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Time evolution of unstable quantum states and a resolution of Zeno's paradox

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The time evolution of quantum states for unstable particles can be conveniently divided into three domains: the very short time where Zeno's paradox is relevant, the intermediate interval where the exponential decay holds more or less, and the very long time where the decay is governed by a power law. In this work, we reexamine several questions relating to the deviations from the simple exponential decay law. On the basis of general considerations, we demonstrate that deviations from exponential decay near $t = 0$ are inevitable. We formulate general resonance models for the decay. From analytic solutions to specific narrow-width models, we estimate the time parameters T_1 and T_2 separating the three domains. The parameter T_1 is found to be much much less than the lifetime Γ^{-1}, while T_2 is much greater than the lifetime. For instance, for the charged pion decay, $T_1 \sim 10^{-14}/\Gamma$ and $T_2 \sim 190/\Gamma$. A resolution of Zeno's paradox provided by the present consideration and its limitations are discussed.

I. INTRODUCTION

Quantum mechanics enables us to calculate the time evolution of a dynamical system provided the Hamiltonian is defined and the initial state suitably specified. The Hamiltonian must include all the interactions to which the system is subjected; and, in a sense, it deals with a closed system, since even when external fields or forces are considered there is no reaction on them by the system.

A particularly interesting class of systems for which time evolution may be studied are the so-called "unstable-particle" systems. By definition an unstable particle is a nonstationary state which undergoes substantial changes in a time much larger than the natural time periods associated with the energy of the system. In this case the "natural" evolution in time and the "decay transition" may be viewed as two separate kinds of time development; and it would be profitable to think of the natural evolution as if it were accounted for by an unperturbed Hamiltonian and the decay transition being brought about by an additional perturbation. Conversely, given a Hamiltonian with a point spectrum and a continuous spectrum, we may introduce perturbations which lead to "decay" of the states which belonged to the point spectrum and which were therefore stationary. In this we can determine the precise time development of the system.

On the basis of classical probability theory we would expect a simple decay process to exhibit a purely exponential behavior with the "lifetime" being given as a fixed parameter. By and large quantum decay processes all exhibit such a behavior; and it seems almost essential if we are to think of the "unstable particle" as an autonomous entity. Such an exponential law may be derived as an approximation to the actual time development. While many careful studies have been devoted to this question, the approximation is essentially the one introduced by Dirac[1] in the calculation of the rate of atomic radiative transitions and treats the continuum of final states of the decay as being unbounded above and below.

In this paper we reexamine the questions relating to the deviation from the exponential decay law of particle decay processes. Although many studies have been devoted to this question,[2,3] we feel such a reexamination is still useful for the following reasons. Most of the works devoted to this question focus on the deviation from exponential decay law at large time, whereas the deviation in the region of very small time is comparatively neglected. For instance, though the known works of Khalfin[3] and others provide a very general argument for the necessity of deviation at large times, there seems to be no such general argument in the literature pointing to the necessity of deviation from exponential law in the region of very small times. One of the objects of this paper is to fill this gap by providing a similarly general argument which shows the necessity of deviation from exponential decay law at small time.

A second motivation for this reexamination stems from a recently formulated conclusion[4] in quantum theory, the quantum Zeno's paradox. It says that an unstable particle when monitored (for its existence) at sufficiently small intervals of time will be found to live longer than the particle monitored infrequently and in the limit of continuous monitoring it will be found not to decay at all. It is evident as explained in some detail in Sec. II that the quantum Zeno's effect is intimately related to the deviation from the exponential decay law at small time, and a study of the latter will provide a better understanding and a possible resolution of the former seemingly paradoxical conclusion.

A third related object of this paper is to formulate general resonance models and to estimate the time parameters T_1 and T_2 which separate the intermediate region of time, where the exponential decay law holds to a chosen degree of approximation, from the regions of small time and larger time where deviation from the exponential law is important. Finally, we also briefly discuss the resolution of the quantum Zeno's paradox provided by the present discussion and the limitations of such a resolution.

II. DEVIATION FROM THE EXPONENTIAL DECAY LAW AT SMALL TIME

To discuss this question it is necessary to start with a brief recapitulation of the quantum-theoretical formalism for describing unstable states. Let $\mathcal{H}$ denote the Hilbert space formed by the unstable (undecayed) states of the system as well as the states of decay products. The time evolution of this total system is then described by the unitary group $u_t = e^{-iHt}$, where H denotes the self-adjoint Hamiltonian operator of the system. For simplicity, we shall assume that there is exactly one unstable state represented by the vector $|M\rangle$ of $\mathcal{H}$. The state $|M\rangle$, being an unstable state, must be orthogonal to all bound stationary states of the Hamiltonian H. Hence $|M\rangle$ is associated with the continuous spectrum of H. On physical grounds we also suppose that the Hamiltonian H has no singular continuous spectrum. (In contrast to this simplifying situation in quantum mechanics, the spectrum of the Liouville operator of a classical dynamical system which is weakly mixing but not mixing must have a singular continuous part.) Thus if F_λ denotes the spectral projections of the Hamiltonian

$$H = \int \lambda \, dF_\lambda \equiv \int \lambda |\lambda\rangle\langle\lambda| d\lambda \ , \tag{1}$$

then the function $\langle M|F_\lambda|M\rangle$ is absolutely continuous, and its derivative

$$\psi(\lambda) = \frac{d}{d\lambda}\langle M|F_\lambda|M\rangle = \langle M|\lambda\rangle\langle\lambda|M\rangle \tag{2}$$

can be interpreted as the energy distribution function of the state $|M\rangle$; i.e., the quantity

$$\int_E^{E+dE} \psi(\lambda) d\lambda$$

is the probability that the energy of the state $|M\rangle$ lies in the interval $[E, E+dE]$.

