{M_2}^{M_2+L} d\lambda\, \phi_\lambda{}^2(k)\phi_\lambda{}^{2*}(l)=&\delta(\vec{k}-\vec{1})-\frac{Z_2 g_0{}^2 f(k)f(l)}{2(k-l+i\epsilon)(k-M_2+i\epsilon)}\frac{1}{\tilde{\eta}^+(M_2+k)}\\ &-\frac{Z_2 f(k)f(l)g_0{}^2}{2(l-k-i\epsilon)(l-M_2-i\epsilon)}\frac{1}{\tilde{\eta}^-(M_2+l)}\\ &-\frac{g_0{}^2 f(k)f(l)Z_2}{i\pi}\int_{M_2}^{M_2+L} d\lambda\frac{1}{(\xi_2-k-i\epsilon)(\xi_2-M_2-i\epsilon)(\xi_2-l+i\epsilon)}\\ &\times\left[\frac{1}{\tilde{\eta}^-(\lambda)}-\frac{1}{\tilde{\eta}^+(\lambda)}\right] .\end{aligned} \tag{C7}$$

The right-hand side of Eq. (C7) can be readily evaluated using the method of contour integration and noting $\tilde{\eta}$ has a zero at Λ_2 such that

$$2(\Lambda_2-2M_2)\overset{s}{=}Z_2\alpha(\Lambda_2-M_2) . \tag{C8}$$

We then get

$$\int_{M_2}^{M_2+L} d\lambda\, \phi_\lambda{}^2(k)\phi_\lambda{}^2(l)\overset{s}{=}\delta(\vec{k}-\vec{1})+F^2(k)F^2(l)+\frac{g_0{}^2 f(k)f(l)}{(\Lambda_2-M_2-k)(\Lambda_2-M_2-l)}\left[\frac{1}{2(\Lambda-2M_2)}\frac{1}{\tilde{\eta}'(\Lambda_2)}\right] . \tag{C9}$$

We now show that the last term in Eq. (C9) is cancelled by the contribution of the discrete state $|\Lambda_2\rangle\rangle$ to the sum (C2). We proceed by noting that the normalization $|c|^2$ defined in Eq. (3.31f) can be written as

$$|c|^2=-\frac{Z_1\alpha(\Lambda_2-M_2)}{Z_2\alpha^2(\Lambda_2-M_1)}\frac{1}{\tilde{\eta}'(\Lambda_2)} . \tag{C10}$$

In deriving this, use has been made of the bound-state condition $\eta(\Lambda_2)=0$ to eliminate K_{Λ_2}. Then, using Eqs. (3.31b) and (3.31d), we readily obtain

$$\phi_{\Lambda_2}{}^2(k)\phi_{\Lambda_2}{}^2(l)\overset{s}{=}-\frac{g_0{}^2 f(k)f(l)}{(\Lambda_2-M_2-l)(\Lambda_2-M_2-k)}\times\frac{1}{2}\frac{1}{(\Lambda-2M_2)}\frac{1}{\tilde{\eta}'(\Lambda_2)} ,$$

where we have made use of the bound-state condition, Eq. (C8). We thus see that this indeed cancels the last term of Eq. (C9), thereby verifying Eq. (C1), at least in the SC limit. The consistency of other matrix elements with the solutions presented in the text can be checked in a similar manner.

APPENDIX D

In this appendix, we prove that the overcomplete basis equations of motion, (3.22), together with the constraint equations are equivalent to the equations of motion, (3.20), for the Källén-Pauli amplitudes.

