

EQUIVALENCE OF SEMICLASSICAL AND QUANTUM MECHANICAL DESCRIPTIONS OF STATISTICAL LIGHT BEAMS

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With the advent of the laser, attention has been focused on the problem of the complete description of the electromagnetic field associated with arbitrary light beams. The classical theory of optical coherence¹ works almost exclusively with two-point correlations; and this theory is adequate for the description of the classical optical phenomena of interference and diffraction in general. More sophisticated experiments on intensity interferometry and photoelectric counting statistics necessitated special higher order correlations. Most of this work² was done using a classical or a semiclassical formulation of the problem. On the other hand, statistical states of a quantized (electromagnetic) field have been considered recently,³ and a quantum mechanical definition of coherence functions of arbitrary order presented. It is the aim of this note to elaborate on this definition and to demonstrate its complete equivalence to the classical description as long as no nonlinear effects are considered.

We begin with an outline of the analytic function representation⁴ of canonical creation and destruction operators. If a and a^\dagger satisfy the relations

$$[a, a^\dagger] = 1,$$

every irreducible representation is equivalent to the Fock representation in terms of the states $\psi(n)$, satisfying

$$a^\dagger a \psi(n) = n \psi(n); \quad (\psi(m), \psi(n)) = \delta_{mn}.$$

The matrix elements of a and a^\dagger in this representation are

tation are

$$(\psi(m), a \psi(n)) = \sqrt{n} \delta_{m, n-1},$$

$$(\psi(m), a^\dagger \psi(n)) = (n+1)^{1/2} \delta_{m, n+1}.$$

One could, however, introduce an overcomplete set of eigenstates of the destruction operator given by

$$|re^{i\theta}\rangle \equiv |z\rangle = \exp(-\frac{1}{2}|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/2}} \psi(n), \quad (1)$$

satisfying

$$a|z\rangle = z|z\rangle; \quad \langle z|a^\dagger = z^* \langle z|; \quad \langle z|z\rangle = 1,$$

for every complex number z . These states are all normalized but not orthogonal⁵; they are complete in the sense that they furnish a resolution of the identity

$$1 = (1/\pi) \iint r dr d\theta |re^{i\theta}\rangle \langle re^{i\theta}|.$$

More generally,

$$\int \frac{d\theta}{2\pi} |re^{i\theta}\rangle \langle re^{i\theta}| = e^{-r^2} \sum_{n=0}^{\infty} \frac{r^{2n}}{n!} \psi(n) \psi^\dagger(n). \quad (2)$$

We can make use of the overcompleteness⁶ of the states to represent every density matrix,

$$\rho = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \rho(n, n') \psi(n) \psi^\dagger(n'),$$

in the "diagonal" form

$$\rho = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \rho(n, n') \frac{(n!n'!)^{1/2}}{(n+n')!} \left\{ \left(\frac{\partial}{\partial r} \right)^{n+n'} \int \frac{d\theta}{2\pi} \exp[r^2 + i(n'-n)\theta] |re^{i\theta}\rangle \langle re^{i\theta}| \right\} \Big|_{r=0}. \quad (3)$$

This form is particularly interesting since if $O = (a^\dagger)^\lambda a^\mu$ be any normal ordered operator (i. e., all creation operators to the left of all annihilation operators), its expectation value in the statistical state represented by the density matrix in the "diagonal" form

$$\rho = \int d^2z \phi(z) |z\rangle \langle z| \quad (4)$$

is given by

$$\text{tr}\{\rho O\} = \text{tr}\{\rho (a^\dagger)^\lambda a^\mu\} = \int d^2z \phi(z) (z^*)^\lambda z^\mu. \quad (5)$$

This is the same as the expectation value of the complex classical function $(z^*)^\lambda z^\mu$ for a probability distribution $\phi(z)$ over the complex plane. The demonstration above shows that any statistical state of the quantum mechanical system may be described by a classical probability distribution over a complex plane, provided all operators are written in the normal ordered form. In other words, the classical complex representations¹ can be put in one-to-one correspondence with

quantum mechanical density matrices. Hermiticity of ρ implies that $\phi(z)$ is a "real" function in the sense that $\phi^*(z^*) = \phi(z)$, but not necessarily positive definite.