The distribution function $\psi(\lambda)$ has the following general properties:

(i) $\psi(\lambda) \geq 0$;

(ii) $\int \psi(\lambda) d\lambda = 1$ corresponding to the normalization condition $\langle M|M\rangle = 1$;

(iii) $\psi(\lambda) = 0$ for λ outside the spectrum of H. It may be noted that, in defining the energy distribution function $\psi(\lambda)$ as we have done above, we have absorbed the customary density-of-states factor or the phase-space factor $\sigma(\lambda)$ in $\psi(\lambda)$.

The above-mentioned conditions are quite general and hold for any state which is orthogonal to the bound states of H. In order that the state may be identified as an unstable particle state with a characteristic lifetime, its energy distribution function should satisfy certain additional conditions. We shall discuss these conditions in Sec. III. But the discussion of the present section will use only properties (i)–(iii) of the energy distribution function.

the nondecay probability $Q(t)$ (or the probability for survival) at the instant t for the unstable state $|M\rangle$ is given by

$$Q(t) = |\langle M|e^{-iHt}|M\rangle|^2 \ . \tag{3}$$

Accordingly, the decay probability $P(t)$ at t is $1 - Q(t)$. The nondecay amplitude $a(t) = \langle M|e^{-iHt}|M\rangle$ may be easily seen to be the Fourier transform of the energy distribution function $\psi(\lambda)$,

$$a(t) = \langle M|e^{-iHt}|M\rangle = \int e^{-i\lambda t} d\langle M|F_\lambda|M\rangle$$
$$= \int e^{-i\lambda t}\psi(\lambda)d\lambda \ . \tag{4}$$

The celebrated Paley-Wiener theorem[5] then shows that if the spectrum of H is bounded below so that $\psi(\lambda) = 0$ for $\lambda < 0$ then $|a(t)|$ and hence $Q(t) = |a(t)|^2$ decreases to 0 as $t \to \infty$ less rapidly than any exponential function $e^{-\Gamma t}$. This is essentially Khalfin's argument proving the necessity of deviation from the exponential decay law at large time.

The following proposition shows that $Q(t)$ must deviate from the exponential decay at sufficiently small time too. Let the spectrum of H be bounded below and assume further that the energy expectation value for the state $|M\rangle$ is finite,

$$\int \lambda \psi(\lambda) d\lambda < \infty \ . \tag{5}$$

Then $Q(t) > e^{-\Gamma t}$ for sufficiently small t.

It may be emphasized that semiboundedness of H is essential for the proof. For otherwise we may consider the energy distribution function $\psi(\lambda) = 1/(1+\lambda^2)$, for which

$$a(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda t}}{1+\lambda^2}\, d\lambda = e^{-|t|} \quad , \tag{6}$$

and $Q(t)$ coincides with the exponential function $e^{-2|t|}$ for all $t > 0$. We shall assume, without loss of generality, that the spectrum of H is confined in the positive semiaxis $[0, \infty]$.

To prove the proposition it is sufficient to show that $Q(t)$ is differentiable and

$$\dot{Q}(0) \equiv \frac{d}{dt} Q(t)\Big|_{t=0} > -\Gamma \quad (\Gamma > 0) \ . \tag{7}$$

We shall in fact show that

$$\dot{Q}(0) = 0 \ . \tag{8}$$

In view of the positivity of the operator H, the energy distribution function $\psi(\lambda) = 0$, for $\lambda < 0$. Thus the condition (5) implies that this function $\lambda\psi(\lambda)$ is absolutely integrable,

$$\int |\lambda|\psi(\lambda) d\lambda < \infty \, . \tag{9}$$

From this it follows that

$$a(t) = \int e^{-i\lambda t}\psi(\lambda) d\lambda \tag{10}$$

is differentiable for all t, and the derivative

$$\dot{a}(t) = \int e^{-i\lambda t}(-i\lambda)\psi(\lambda) d\lambda \tag{11}$$

is continuous. Now

$$a^*(t) = a(-t) \ , \tag{12}$$

so that

$$\frac{d}{dt} a^*(t)\Big|_{t=-s} = -\frac{d}{dt} a(t)\Big|_{t=-s} = -\dot{a}(-s) \ . \tag{13}$$

Since $Q(t) = a(t)a^*(t)$,

$$\frac{d}{dt} Q(t)\Big|_{t=s} = a(-s)\dot{a}(s) - a(s)\dot{a}(-s) \ . \tag{14}$$

In particular,

$$\dot{Q}(0) = \dot{a}(0_+) - \dot{a}(0_-) = 0 \ , \tag{15}$$

since $a(0) = 1$ and $\dot{a}(t)$ is continuous so that $\dot{a}(0_+) = \dot{a}(0_-)$.

The preceding proposition shows that at sufficiently small time the nondecay probability $Q(t)$ falls off less rapidly than would be expected on the basis of the exponential decay law. Thus if the unstable system is monitored for its existence at sufficiently small intervals of time, it would appear to be longer lived than if it were monitored at intermediate intervals where the decay law is exponential. The quantum Zeno's paradox states that in the limit of continuous monitoring the particle will be found not to decay at all. This conclusion in the present special case of a one-dimensional subspace of undecayed (unstable) states follows in fact as an immediate corollary of the preceding proposition. Following the discussion of Ref. 4, it can be easily seen that if the system prepared initially in the unstable state $|M\rangle$ is (selectively) monitored on its survival at the instants $0, t/n, \ldots, (n-1)t/n, t$, then the probability for its survival is given by

$$Q\left(\frac{t}{n}\right)^n \ .$$

Since $Q(t)$ is continuously differentiable and $\dot{Q}(0) = 0$, it can be easily shown that

$$\lim_{n\to\infty} Q\left(\frac{t}{n}\right)^n = 1 \tag{16}$$

independent of t. It is evident that the survival probability under discrete but frequent monitoring will be close to 1 provided that t/n is sufficiently small, so that the departure from the exponential decay law remains significant. It is thus important to estimate the time scale for which the small-time deviation from the exponential decay law is prominent.