To this end, we first write the amplitudes introduced in Eqs. (3.21) in terms of the Källén-Pauli amplitudes as

$$\phi_\lambda{}^i(l)=\sqrt{Z_i}\psi_\lambda(l) + \int d^3p\, F^i(p)\psi_\lambda(l,p)\ , \tag{D1a}$$

$$\phi_\lambda(k,l)=g_k\psi_\lambda(l) + \int d^3p\, G^*_{kp}\psi_\lambda(l,p)\ , \tag{D1b}$$

$$\chi_\lambda{}^i= \int d^3p\, F^i(p)\psi_\lambda(p)\ , \tag{D1c}$$

and

$$\chi_\lambda(k)= \int d^3p\, G^*_{kp}\psi_\lambda(p)\ . \tag{D1d}$$

It is simple to verify that Eqs. (D1) are equivalent to Eqs. (3.32) together with the constraint conditions (3.23).

To derive the equations of motion for the Källén-Pauli amplitudes, we first rewrite Eq. (3.22c), using Eqs. (D1a), (D1c), and (3.32a) as

$$(\lambda-M_i-m_0)\int d^3p\, F^i(p)\psi_\lambda(p)+\sqrt{Z_i}\int g_0 f(p)\psi_\lambda(p)= \int d^3l\, d^3p\, F^i(p)\psi_\lambda(l,p)\ . \tag{D2a}$$

By writing $(\lambda-M_i-m_0)$ on the left-hand side of Eq. (D2a) as $(\lambda-M_i-m_0+p-p)$, Eq. (D2a) can be rewritten as

$$\int d^3p\, F^i(p)\left[(\lambda-m_0-p)\psi_\lambda(p)-g_0\int d^3l\, f(l)\psi_\lambda(l,p)\right]=0\ . \tag{D2b}$$

In a similar manner, we can rewrite Eq. (3.22d) in the form

$$\int d^3p\, G^*_{kp}\left[(\lambda-m_0-p)\psi_\lambda(p)-g_0\int d^3l\, f(l)\psi_\lambda(l,p)\right]=0\ . \tag{D3}$$

Multiplying Eq. (D3) by G_{kq} and integrating over k we have, using Eq. (A11) together with Eq. (D2b), the required equation of motion

$$(\lambda-m_0-p)\psi_\lambda(p)=g_0\int d^3l\, f(l)\psi_\lambda(l,p)\ . \tag{D4}$$

To derive the equation of motion for $\psi_\lambda(k,l)$, we rewrite Eq. (3.22a) using Eqs. (D1a) and (D1c) together with the definition of $F^i(p)$ as

$$\int d^3p[(\lambda-p-l)F^i(p)\psi_\lambda(l,p)-g_0\sqrt{Z_i}f(p)\psi_\lambda(l,p)] = g_0 f(l)\int d^3p\, F^i(p)\psi_\lambda(p)-\sqrt{Z_i}(\lambda-m_0-l)\psi_\lambda(l)-\sqrt{Z_i}(m_0-M_i)\psi_\lambda(l)\ . \tag{D5a}$$

Then, using Eq. (D4) and the fact that $\alpha(M_i)=0$ on the right-hand side of Eq. (D5a), we readily obtain

$$\int d^3p\, F^i(p)[(\lambda-p-l)\psi_\lambda(l,p)-g_0 f(l)\psi_\lambda(p)-g_0 f(p)\psi_\lambda(l)]=0\ . \tag{D5b}$$

Again, starting from Eq. (3.22b) and carrying out the same procedure that was used in obtaining Eq. (D5b), we get

$$\int d^3p(\lambda-p-l)\psi_\lambda(l,p)G^*_{kp}=g_0\int d^3p\, f(l)\psi_\lambda(p)G^*_{kp}-g_k\left[-\alpha^-(k)-g_0{}^2\int\frac{d^3p\, f^2(p)}{k-p}\right]\psi_\lambda(l)\ . \tag{D6a}$$

Rewriting the last term on the right-hand side of Eq. (D6a) as

$$\int d^3p\, g_0 f(p)\left[\delta(\vec{k}-\vec{p})+\frac{g_0 f(p) g_k}{k-p}\right]\psi_\lambda(l)=\int d^3p\, g_0 f(p) G^*_{kp}\psi_\lambda(l)\ ,$$

we readily see that Eq. (D6a) takes the form

$$\int d^3p\, G^*_{kp}[(\lambda-p-l)\psi_\lambda(l,p)-g_0 f(l)\psi_\lambda(p)-g_0 f(p)\psi_\lambda(l)]=0\ . \qquad \text{(D6b)}$$

Then, from Eqs. (D5b) and (D6b) we obtain the equation of motion for $\psi_\lambda(l,p)$ in exactly the same manner that we obtained Eq. (D4).

