Heisenberg Equations of Motion for the Charged Spin-$\frac{3}{2}$ Fields*

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We show that contrary to recent claims, the Lagrange and Heisenberg equations of motion are the same for the quantized charged spin-$\frac{3}{2}$ field in the presence of a minimal external electromagnetic interaction. For dynamical systems with constraints depending upon external fields, the explicit dependence of the dynamical variables upon these external fields must be taken into account.

The quantization of fields in the presence of constraints involves serious problems which in some cases can be overcome using the technique of Schwinger[1] which is a special case of the general theory of Dirac.[2] A common difficulty that arises after the fields are quantized using such techniques is that the Lagrangian equations of motion are not necessarily consistent with the Heisenberg equations of motion. In the following we show that when the charged spin-$\frac{3}{2}$ field in the presence of a minimal external electromagnetic interaction is quantized using these techniques,[3] the Lagrange and Heisenberg equations of motion for the quantized fields are consistent. This demonstration of consistency refutes several recent claims.[4,5]

Because the external electromagnetic field is an explicit function of space-time, we must use a slightly different notation for derivatives than is customary in quantum field theory. We use $d_\mu = d/dx^\mu$ to indicate a total derivative and $\partial_\mu = \partial/\partial x^\mu$ to designate a partial derivative. Hamilton's equation, for example, is then written

$$\dot{v} \equiv d_0 v = -i[v, H] + \partial_0 v \,, \tag{1}$$

where $[A, B]$ is the commutator and H is the Hamiltonian. Often in quantum field theory, none of the objects are explicit functions of space-time. In that case all the terms of the form $\partial_\mu v$ are zero so the symbol ∂_μ becomes free and so may be used to indicate a total derivative rather than a partial derivative with no resulting confusion.

With the exception of derivatives, our notation is that of Bjorken and Drell.[6] The space-time coordinates are denoted by $x^\mu = (t, x^1, x^2, x^3)$ and we use the metric tensor $g^{\mu\nu}$ where $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$. The Dirac γ matrices γ^μ satisfy $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$. Greek indices range from 0 through 3, Roman indices range from 1 through 3, and all repeated indices are summed over the range of the index.

The spin-$\frac{3}{2}$ field can be conviently represented by the Rarita-Schwinger[7] vector spinor ψ^λ which, in the presence of a minimal external electromagnetic interaction, obeys field equations that can be obtained from the Lagrangian density[4]

$$L = +[\bar{\psi}_\mu(D_\sigma\gamma^\sigma + m)\psi^\mu - \bar{\psi}_\mu(D_\nu\gamma^\mu + D^\mu\gamma^\nu)\psi_\nu + \bar{\psi}_\mu\gamma^\mu(D^\rho\gamma_\rho - m)\gamma^\nu\psi_\nu] \,, \tag{2}$$

where $D^\mu = -id^\mu - eqA^\mu$, e is the charge on an electron ($e < 0$), and q is the charge operator acting on the spin-$\frac{3}{2}$ field. The field equation obtained from (2) is

$$(D_\sigma\gamma^\sigma + m)\psi^\mu - (D^\nu\gamma^\mu + D^\mu\gamma^\nu)\psi_\nu + \gamma^\mu(D^\rho\gamma_\rho - m)\gamma^\nu\psi_\nu = 0 \,. \tag{3}$$

Taking $\mu = 0$ in (3) yields the primary constraint

$$-D^i\psi_i + D^i\gamma_i\gamma^j\psi_j - m\gamma^i\psi_i = 0 \,. \tag{4}$$

At this point it is convenient to introduce the new field variables

$$\phi_j = P_{jh}\psi^h \,, \tag{5}$$

and

$$\chi = \gamma_j\psi^j \,, \tag{6}$$

where

$$P_{jh} = g_{jh} - \tfrac{1}{3}\gamma_j\gamma_h \,. \tag{7}$$

From the definition of P_{jh} we readily verify that it satisfies the following relations:

$$\begin{aligned} \gamma^j P_{jh} &= P_{jh}\gamma^h \\ &= 0 \,, \\ P_{jh}P^h{}_r &= P_{jr} \,, \\ P_{jh}^\dagger &= P_{hj} \,. \end{aligned} \tag{8}$$

In terms of ϕ_i and χ, the primary constraint (4) becomes

$$\chi = -\tfrac{3}{2}(\tfrac{3}{2}m - D^i\gamma_i)^{-1}D^k\phi_k \,. \tag{9}$$

Taking $\mu = i$ in the field equation (3), rewriting it in terms of ϕ_i and χ, multiplying by P_{ri}, and summing over i allows the equation of motion for