III. RESONANCE MODELS FOR DECAY AMPLITUDES

To estimate the parameters T_1 and T_2 which separate the intermediate-time domain where the exponential decay law holds from small- and large-time domains where deviations are prominent, we need to make a more specific assumption about the energy distribution function $\psi(\lambda)$ of the unstable state $|M\rangle$. In fact, so far we have assumed only very general properties of $\psi(\lambda)$ that are not sufficient to warrant the identification that $|M\rangle$ represents an unstable state which behaves as a more or less autonomous entity with a characteristic lifetime.

To formulate this resonance requirement we shall rewrite the nondecay amplitude as a contour integral. To this end, we consider the resolvents $R(z) \equiv (H - zI)^{-1}$ of the Hamiltonian H. They form a (bounded) operator-valued analytic function of z on the whole of the complex plane except for the cut along the spectrum of H, which we take to be the real half axis $[0, \infty]$. Under mild restrictions on the state $|M\rangle$, for instance, under the condition that $|M\rangle$ lies in the domain of H^2, we have the formula

$$e^{-iHt}|M\rangle = \frac{1}{2\pi i}\int_C e^{-izt}R(z)|M\rangle dz \ , \qquad (17)$$

where C is the contour shown in Fig. 1.[6] The nondecay probability then has the representation

$$a(t) = \langle M|e^{-iHt}|M\rangle = \frac{1}{2\pi i}\int_C e^{-izt}\beta(z)dz \ , \qquad (18)$$

where

$$\beta(z) = \langle M|R(z)|M\rangle \ . \qquad (19)$$

The function $\beta(z)$ is uniquely determined by the energy distribution function $\psi(\lambda)$ of $|M\rangle$ through the formula

$$\beta(z) = -\int_0^\infty \frac{\psi(\lambda)}{\lambda - z}d\lambda \ , \qquad (20)$$

and in turn determines the distribution function $\psi(\lambda)$ through the formula

$$\psi(\lambda) = \lim_{\epsilon\to 0+}\frac{-1}{2\pi i}[\beta(\lambda+i\epsilon)-\beta(\lambda-i\epsilon)] \ . \qquad (21)$$

The function $\beta(z)$ is analytic in the cut plane and is free of zeros there. We may thus introduce

$$\gamma(z) = 1/\beta(z) \ , \qquad (22)$$

which is analytic and free of zeros in the cut plane. The nondecay probability is that given by

$$a(t) = \frac{i}{2\pi}\int_C \frac{e^{-izt}}{\gamma(z)}dz \ . \qquad (23)$$

The above representation for $a(t)$ is quite general and does not yet incorporate the important resonance condition alluded to earlier. The resonance condition may be formulated as the requirement that the analytic continuation of $\gamma(z)$ in the second sheet possesses a zero at $z=E_0-\frac{1}{2}i\Gamma$ with $E_0 \gg \Gamma > 0$. Under this condition the above representation for $a(t)$ shows that it will have a dominant contribution $e^{-iEt}e^{-\Gamma t/2}$ from the zero of $\gamma(z)$ in the second sheet and certain correction terms to the exponential decay law arising from a "background" integral. An investigation of the corrections to the exponential decay law then amounts to

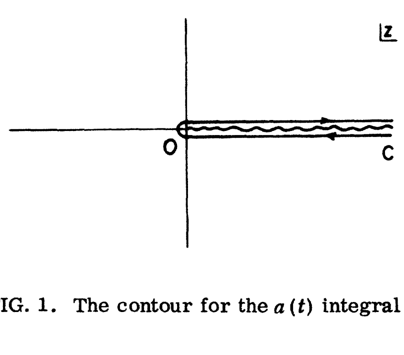

FIG. 1. The contour for the $a(t)$ integral in (23).

an investigation of the background integral in (23). This approach to studying the deviation from the exponential decay law has been adopted in the past.[7] Here we investigate the detailed properties of the background integral by making a specific choice for $\gamma(z)$.

To facilitate the choice and to relate our results to investigations on the Lee model[8] and the related Friedrichs model,[9] we note that one can write (suitably subtracted) dispersion relations for $\gamma(z)$. For instance, if $\gamma(z)$ has the asymptotic behavior

$$|\gamma(z)-z| \xrightarrow[|z|\to\infty]{} z^n \qquad (24)$$

with $n \leq 0$, then

$$\gamma(z) = z - \lambda_0 + \frac{1}{\pi}\int_0^\infty \frac{|f(\lambda)|^2}{\lambda - z}d\lambda \ , \qquad (25)$$

with

$$|f(\lambda)|^2 = \frac{1}{2i}\lim_{\epsilon\to 0_+}[\gamma(\lambda+i\epsilon)-\gamma(\lambda-i\epsilon)] \ . \qquad (26)$$

On the other hand, if $\gamma(z)$ satisfies (24) with $0<n<1$, then $\gamma(z)$ satisfies the once-subtracted dispersion relation. With the subtraction at $z=E_s$,

$$\gamma(z) = z - E_s + \gamma(E_s) + \frac{z-E_s}{\pi}\int_0^\infty \frac{|f(\lambda)|^2}{(\lambda-z)(\lambda-E_s)}d\lambda \ . \qquad (27)$$

It may be noted that the form (25) for $\gamma(z)$ is the one obtained in various model-theoretic descriptions of unstable states.[1,8,9] All such descriptions picture the unstable state $|M\rangle$ as a normalized stationary state of an unperturbed Hamiltonian H_0 associated with a point spectrum of H_0 embedded in the continuous spectrum. The decay transition is caused solely by a perturbation H_I, under suitable assumptions about H_I, for instance that the transition amplitude of H_I from the states associated with the continuous spectrum of H_0 into themselves may be neglected in the evaluation of $a(t)$. The nondecay amplitude can be shown to be given by (23) and (25) or (27), where

$$|f(\lambda)|^2 = |\langle\lambda|H_I|M\rangle|^2 \ , \qquad (28)$$

with $|\lambda\rangle$ being the continuum eigenkets of H_0

Next define

$$k = z^{1/2}e^{i\pi/4} \qquad (29)$$

and write

$$\gamma(z) = \tilde{\gamma}(k) = e^{-i\pi/2}(k-k_+)(k-k_-)\xi(k) \ , \qquad (30)$$

with resonance poles as stated earlier at

$$z = E_0 - \tfrac{1}{2}i\Gamma \ \text{ and } \ z = e^{2\pi i}E_0 + \tfrac{1}{2}i\Gamma \ . \qquad (31)$$

