PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Noninvariance Groups in Particle Physics*

JACOB G. KURIYAN
Syracuse University, Syracuse, New York
and
Institute for Advanced Study, Princeton, New Jersey†

AND

E. C. G. SUDARSHAN
Syracuse University, Syracuse, New York
(Received 28 March 1967)

The concept of a noninvariance group is introduced by considering some classical systems. This is then used to study properties of strongly interacting particles. A logically consistent way of deriving static properties such as magnetic moments and electromagnetic mass differences, as well as predictions regarding scattering cross sections, is described.

INTRODUCTION

IN the past few years, group theory has emerged as a powerful tool in the study of elementary particles. The conventional approach in applications of group theory to particle physics is to postulate that the particles form a multiplet which furnishes a unitary irreducible representation (UIR) of a symmetry group K, and that K is an invariance group of the interaction Hamiltonian. The consequences of this postulate can be worked out systematically and checked against experiment.

If the invariance group is a small one, e.g., $SU(2)$ as isospin symmetry group, then the symmetry may be a fairly accurate description of nature. However, this will enable us to relate only a small number of particles with one another. In order that a larger number should be included in a multiplet, one is necessarily forced to consider higher symmetry groups such as $SU(3)$, $G(2)$, etc. Deviations from such symmetries are attributed to symmetry-breaking interactions. For groups such as $SU(6)$, this symmetry-breaking interaction has to be invoked quite early in the game, and the invariance one talks of is, at best, only a crude approximation.

One could, on the other hand, postulate a lower symmetry for the Hamiltonian and imbed this group in a larger one which is not an invariance group, but whose structure is determined by the dynamics of the physical system under consideration. This procedure has the attractive feature that one is able to relate various multiplets without invoking a larger symmetry breaking. The approach based on this noninvariance group[1] (henceforth called NIG) will be pursued in this paper.

In order to understand this approach, it is best to consider how the NIG's enter into the study of "classical" systems. We may then use this analogy as a guide in applying the approach to strongly interacting particles. Historically, this is the way the subject has developed, and we shall not deviate from this pattern.

In Sec. I we introduce the concept of the NIG by considering some examples from classical physical systems. By analogy, we apply NIG techniques to particle physics. In Sec. II, $SU(4)$, $SL(4,R)$, and $T_9\times SU(2)\otimes SU(2)$ are considered as NIG's. In Sec. III, $SU(6)$ is considered as a candidate for an NIG. In Sec. IV, we summarize the results of this approach.

I. THE NOTION OF NONINVARIANCE GROUPS

We recall that a sequence of states with common relevant properties is usually the clue to the identification of a simple underlying dynamical structure. For example, the doublet structure of the levels of atoms with one optical electron is a manifestation of the two-valued electron spin orientation; the singlet and triplet states of "two-electron" spectra relate to the composition of the spins of the two electrons. Further, the sequence of levels with splitting corresponding to the Landé interval rule furnishes evidence for a total orbital angular momentum **L** and total spin **S** which couple to form the total angular momentum **J**. In these cases, the sequence of levels of the atomic system was simply understood in terms of a dynamical substructure described by the $O(3)\otimes O(3)$ group, which was, however, not an invariance group of the Hamiltonian. We could proceed further and try to identify the set of all states of a dynamical system as a realization of a group; unless the Hamiltonian is a constant,

* Submitted in partial fulfillment of the Ph.D. degree at Syracuse University, 1966. Research supported in part by U. S. Atomic Energy Commission and the National Science Foundation.

† Present address.

[1] N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); Phys. Letters **19**, 322 (1965). A. O. Barut and A. Böhm have also considered this approach, and references to their work are given in Barut's lectures, in *Proceedings of the Seminar on High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Agency, Vienna, 1965); E. C. G. Sudarshan, Conference on Symmetries, 1965 (unpublished); Syracuse University Report No. NYO-3399-55 (unpublished), E. C. G. Sudarshan, invited paper at the Eastern Theoretical Physics Conference, Stony Brook, New York, 1965 (unpublished).

it is evident that this group cannot leave it invariant. We may therefore refer to such a group as a NIG.

For the hydrogen atom one could explicitly construct the generators of $O(3)$ and the Lenz vector [together they constitute an $O(4)$ group], which commute with the Hamiltonian. The representations[2] of $O(4)$ that one uses to classify the (negative) energy levels of the hydrogen atom are the degenerate representations—and the $O(4)$ labels uniquely specify an energy level.[3] For the positive energy states, one could use, similarly, the representations of the noncompact group $O(3,1)$. In the strong-coupling limit ($g \to \infty$) or the free-particle limit, the symmetry group is $E(3)$, the Euclidean group in three dimensions.