ϕ_r to be written as

$$d_0\phi_r = -i\gamma_0(D_i\gamma^i + m)\phi_r + \tfrac{1}{3}2i\gamma_0(-D_r + \gamma_r D_i\gamma^i - m\gamma_r)\chi + ieqA_0\phi_r + i\gamma_0 P_{ri} D^i G \ . \qquad (10)$$

Here

$$G = ieq(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF_{kl}\sigma^{kl})^{-1} \times [\gamma^\mu F_{\mu j}\phi^j - \tfrac{1}{3}iF_{ij}\sigma^{ij}\chi - \tfrac{2}{3}\gamma_0\gamma^j F_{0j}\chi] \ , \qquad (11)$$

and

$$\sigma^{kl} = \tfrac{1}{2}i[\gamma^k, \gamma^l] \ .$$

To verify that the Heisenberg equations are satisfied we will need the identity

$$P^{ir}D_iD_r = -\tfrac{2}{3}(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{ir}\sigma_{ir}) + (\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{ir}\sigma_{ir}) \ , \qquad (12)$$

where $F^{\mu\nu} = d^\nu A^\mu - d^\mu A^\nu = \partial^\nu A^\mu - \partial^\mu A^\nu$.

Using the Schwinger quantization procedure, we are led to the quantization conditions[3]

$$\{\phi_i(x,t), \phi_j(x',t)\} = 0 \qquad (13)$$

and

$$\{\phi_i(x,t), \phi_j^\dagger(x',t)\} = -P_{ir}[g^{rk} - D^r(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{ab}\sigma_{ab})^{-1}D^k]P_{kj}\delta^3(x-x') \ . \qquad (14)$$

In the presence of a minimal external electromagnetic interaction, the Heisenberg equations of motion take the form

$$d^\nu\phi_j = -i[\phi_j, P^\nu] + \partial^\nu\phi_j \ . \qquad (15)$$

The commutation relations (14) already show that the quantities ϕ_j depend explicitly on time, but the dependence is too complicated for us to display in a direct fashion. At first sight it might seem strange that the ϕ_j are explicit functions of time, but this comes about because of our primary constraint. If we arbitrarily decide that ψ_1 and ψ_2 are not explicit functions of time, then we can use the primary constraint (4) to express ψ_3 as a function of ψ_1, ψ_2, and the electromagnetic field. By virtue of the fact that the electromagnetic field is an explicit function of time, ψ_3 also contains explicit time dependence. So at least one of the ψ_i's must contain explicit time dependence. Because we want to treat all of the ψ_i's and, consequently, all of the ϕ_i's on the same footing, all of the ψ_i's and all of the ϕ_i's contain explicit time dependence.

From the action principle we find that P^ν, the generator of space-time translations, is given by

$$P^\nu = -i\int d^3x[\phi^{i\dagger}d^\nu\phi_i + \tfrac{2}{3}\chi^\dagger d^\nu\chi] \ . \qquad (16)$$

Using the primary constraint (9) we calculate

$$d^\nu\chi = -\tfrac{3}{2}(\tfrac{3}{2}m - D^j\gamma_j)^{-1}[D^i d^\nu\phi_i + eq(\partial^\nu A^i)(-\phi_i + \tfrac{2}{3}\gamma_i\chi)] \ . \qquad (17)$$

With (17) and $\chi^\dagger$ as determined from (9), P^ν takes the form

$$P^\nu = -i\int d^3x\{\phi_k^\dagger[g^{ki} + \tfrac{3}{2}D^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}D^i]d^\nu\phi_i + \tfrac{3}{2}eq\phi_k^\dagger D^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^\nu A^i)(-\phi_i + \tfrac{2}{3}\gamma_i\chi)\} \ . \qquad (18)$$

We are now in a position to verify that ϕ_i and χ obey the Heisenberg equations of motion. Using (18) and the identity $[A, BC] = \{A, B\}C - B\{C, A\}$ we calculate

$$\begin{aligned}-i[\phi_j(x',t), P^\nu] = -\int d^3x\Big\{&\{\phi_j(x',t), \phi_k^\dagger\}[g^{ki} + \tfrac{3}{2}D^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}D^i]d^\nu\phi_i \\ &- \phi_k^\dagger[g^{ki} + \tfrac{3}{2}D^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}D^i]\{d^\nu\phi_i, \phi_j(x',t)\} \\ &+ \{\phi_j(x',t), \phi_k^\dagger\}\tfrac{3}{2}eqD^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^\nu A^i)(-\phi_i + \tfrac{2}{3}\gamma_i\chi) \\ &- \tfrac{3}{2}eq\phi_k^\dagger D^k(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^\nu A^i)\{-\phi_i + \tfrac{2}{3}\gamma_i\chi, \phi_j(x',t)\}\Big\} \ . \qquad (19)\end{aligned}$$