In the k plane they are at

$$k_\pm \simeq \pm k_0 + \delta \ , \qquad (32)$$

where $k_0 = E_0^{1/2} e^{i\pi/4}$, $\delta = \Delta^{1/2} e^{-i\pi/4}$ with $\Delta^{1/2} = \Gamma/4E_0^{1/2}$. (See Fig. 2.) Substituting (30) into (23) and deforming the contour we may write

$$a(t) = \frac{i}{2\pi}\int_C \frac{e^{-k^2t}2k\,dk}{(k-k_+)(k-k_-)\xi(k)} = a_+(t) + a_1(t) + a_2(t)\ , \qquad (33)$$

with

$$a_+(t) = \frac{1}{\xi(k_+)}\frac{k_+}{k_0} e^{-iE_0 t} e^{-\Gamma t/2}\ , \qquad (34)$$

$$a_1(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-k^2t}2k\,dk}{(k-k_+)(k-k_-)\xi(k)} \simeq \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-k^2t}2k\,dk}{(k^2-k_0^2)\xi(k)}\left(1+\frac{2\delta k}{k^2-k_0^2}\right)\ , \qquad (35)$$

and $a_2(t)$ a contribution which can be dropped owing to a suitable cancellation. These three parts are associated with the deformed contour

$$C \to S = S_+ + S_1 + S_2$$

illustrated in Fig. 2. Note that we do not have to include any contribution from k_-.

To proceed further one has to make the specific choices for $\xi(k)$. We may now restate our problem in the following fashion. Given an amplitude of the form (33) with a suitable choice for ξ, how does the decay probability behave as a function of time? What are the characteristic times T_1 and T_2 for the system? How sensitive are these conclusions in relation to the specific forms assumed for ξ? In the following section, we take up a study of these questions.

IV. SPECIFIC DECAY MODELS AND A RESOLUTION OF ZENO'S PARADOX

In the Appendix, two specific choices of ξ are considered. For model I,

$$\xi(z) = 1\ . \qquad (A1)$$

This leads to a dispersion relation of the form of Eq. (27). For model II,

$$\xi(z) = \frac{\sqrt{-z} - B^{1/2}}{\sqrt{-z} - (B^{1/2} + 2\Delta^{1/2})}\ . \qquad (A19')$$

This leads to the dispersion relation of the form of Eq. (25). The details of both solutions are given in the Appendix. We proceed to look at several aspects of these solutions.

A. The large-t power law and its geometric interpretation

From (A12) and (A33), the large-t behavior of the survival amplitude for both models is given by

$$|a(t)| \sim \text{const} \times \frac{1}{t^{3/2}}\ . \qquad (36)$$

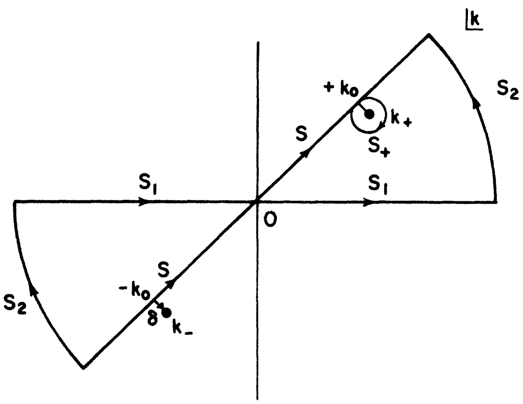

FIG. 2. Contours defining the integrals (34) and (35).

A slower than exponential decay, as mentioned in Sec. II is expected from the general argument of Khalfin though it could be like $\exp(-t^{1-\epsilon})$. On the other hand, the specific $t^{-3/2}$ law is not only a particular property of these special models, but a reflection of the kinematics of the decay process. We may see this as follows. We write $|f(E)|^2 \equiv |\tilde f(E)|^2\sigma(E)$, where $\sigma(E)$ is the phase-space weight factor. Then from (23) and (26),

$$a(t) = \frac{1}{\pi}\int_0^\infty dE\,\frac{|\tilde f(E)|^2}{|\gamma(E+i\epsilon)|^2}\sigma(E)e^{-iEt} \qquad (37)$$

$$\simeq \frac{1}{\pi}\int_0^{1/t} dE\,\frac{|\tilde f(E)|^2}{|\gamma(E+i\epsilon)|^2}\sigma(E)e^{-iEt} \sim \frac{1}{\pi}\int_0^{1/t} dE\,\sigma(E)e^{-iEt} \simeq \frac{1}{\pi}\int_0^\infty dE\,\sigma(E)e^{-iEt} \qquad (38)$$

for very large times, because of the rapid variation of the phase factor, provided the functions $\tilde f(E)$ and $\gamma(E+i\epsilon)$ behave gently near zero. The phase-space factor $\sigma(E)$ has a power-law behavior in the neighborhood of the origin. For a nonrelativistic system $E = k^2/2m$,

$$\sigma(E) = 4\pi k^2\frac{dk}{dE} \sim \sqrt{E}\ , \qquad (39)$$

while for a relativistic system $E = (k^2+m^2)^{1/2} - m$, as $E \to 0$,

$$\sigma(E) = 4\pi k(E+m) \sim \sqrt{E}\ . \qquad (40)$$

Hence, in both cases we may recognize (38) to behave like

$$\int_0^\infty dE\sqrt{E}\,e^{-iEt} = t^{-3/2}\int_0^\infty du\sqrt{u}\,e^{-iu}\ . \qquad (41)$$

Thus the inverse-cube dependence of the probability of nondecay $Q(t)$ may be related to the structure of the phase-space factor, provided the form factor $\bar{f}(E)$ is gently varying.