We can incorporate the dynamical information that there are various bound states by classifying a finite number of bound states in one representation of $O(5)$, or all except the first n levels in one UIR of $O(4,1)$.[4] The groups $O(4,1)$ and $O(5)$ contain transformations which take states with one energy into states with another energy and are not invariance groups of the Hamiltonian. These have been called the noninvariance groups.[1]

Detailed consideration of this system yields the following two results:

(a) When *all* the states of the dynamical system constitute a single irreducible representation of the NIG, the generalized enveloping algebra of the NIG is identical to the algebra of all dynamical variables. If only a subspace constitutes an irreducible representation, the enveloping algebra is the restriction of the algebra of dynamical variables to this subspace. The NIG is thus equivalent to the dynamical system.[5]

(b) Varying the energy eigenvalue of the Hamiltonian of the hydrogen atom (which corresponds to considering the bound or continuum states) and then restricting the values of the only arbitrary parameter that occurred in the expression for the noninvariant generators (which corresponds to considering various kinds of orbits: elliptical, hyperbolic, etc.), we obtained various kinds of NIG: $O(4,1)$, $O(3,2)$, and $O(5)$. Even though the NIG was constructed from the primitive dynamical variables and was artificial in a sense, we learned the important fact that the dynamics of the system governed the structure of the NIG.

[2] E. C. G. Sudarshan and N. Mukunda, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1965), Vol. **VIIIB**.

[3] V. Fock, Z. Physik **98**, 145 (1935); W. Pauli, *ibid.* **36**, 336 (1926); V. Bargmann, *ibid.* **99**, 576 (1936).

[4] J. Dixmier, Bull. Soc. Math. France **89**, 9 (1961); S. Ström, Arkiv Fysik **30**, 455 (1965); J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, Institute for Advanced Study Report (to be published).

[5] In the present case this is not strictly true since the NIG generalized enveloping algebra does not connect the negative-energy states with the positive-energy states (but q or p connect bound and unbound states). The NIG generalized enveloping algebra treats the bound states alone, for example, as the dynamical system.

In particle physics, on the other hand,[6] we know very little about the Hamiltonian for a description of strongly interacting particles. However, it is possible to introduce some kind of group structure and reason that the choice of the NIG will decide the dynamics of the system. That is, we suppose, on the basis of our analysis of the classical system, that the choice of the NIG implies a definite, though at present largely unknown, postulate on the dynamics of strongly interacting particles. In this sense, one could call these NIG "dynamical groups."

We can get some guidance in the choice of interactions for the particle spectrum described using a NIG by considering the treatment of interactions in atomic physics. The interaction with the radiation field is given by the gauge-invariant coupling; however, it is adequate in most cases to consider a dipole approximation with additional correction terms of quadrupole type, and so on. In addition to this most important coupling, there are other interactions like the Zeeman effect which are introduced by other interaction structures with corresponding parameters specifying the strength of the coupling. The selection rules in electric dipole transitions are quite different from those for the Zeeman effect. The situation for the pion-nucleon interaction can be viewed in the same fashion. The actual pion-nucleon interaction is not exclusively a p-wave coupling, but it is well approximated in the low-energy domain by the p-wave coupling. The electromagnetic and weak interactions are to be suitably specified in relation to the NIG, but with new characteristic coupling parameters. In contrast with the atomic case, we have no primary interaction structure which we are to approximate; hence the simplest choice of interactions should be considered first. The natural choice is for the coupling matrices to be linear in the generators.

This is the starting point of the application of the concept of a NIG to a dynamical system involving strongly interacting particles. The NIG we seek must have the *symmetry group* as a *subgroup*. Secondly, the noninvariant generators must be specified in some fashion; note that this is a *dynamical postulate*.

II. NONINVARIANCE GROUPS IN PARTICLE PHYSICS

At first, we consider a kind of "static" model involving nonstrange particles.[7] The invariance group of the Hamiltonian which describes such a system is

$$K = SU(2)^I \otimes SU(2)^J,$$

where I and J stand for isospin and spin, respectively. Low-lying isobars (baryons and baryon resonances)

[6] For further details see Ref. 1.

[7] J. G. Kuriyan and E. C. G. Sudarshan, Phys. Letters **21**, 106 (1966). N. Mukunda, in *Conference on Noncompact Groups in Particle Physics, Milwaukee, 1966* (W. A. Benjamin, Inc., New York, 1966).

described by such a Hamiltonian fall into multiplets which furnish a UIR of K. Pionic current sources form a multiplet of this invariance group. In this representation space the mesonic currents transform as tensor operators of the group K.