From (13), ϕ_j anticommutes with ϕ_i and its space derivatives, so the fourth term in (19) is zero. Since $d^\nu\phi_i$ can be expressed in terms of ϕ_j and its space derivatives [Eq. (10)] $d^0\phi_i$ as well as $d^k\phi_i$ anticommutes with ϕ_j. Consequently, the second term in (19) is also zero. Using the adjoint of (14) and the identity (12), we find

$$-i[\phi_j(x',t), P^\nu] = \int d^3x[\delta^3(x-x')d^\nu\phi_j - \delta^3(x-x')eqP_{jn}D^n(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^\nu A^i)(\phi_i - \tfrac{2}{3}\gamma_i\chi)] \ . \qquad (20)$$

If we carry out the integration and then make the substitution $x' \to x$ we obtain

$$d^{\nu}\phi_j = -i[\phi_j, P^{\nu}] + eqP_{jn}D^n(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^{\nu}A^i)(\phi_i - \tfrac{2}{3}\gamma_i\chi) . \tag{21}$$

By comparing (21) and (15), we see that consistency demands that

$$\partial^{\nu}\phi_j = eqP_{jn}D^n(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^{\nu}A)(\phi_i - \tfrac{2}{3}\gamma_i\chi) , \tag{22}$$

yielding the dependence of ϕ_j on the external field A^i. At this point we can calculate $\partial^{\nu}\chi$ from the equation

$$\partial^{\nu}\chi = \frac{\partial\chi}{\partial A^i}\,\partial^{\nu}A^i + \frac{\partial\chi}{\partial\phi_i}\,\partial^{\nu}\phi_i .$$

Using (9) and (22) leads to the result

$$\partial^{\nu}\chi = eq(\tfrac{3}{2}m + D^i\gamma_i)(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^{\nu}A^j)(\phi_j - \tfrac{2}{3}\gamma_j\chi) . \tag{23}$$

We have deduced (22) and (23) by assuming that ϕ_j satisfies the Heisenberg equations. Equation (23) permits us to verify that χ obeys the Heisenberg equations of motion. Using the primary constraint (9)

$$-i[\chi(x,t), P^{\nu}] = \tfrac{3}{2}i(\tfrac{3}{2}m - D^j\gamma_j)^{-1}D_k[\phi^k(x,t), P^{\nu}] .$$

Using (21), (12), and

$$(\tfrac{9}{4}m^2 - D^rD_r + \tfrac{1}{2}eqF^{rs}\sigma_{rs}) = (\tfrac{3}{2}m - D^r\gamma_r)(\tfrac{3}{2}m + D^s\gamma_s) ,$$

we find

$$-i[\chi(x,t), P^{\nu}] = -\tfrac{3}{2}(\tfrac{3}{2}m - D^j\gamma_j)^{-1}D_k d^{\nu}\phi^k + \tfrac{3}{2}eq(\tfrac{3}{2}m - D^j\gamma_j)^{-1}(\partial^{\nu}A^i)(\phi_i - \tfrac{2}{3}\gamma_i\chi)$$
$$-eq(\tfrac{3}{2}m + D^i\gamma_i)(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^{\nu}A^j)(\phi_j - \tfrac{2}{3}\gamma_j\chi) . \tag{24}$$

From the formula $\partial^{\nu}R^{-1} = -R^{-1}[\partial^{\nu}R]R^{-1}$, we readily verify that the second term of (24) becomes

$$\tfrac{3}{2}eq(\tfrac{3}{2}m - D^j\gamma_j)^{-1}(\partial^{\nu}A^i)(\phi_i - \tfrac{2}{3}\gamma_i\chi) = \{d^{\nu}[-\tfrac{3}{2}(\tfrac{3}{2}m - D^j\gamma_j)^{-1}D_k]\}\phi^k . \tag{25}$$

So with (25), (24) can be rewritten

$$d^{\nu}\chi = -i[\chi, P^{\nu}] + eq(\tfrac{3}{2}m + D^i\gamma_i)(\tfrac{3}{2}m^2 + \tfrac{1}{2}eqF^{rs}\sigma_{rs})^{-1}(\partial^{\nu}A^j)(\phi_j - \tfrac{2}{3}\gamma_j\chi) . \tag{26}$$

Equation (26) is in agreement with the Heisenberg equations of motion since by (23) the last term in (26) is $\partial^{\nu}\chi$.

From the above we conclude that the quantized spin-$\frac{3}{2}$ field in the presence of a minimal external electromagnetic interaction does obey the Heisenberg equations of motion although the equations take the general form of (1). The form of (1) might have been expected from the general result of the paper of Johnson and Sudarshan that the kinematics of the spin-$\frac{3}{2}$ field necessarily involves the dynamics.[3]

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