This power-law dependence has a simple geometrical meaning: The "unstable particle" as such is not a new state, but a certain superposition of the decay products. These latter states have a continuum of energy eigenvalues. The precise manner in which the superposition is constituted depends on our definition of the unstable particle, and the development of this wave packet as a function of time depends on the dynamics of the system. But eventually the packet spreads so that the decay products separate sufficiently far to be outside each other's influence. Once this state is reached the further expansion is purely kinematic, the amplitude decreasing inversely as the square root of the cube of time. Consequently, the overlap amplitude $a(t)$ also behaves thus. The requirement of gentle variation of the form factor is precisely that the corresponding interaction becomes negligible beyond some large but finite distance.

In view of this geometric interpretation we expect that any unstable system with well-behaved interactions would exhibit such a power law rather than an exponential law.

B. Two types of t dependence near $t = 0$

The short-time behavior of the probability $Q(t)$ given by two models are quite different. For model I, from (A17) and (A18),

$$a(t) \to 1 - \text{const} \times e^{i\pi/4} t^{1/2} , \tag{42}$$

which leads to the decay rate as $t \to 0$,

$$\dot{Q}(t) \propto -\frac{1}{\sqrt{t}} \to \infty . \tag{43}$$

For model II, from (A34) and (A36),

$$a(t) \to 1 - i \times \text{const} \times t - \text{const} \times e^{-i\pi/4} t^{3/2} \tag{44}$$

and

$$\dot{Q}(t) \propto -t^{1/2} \to 0 . \tag{45}$$

Model II is an example of the proposition considered in Sec. II, where the energy expectation value for the resonance state $\langle M|H|M\rangle$ is finite. From general arguments, we already concluded that as $t \to 0$, the decay rate should approach 0. Equation (45) is in agreement with this conclusion. On the other hand, if $\langle M|H|M\rangle$ does not exist, such as in model I, as $t \to 0$, the rate of decay is undefined. So the exponential law again does not hold. We see that in no case could the exponential law hold to arbitrarily small values of t. The conclusion that we have arrived at only depends on the basic notions of quantum mechanics; it is therefore quite general.

C. Repeated measurements in short-time and long-time limits

From the above discussions, we are led to two possibilities regarding the leading-term behavior of $Q(t)$ as $t \to 0$:

$$Q(t) \to 1 - \frac{\alpha}{\beta} t^{\beta} \text{ and } \dot{Q}(t) \to -\alpha t^{\beta-1}, \quad \beta \neq 1 . \tag{46}$$

Since $0 \leq Q(t) \leq Q(0)$, $\alpha > 0$ and $\beta > 0$ [we are not considering nonpolynomial dependences such as $t^{\beta}(\log t)^{\gamma}$], the ranges $\beta < 1$ and $\beta > 1$ behave quite differently. In one case the rate is becoming larger as $t \to 0$, and in the other case it is vanishing.

Now consider as in Sec. II, the n measurements at times $t/n, 2t/n, \ldots, t$. In the limit of $n \to \infty$, the time interval t/n tends to zero. Hence, for arbitrarily small t as $n \to \infty$,

$$Q_n(t) \to \left[1 - \frac{\alpha}{\beta}\left(\frac{t}{n}\right)^{\beta}\right]^n \to \begin{cases} 1 & \beta > 1 \\ 0 & \beta < 1 . \end{cases} \tag{47}$$

The first case corresponds to Zeno's paradox in quantum theory. In the second case the limit as $n \to \infty$ does not exist and the paradox does not obtain: continuous observation is forbidden.

It is also interesting to ask what happens in the long-time limit. We have seen that with reasonable dynamics, the asymptotic form is purely kinematic. What about repeated measurement? The wave packet has expanded beyond the range of interaction in accordance with the $t^{-3/2}$ amplitude law: The measurement collapses this packet to the size of the original packet we call the unstable particle, and the time evolution begins again. We then have the behavior $(t/n)^{-3n/2}$. We attenuate the unstable-particle amplitude by repeated observation. Naturally there is now no question of continuous observation.

D. Laboratory observations on unstable particles and a possible resolution of Zeno's paradox

In these discussions we have dealt with the uninterrupted time development of an unstable particle. What can we conclude from this about laboratory observations on unstable particles? Is it proper to apply these considerations for particles that cause a track in a bubble chamber?

The uninterrupted time evolution was, we saw above, characterized by three regions: (i) $0 < t < T_1$, the small-time region where $Q(t) \simeq 1 - (\alpha/\beta)t^{\beta}$, $\beta > 0$; (ii) $T_1 < t < T_2$ the intermediate-time region where an exponential law holds; (iii) $t > T_2$ the large-time region where there is an inverse-power-law behavior. Out of these the intermediate-time region alone satisfies the simple composition law

$$Q(t_1)Q(t_2) = Q(t_1 + t_2) \ . \tag{48}$$

In this domain, therefore, a classical probability law operates, and the results for the two-step measurement are the same as for the one-step measurement.

On the other hand, if the particle is making a track or otherwise interacting with a surrounding medium and is thus an open system, the considerations we have made do not apply. Instead we would have to account for the interpretation of the evolution by the interaction and a consequent reduction of the wave packet. The nondecay probability is now defined by the composition law:

$$Q(t_1, t_2, \ldots, t_n) = Q(t_1)\, Q(t_1)Q(t_2)\cdots Q(t_n) \ . \tag{49}$$

Hence, if $t_1 = t_2 = \cdots = t_n = \tau$, we can write

$$Q(n\tau) = [Q(\tau)]^n \ , \tag{50}$$

so that for times large compared with τ the dependence is essentially exponential, independent of the law of quantum evolution $Q(t)$.[10] If the interruptions do not occur at equal intervals but are randomly distributed, the behavior would be more complex but this has been considered by Ekstein and Siegert[11] and furthered by Fonda *et al.*[12] The pure exponential behavior is somewhat altered but the power-law dependence of the long-time behavior of the uninterrupted time evolution is no longer obtained.