---

*On leave from the Institute of Physics, National Taiwan Normal University, Taipei, Taiwan, Republic of China.

[1]J. C. Houard and B. Jouvet, Nuovo Cimento 18, 466 (1960).

[2]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

[3]J. D. Bjorken, Ann. Phys. (N.Y.) 24, 174 (1963); G. S. Guralnik, Phys. Rev. 136, B1414 (1964); I. Białynicki-Birula, *ibid.* 130, 465 (1963).

[4]K. Kikkawa, Prog. Theor. Phys. 56, 947 (1976); T. Eguchi, Phys. Rev. D 14, 2755 (1976); 17, 611 (1978); H. Terazawa *et al., ibid.* 15, 480 (1977); T. Saito and K. Shigemoto, Prog. Theor. Phys. 57, 242 (1977); 57, 643 (1977); G. Rajasekaran and V. Srinivasan, Pramāna 10, 33 (1978); G. Konisi and W. Takahasi, Phys. Rev. D 23, 380 (1981); H. P. Dürr and H. Saller, *ibid.* 22, 1176 (1980) (see also references cited therein); K. Shizuya, *ibid.* 21, 2327 (1980).

[5]J. Lemmon and K. T. Mahanthappa, Phys. Rev. D 13, 2907 (1976).

[6]D. Lurié and A. J. Macfarlane, Phys. Rev. 136, B816 (1964).

[7]R. Rajaraman, Pramāna 11, 491 (1978).

[8]In certain types of non-Abelian gauge theories, it seems possible to satisfy the compositeness conditions without reducing the theory to a free theory provided a certain protocol for renormalization is followed. There is, in these cases, no necessity of the strong-coupling limit. See, e.g., C. C. Chiang, C. B. Chiu, E. C. G. Sudarshan, and X. Tata, Phys. Rev. D 25, 1136 (1982).

[9]C. C. Chiang, C. B. Chiu, and E. C. G. Sudarshan, Phys. Lett. 103B, 371 (1981).

[10]T. D. Lee, Phys. Rev. 95, 1329 (1954); K. O. Friedrichs, Commun. Pure Appl. Math. 1, 361 (1948); G. Källén and W. Pauli, K. Dan. Vidensk. Selsk. 30, No 7 (1955).

[11]See, for example, E. C. G. Sudarshan, *Field Theory, Quantization and Statistical Physics,* edited by E. Tarapugui (Reidel, Dordrecht, 1981), p. 237–245.

[12]For a repulsive separable-potential model, it is possible to have a discrete state beyond the cutoff but none below the $N\theta$ threshold.

[13]M. Bolsterli, Phys. Rev. 166, 1760 (1968); C. A. Nelson, J. Math. Phys. 13, 1051 (1972). The scattering amplitudes for the $V\theta$ sector of the Lee model were first discussed by R. D. Amado, Phys. Rev. 122, 696 (1961).

[14]In the notation of the second paper of Ref. 13, the same M matrix as given in Eq. (9) of the said paper results when the $\phi_\lambda{}^2(k)$ contributions are removed, with the replacements $Z\rightarrow Z_1$ and $F(k)\rightarrow F^1(k)$.

[15]See, for example, T. Eguchi, 1978 article quoted in Ref. 4.

[16]X. Tata, E. C. G. Sudarshan, C. B. Chiu, and C. C. Chiang, following paper, Phys. Rev. D 26, 2114 (1982).