These considerations generalize in a straightforward manner to an arbitrary (countable) number of degrees of freedom, finite or infinite.⁷ The states are now represented by a sequence of complex numbers $\{z\}$; and the Fock representation basis is labeled by a sequence of non-negative integers $\{n\}$ and density matrices by functions of two such sequences $\rho(\{n\}, \{n'\})$. Any such state can be put into one-to-one correspondence with classical probability distributions in a sequence of complex variables $\phi(\{z\})$ such that the expectation value of any normal ordered operator $O(\{a^\dagger\}, \{a\})$ is given by

$$\text{tr}\{O(\{a^\dagger\}, \{a\})\rho\} = \prod_{\lambda} \int d^2z_{\lambda} O(\{z^*\}, \{z\}) \phi(\{z\}), \quad (6)$$

where the "real" function $\phi(\{z\})$ is given by

$$\phi(\{z\}) = \prod_{\lambda} \left[\sum_{n_{\lambda}=0}^{\infty} \sum_{n'_{\lambda}=0}^{\infty} \frac{\rho(\{n\}, \{n'\}) (n_{\lambda}! n'_{\lambda}!)^{1/2}}{(n_{\lambda} + n'_{\lambda})! (2\pi r_{\lambda})} \exp[r_{\lambda}^2 + i(n'_{\lambda} - n_{\lambda})\theta_{\lambda}] \left\{ \left(\frac{\partial}{\partial r_{\lambda}} \right)^{n_{\lambda} + n'_{\lambda}} \delta(r_{\lambda}) \right\} \right]. \quad (6)$$

Consequently the description of statistical states of a quantum mechanical system with an arbitrary (countably infinite) number of degrees of freedom is completely equivalent to the description in terms of classical probability distributions in the same (countably infinite) number of complex variables. In particular, the statistical states of the quantized electromagnetic field may be described uniquely by classical complex linear functions on the classical electromagnetic field. This functional will be "real" reflecting the Hermiticity of the density matrix; and leads in either version to real expectation values for Hermitian (real) dynamical variables.

Several additional remarks are in order. Firstly, since the states $|z\rangle$ are eigenfunctions of the annihilation operators, the analogous states for a quantized field are eigenfunctions of the positive-frequency (annihilation) part of the field. The corresponding classical theory should then work with positive-frequency parts of the classical field; but this is precisely what is involved in the concept of the (classical) analytic signal.¹ Secondly, while thermal beams are usually represented by Gaussian classical probability func-

tions corresponding to a density matrix diagonal in the occupation numbers $\{n\}$ given by the grand canonical ensemble for the blackbody radiation, there are other probability functions! A particular one may not be diagonal in the occupation number sequence $\{n\}$ and this implies, in accordance with Eq. (6), that not all phase-angle sequences $\{\theta\}$ have equal weight. In such a case the expectation values of operators with unequal number of creation and destruction operators need not all vanish. We note in passing, that Eq. (6) for $\phi(\{z\})$ in terms of $\rho(\{n\}, \{n'\})$ can be inverted to yield

$$\rho(\{n\}, \{n'\}) = \prod_{\lambda} \int d^2z_{\lambda} \phi(\{z\}) e^{-|z_{\lambda}|^2} \frac{z_{\lambda}^{n_{\lambda}} (z_{\lambda}^*)^{n'_{\lambda}}}{(n_{\lambda}! n'_{\lambda}!)^{1/2}}. \quad (7)$$

If we do this for the Gaussian functions, we obtain the Bose-Einstein distribution, diagonal in the occupation number sequences corresponding to the equal weightage of all phase angles. It is worth pointing out that this result reproduces the Purcell-Mandel derivation⁸ for photoelectric counting statistics. The method of inverting the expectation values to obtain the probability dis-

tribution as the Fourier transform of the characteristic function can be generalized in the present case, using probability functions and characteristic functionals.⁹ The methods developed here are thus adequate to determine the quantum mechanical density matrix, provided all the correlation functions are given.¹⁰

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¹For a comprehensive review, see M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1959), Chap. X.

²See L. Mandel (to be published), for a systematic account.

³R. J. Glauber, *Phys. Rev. Letters* **10**, 84 (1963).

⁴V. Bargmann, *Commun. Pure Appl. Math.* **14**, 187 (1961); I. E. Segal (to be published); J. R. Klauder, *Ann. Phys. (N.Y.)* **11**, 123 (1960); S. S. Schweber,

J. Math. Phys. **3**, 831 (1962).

⁵These states are easily represented in a Schrödinger coordinate representation

$$|0\rangle \equiv \psi(0) \rightarrow \pi^{-1/4} \exp(-\frac{1}{2}x^2),$$

$$|z\rangle = \exp(za^\dagger)\psi(0) \rightarrow \pi^{-1/4} \exp(-\frac{1}{2}[x+z]^2),$$

as pointed out by Dr. C. Ryan.

⁶For a general theory of representation by an overcomplete family of states, see J. R. Klauder (to be published).

⁷All strange representations of the infinite canonical ring are deliberately ignored here.

⁸E. M. Purcell, *Nature* **178**, 1449 (1956). L. Mandel, *Proc. Phys. Soc. (London)* **71**, 1037 (1958); **74**, 233 (1959). See also reference 2.

⁹I. E. Segal, *Com. J. Math.* **13**, 1 (1961). See also E. Hopf, *J. Rat. Mech. Anal.* **2**, 587 (1953). These have to be generalized here by admitting indefinite linear functionals.

¹⁰The only case where all correlation functions are known is for the important but familiar example of the blackbody radiation. We hope that this circumstance is not time independent!