We wish to call particular attention to this result: The long-time behavior of the closed and open systems are essentially different. Classical probabilistic notions do not apply to the closed system. The reason is not far to seek: Classical intuition is related to probabilities which are the directly "observed" quantities. But probabilities do not propagate: Propagation is for the amplitude. Despite this, it is difficult if not impossible to observe the differences between the two. To be able to see the difference we must reach the third domain $t > T_2$, but since T_2 is much much larger than the mean lifetime, by the time this domain is reached the survival probability is already many orders of magnitude smaller than unity. For both models considered, we found

$$T_2 \sim \frac{5}{\Gamma}\ln\left(\frac{E_0}{\Gamma}\right) + \frac{3}{\Gamma}\ln\left(5\ln\frac{E_0}{\Gamma}\right) \ . \tag{A15}$$

Take the example of the decay of a charged pion, $\pi \to \mu\nu$

$$\Gamma = (3\times 10^{-8}\ \text{sec})^{-1}$$

and

$$E_0 = m_\pi - m_\mu = 34\ \text{MeV} = (2\times 10^{-23}\ \text{sec})^{-1} .$$

This leads to $T_2 \sim 190/\Gamma$. So, by the time the power law is operative, $Q(t) < 10^{-80}$. Clearly this is outside of the realm of detection.

In the small-time domain we have other physical considerations that may prevent the conditions for Zeno's paradox from manifesting. This is ultimately to be traced to the atomic structure of matter and therefore to our inability to monitor the unstable system continuously. For example, in our model II, where Zeno's paradox is operative, in the Appendix one finds $T_1 \sim 10^{-14}/\Gamma \sim 10^{-21}$ sec for charged-pion decay. On the other hand, we have check points at interatomic distances, a time of the order of $10^{-8}/(3\times 10^{10}) \simeq 3\times 10^{-19}$ sec. We have no way of monitoring the natural evolution of a system for times finer than this.

This resolution of Zeno's paradox is quite satisfactory as resolutions go in modern physics, but it raises a more disturbing question: Is the continued existence of a quantum world unverifiable? Is the sum total of experience (of the quantum world) a sequence of still frames that we insist on endowing with a continuity?[13] Is this then the resolution of Zeno's paradox?

Note added in proof. After this work was completed, we learned from L. Khalfin that he had previously considered in some detail the small-t behavior of an unstable quantum system. For instance, he also had arguments similar to that leading to Eq. (15). [See L. A. Khalfin, Zh. Eksp. Teor. Fiz. Pis'ma Red. 8, 106 (1968) [JETP Lett. 8, 65 (1968), and some earlier references quoted therein.] We thank A. Goldhaber, who first called our attention to the small-t behavior work of Khalfin. Both Khalfin's and our considerations apply to the case where the subspace of undecayed state is one dimensional. Another argument showing the necessity of the departure from the exponential decay law for small time has been considered by us [see B. Misra and K. B. Sinha, Helv. Phys. Acta 50, 99 (1977)], which applies even when the subspace of undecayed states is infinite dimensional. We also want to mention a paper by L. Fonda, in Proceedings of the XIII Winter School of Theoretical Physics, Karpacz, Poland, 1976 (unpublished). In particular, assuming that the interaction between the experimental apparatus and the measured system is a Poisson process, Degasperis *et al.* have derived an integral expression relating the measured lifetime to the nondecay probability for the undisturbed system and the mean frequency of the interactions. [See A. Degasperis, L. Fonda, and G. C. Ghirardi, Nuovo Cimento 21A, 471 (1974).] In the limit of infinitely frequent interactions this leads to nondecay. On the other hand, assuming the measuring interaction takes place at regular intervals, J. Rau of Ref. 10 derived another law for the measured lifetime, which also leads to nondecay in the same limit. The theorem on Zeno's paradox given in Ref. 4 proves the nondecay result quite generally, which includes the above-mentioned results as special cases. We also would like to mention a paper closely related to our work by R. G. Winter, Phys. Rev. 123, 1503 (1961). In this work, a simple barrier-penetration problem was studied to elucidate the time development of quasistationary states in the small, the intermediate, and the large time regions. Some interference phenomena are stressed. We thank Dr. Winter for calling our attention to his interesting work.

APPENDIX

Model I

Consider the case

$$\xi(k) = 1 \ . \tag{A1}$$

Now Eq. (30) reads

$$i\tilde{\gamma}(k) = (k - k_+)(k - k_-) \ ,$$

or from (29), (31), and (32),

$$\gamma(E) = E - (E_0 + \Delta) + 2i\sqrt{E\Delta} \ . \tag{A2}$$

We assume the analytic continuation of $\gamma(E)$ is given by

$$\gamma(z) = z - (E_0 + \Delta) + 2\sqrt{-z\Delta} \ . \tag{A3}$$

One finds that $\gamma(z)$ satisfies the once-subtracted dispersion relation,

$$\gamma(z) = z - (\sqrt{E_0} + \sqrt{\Delta})^2 + \frac{z + E_0}{\pi} \int_0^\infty dE \, \frac{2\sqrt{\Delta E}}{(E - z)(E - E_0)} \ , \tag{A4}$$

where the subtraction is at $z = -E_0$. This is in accord with (27).

From (34), for the present case the pole term is given by

$$a_+(t) = \frac{k_+}{k_0} e^{-iE_0 t - \Gamma t/2} \ . \tag{A5}$$

From (35),

$$a_1(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{2ke^{-k^2 t}}{k^2 - k_0^2} \left(1 + \frac{2\delta k}{k^2 - k_0^2}\right) dk \ . \tag{A6}$$

To evaluate $a_1(t)$, we use the identity[14]

$$\int_0^\infty d\lambda \, \frac{e^{-\lambda t}\lambda^{1/2}}{\lambda + a} = \left(\frac{\pi}{t}\right)^{1/2} - \pi\sqrt{a}\, e^{at} \operatorname{erfc}[(at)^{1/2}] \ . \tag{A7}$$

Quantities "erf" and "erfc" are,[15] respectively, the error function and complementary function, with

$$\operatorname{erfc}(y) \equiv 1 - \operatorname{erf}(y)$$

and (A8)

$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx \ .$$

After some algebra we get,

$$a_1(t) = \frac{i\delta}{p}\left[(1 + 2p^2 t)e^{p^2 t} \operatorname{erfc}(p\sqrt{t}) - 2p\left(\frac{t}{\pi}\right)^{1/2}\right] \ , \tag{A9}$$

where $p = e^{-i\pi/2}k_0 = e^{-i\pi/4}E_0^{1/2}$.

The complementary error function has the asymptotic expansion[15]

$$\operatorname{erfc}(y) = \frac{e^{-y^2}}{\sqrt{\pi}y}\left(1 - \frac{1}{2y^2} + \frac{3}{4}\frac{1}{y^4} + \cdots\right) . \tag{A10}$$

Substituting (A10) into (A9), to leading order in the inverse power of t, gives

$$a_1 \sim \frac{\delta i}{\sqrt{\pi}\, p^4} \frac{1}{t^{3/2}} \ . \tag{A11}$$

This approximation is reasonable typically for $|p\sqrt{t}| \gtrsim y_1 = 5$. Notice that the form of (A11) is in agreement with our general result in the text. Dropping factors of the order of unity, (A11) gives

$$|a_1| \sim \left(\frac{\Gamma}{E_0}\right)^{5/2} \frac{1}{(\Gamma t)^{3/2}} \ ,$$

for (A12)

$$t \gtrsim T_1 \equiv \frac{y_1^2}{E_0} \ .$$

At T_1, the ratio of this background contribution to the pole contribution is given by

$$R = \frac{|a_1|}{|a_+|} \sim \frac{\Gamma}{E_0 y_1^3} \ , \tag{A13}$$

where $\Gamma \ll E_0$ was used. Take the example of a charged pion. It decays into the $\mu\nu$ state. Here

$$E_0 \equiv m_\pi - m_\mu = 34 \text{ MeV} = (2 \times 10^{-23} \text{ sec})^{-1} \ ,$$

$$\Gamma = (3 \times 10^{-8} \text{ sec})^{-1}, \text{ and } \Gamma/E_0 \sim 10^{-15} \ .$$

So $T_1 \sim y_1^2/E_0 \sim 2 \times 10^{-14}/\Gamma \sim 10^{-21}$ sec. At this moment $R \sim 10^{-17}$.

Because of the exponential falloff of the pole term and the $t^{-3/2}$ falloff of the background, after a certain long time interval has elapsed, R will eventually be comparable to unity. Beyond this point, say $t = T_2$, the background contribution will quickly dominate. Making use of $\Gamma \ll E_0$, (A5) and (A12) lead to

$$T_2 \sim \frac{5}{\Gamma} \ln\left(\frac{E_0}{\Gamma}\right) + \frac{3}{\Gamma} \ln\left(5 \ln \frac{E_0}{\Gamma}\right) \ . \tag{A14}$$

For a charged pion,

$$T_2 \sim 180/\Gamma \ . \tag{A15}$$

At this T_2 the amplitude is down by $e^{-90} \sim 10^{-39}$.

For $t \ll T_1$, one can expand (A5) and (A9) in powers of $y = p\sqrt{t}$. Using the identity[15]

$$\operatorname{erf}(y) = \frac{2y}{\sqrt{\pi}}\left(1 - \tfrac{1}{3}y^2 + \cdots\right) , \tag{A16}$$

we arrive at

$$a(t) \equiv a_+(t) + a_1(t)$$
$$\sim 1 - 4i\delta\left(\frac{t}{\pi}\right)^{1/2} = 1 - 4e^{i\pi/4}\frac{\Delta^{1/2}}{\sqrt{\pi}}\sqrt{t} \ . \tag{A17}$$

In turn, as $t \to 0$,

$$\dot{Q}(t) \equiv \frac{d}{dt}|a|^2 \sim -2\left(\frac{2\Delta}{\pi}\right)^{1/2}\frac{1}{t^{1/2}} \ . \tag{A18}$$

Model II

Consider the parametrization

$$\xi(k) = \frac{k-b}{k-c} . \tag{A19}$$

This will lead to the form of the Lee model, provided

$$c = b + 2\delta \quad \text{and} \quad b = B^{1/2}e^{-i\pi/4} , \tag{A20}$$

where B is a parameter chosen to be real and positive. For the complex variable z, (A19) can also be written as

$$\xi(z) = \frac{\sqrt{-z} - B^{1/2}}{\sqrt{-z} - (B^{1/2} + 2\Delta^{1/2})} \ . \tag{A19$'$}$$

The $(k-b)$ factor so specified gives rise to a "virtual state" at $E = e^{-i\pi/2}B$. Since we are interested in the resonance-decay problem, we will push this virtual state far away from the real axis. So whenever needed we will only specialize to the case

$$B \gtrsim E_0 \ . \tag{A21}$$

For present ξ, (30) leads to

$$i\tilde{\gamma}(k) = (k-k_+)(k-k_-)\frac{k-b}{k-(b+2\delta)}$$
$$= k^2 + (2b\delta + k_+k_-)$$
$$+ \frac{2\delta[b(b+2\delta) + k_+k_-]}{k-(b+2\delta)} \ , \tag{A22}$$

or

$$\gamma(z) = \tilde{\gamma}(k) = z - (E_0 + \Delta + 2\sqrt{B\Delta})$$
$$- \frac{2[E_0 + \Delta + (B + 2\sqrt{\Delta B})]\Delta^{1/2}}{z^{1/2}e^{i\pi/2} - (\sqrt{B} + 2\sqrt{\Delta})} \ . \tag{A23}$$

This expression can be rewritten as

$$\gamma(z) = z - (E_0 + \Delta 2\sqrt{B\Delta})$$
$$+ \frac{1}{\pi}\int_0^\infty \frac{2[E_0 + (\sqrt{\Delta} + \sqrt{B})^2](\Delta E)^{1/2}dE}{[E + (\sqrt{B} + 2\sqrt{\Delta})^2](E - z)} \tag{A24}$$

in agreement with the general form of (25).

From (34), for the present case the pole term is given by

$$a_+(t) = \frac{k_+ - (b+2\delta)}{k_+ - b}\,\frac{k_+}{k_1}\,e^{-iE_0t}e^{-\Gamma t/2} \ . \tag{A25}$$

To evaluate $a_1(t)$, we write

$$\frac{1}{\xi} \equiv \frac{k-(b+2\delta)}{k-b} = 1 - \frac{2\delta}{k} - \frac{2\delta b}{k(k-b)} \ ,$$

and define

$$a_1(t) \equiv a_1'(t) + a_1''(t) + a_1'''(t) \ , \tag{A26}$$

where

$$a_1'(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{2ke^{-k^2t}}{k^2-k_0^2}\left(1 + \frac{2\delta k}{k^2-k_0^2}\right)dk \ ,$$

$$a_1''(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{2ke^{-k^2t}}{k^2-k_0^2}\left(-\frac{2\delta}{k}\right)\left(1 + \frac{2\delta k}{k^2-k_0^2}\right)dk \ ,$$

and

$$a_1'''(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{2ke^{-k^2t}}{k^2-k_0^2}\left(\frac{-2\delta b}{k(k-b)}\right)\left(1 + \frac{2\delta k}{k^2-k_0^2}\right)dk \ .$$

From (A6) and (A9),

$$a_1'(t) = \frac{i\delta}{p}\left[(1 + 2p^2t)e^{p^2t}\operatorname{erfc}(p\sqrt{t}) - 2p\left(\frac{t}{\pi}\right)^{1/2}\right], \tag{A27}$$

where $p = e^{-i\pi/2}k_0 = e^{-i\pi/4}E_0^{1/2}$.

For the a_2'' term, we use the identity[14]

$$\int_0^\infty \frac{e^{-t\lambda}}{\lambda^{1/2}(\lambda + a)}d\lambda = \frac{\pi}{\sqrt{a}}e^{at}\operatorname{erfc}(\sqrt{at}) \ . \tag{A28}$$

To leading order in $|\delta/k_0|$, it gives

$$a_1''(t) \approx -\frac{2\delta i}{p}e^{p^2t}\operatorname{erfc}(p\sqrt{t}) \ . \tag{A29}$$

For a_1''' term, again to leading order in $|\delta/k_0|$, we write

$$a_1'''(t) = -\frac{2\delta b^2 i}{\pi(k_0^2 - b^2)}\int_{-\infty}^{\infty}e^{-k^2t}\left(\frac{1}{k^2-k_0^2} - \frac{1}{k^2-b^2}\right)dk$$
$$= \frac{2b^2\delta i}{b^2+p^2}\left(\frac{1}{p}e^{p^2t}\operatorname{erfc}(p\sqrt{t}) - \frac{1}{p_1}e^{p_1^2t}\operatorname{erfc}(p_1\sqrt{t})\right) , \tag{A30}$$

where $p_1 = e^{i\pi/2}b = e^{i\pi/4}B^{1/2}$. Combining (A27), (A29), and (A30) gives

$$a_1(t) = \frac{i\delta}{p}\left[(2p^2t-1)e^{p^2t}\,\mathrm{erfc}(p\sqrt{t}) - 2p\left(\frac{t}{\pi}\right)^{1/2}\right] + \frac{2b^2\delta i}{b^2+p^2}\left(\frac{1}{p}e^{p^2t}\,\mathrm{erfc}(p\sqrt{t}) - \frac{1}{p_1}e^{p_1{}^2t}\,\mathrm{erfc}(p_1\sqrt{t})\right). \qquad \text{(A31)}$$

Substituting (A10) into (A31), to leading order in $1/t$ one finds

$$a_1 \approx \frac{\delta i}{\sqrt{\pi}\,p^2}\left(\frac{1}{p^2}+\frac{1}{b^2}\right)\frac{1}{t^{3/2}}\ . \qquad \text{(A32)}$$

We assume, as discussed earlier, $B \gtrsim E_0$. Dropping a factor of the order of unity, (A32) gives

$$|a_1| \sim \left(\frac{\Gamma}{E_0}\right)^{5/2}\frac{1}{(\Gamma t)^{3/2}}\ , \quad \text{for} \quad t \gtrsim T_1 = \frac{y_1{}^2}{E_0}\ . \qquad \text{(A33)}$$

This is identical to the result of model I. Again $T_1 \sim 10^{-21}$ sec. At this moment $R \sim 10^{-17}$.

For $t \ll T_1$, from (A16), (A25), and (A31) we arrive at

$$a(t) \sim 1 - i(E_0 + 2\sqrt{B\Delta} + \Delta)t - \frac{8}{3\sqrt{\pi}}e^{-i\pi/4}\Delta^{1/2}(E_0+B)t^{3/2}\ . \qquad \text{(A34)}$$

This leads to, as $t \to 0$,

$$Q(t) \sim 1 - \frac{8}{3}\left(\frac{2\Delta}{\pi}\right)^{1/2}(E_0+B)t^{3/2} \qquad \text{(A35)}$$

and

$$\dot{Q}(t) \sim -4\left(\frac{2\Delta}{\pi}\right)^{1/2}(E_0+B)t^{1/2} \to 0\ . \qquad \text{(A36)}$$

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