For the case at hand, we are considering a p-wave isotriplet of pions and so the tensor operator is of rank 1 with respect to the two $SU(2)$ subgroups. It has long been known that when the pion-nucleon coupling constant tends to infinity, an infinite sequence of bound states of the Hamiltonian (isobar levels) is generated. These have $I=J=$ (half an odd integer). That is to say, the form of the UIR is of the type (I,I) of the invariance group K, with $I=J=\frac{1}{2}, \frac{3}{2}, \cdots$. Our earlier analysis of groups enables us to conclude that one such NIG could be $SU(4)$, which, when reduced with respect to $O(4)$, yields the above class of representations.

We are still left with the most important aspect of the problem, to identify the noninvariant generators. We draw inspiration from a recent paper of Cook, Gobel, and Sakita (CGS).[8] They considered the scattering of mesons and baryons

$$M_\alpha+B_i\to\bar{M}_\beta+B_j,$$

where M_α and M_β are mesons and B_i and B_j are isobar states. The bar stands for the charge-conjugate meson.

The Chew-Low equation for this process is given by

$$\begin{aligned}t_{\beta\alpha}{}^{ji}(w)&=-g^2\sum_k\left(\frac{A_\beta{}^{jk}A_\alpha{}^{ki}}{M_k-M_i-w}+\frac{A_\alpha{}^{jk}A_\beta{}^{ki}}{M_k-M_j+w}\right)\\&\quad-\sum_{\mathbf{p},k}\left\{\frac{t_{\gamma\beta}{}^{kj}(w_p)^*t_{\gamma\alpha}{}^{ki}(w_p)}{M_k+w_p-M_i-w}+\frac{t_{\gamma\alpha}{}^{kj}(w_p)^*t_{\gamma\beta}{}^{ki}(w_p)}{M_k+w_p-M_j+w}\right\}\\&\quad+\text{ two or more meson intermediate states},\end{aligned}$$

where M_i is the energy of the ith isobar, $gA_\alpha{}^{ij}$ is the (ij)th matrix element of the source of meson α with energy w between isobar states, and g is a parameter that represents the strength of the coupling. In the strong-coupling limit A_α is *finite* and the masses of the isobars are degenerate. We now appeal to strong-coupling theory to guarantee an expansion of the coupling matrices and the isobar masses in inverse powers of the coupling strength g^2, i.e.,

$$M_i=M_0+\gamma_i/g^2.$$

In the expression for the scattering amplitude, we notice that the first two terms correspond to the driving terms (pole or renormalized Born terms), while the next two are the elastic-scattering (or one-meson intermediate-state) terms. CGS consider the pole terms and expand them in powers of $1/g^2$ to obtain

$$\begin{aligned}t_{\beta\alpha}{}^{ji}(w)|_{\text{pole}}\sim&-(g^2/w)[A_\beta,A_\alpha]^{ji}\\&+w^{-2}|([A_\beta,[\gamma,A_\alpha]])^{ji}+O(g^{-2}).\end{aligned}$$

Unitarity restricts the scattering amplitude to be finite in the physical region. In order that

$$t_{\beta\alpha}{}^{ji}(w)|_{\text{pole}}$$

be finite in the strong-coupling limit ($g^2\to\infty$), CGS argued that it is necessary to have

$$[A_\alpha,A_\beta]^{ij}=0.$$

They then deduced that in the strong-coupling limit the noninvariant generators A_α commuted among themselves. Thus the noninvariance group in the strong-coupling limit was the semidirect product $A\times K$, of K by an Abelian group A generated by the meson-coupling matrices.

For the charge-symmetric pseudoscalar theory, the group K is given by $SU(2)^I\otimes SU(2)^J$. Instead of A_α we shall use $Q_{i\beta}$ as the meson-coupling matrix that transforms as a vector with respect to both $SU(2)^I$ (isotriplet of pions) and $SU(2)^J$ (p-wave pseudoscalar mesons). i and β take the values 1, 2, 3. CGS's important dynamical assumption was to identify the pion-coupling matrices as the noninvariant generators. The NIG that they obtained in the strong-coupling limit was

$$T_9\times(SU(2)^I\otimes SU(2)^J),$$

where T_9 stands for the Abelian group generated by $Q_{i\alpha}$.

Various possible UIR of this group G provide the allowed isobar (multiplet) spectrum. The group concerned is a noncompact one (because T_9 is Abelian), and all UIR are infinite-dimensional. This implies that there exist an infinite number of isobars in every UIR, a well-known result in strong-coupling theory. One could argue that this undesirable feature is a consequence of the unnaturally high value of the coupling constant, which tends to push a large number of poles of the scattering amplitude onto the physical sheet. A more realistic limit would, perhaps, result in fewer of the poles approaching the physical sheet. Hence there seems to be some justification in the assumption that the low-lying isobars are the only ones of interest.

The plausible assumption of CGS is that once K is chosen the strength of the coupling decides the structure of the group G. This leads them naturally to a conjecture concerning the intermediate-coupling models, where the coupling constant, in the limit, attains a value lower than in the previous case.

At this point we shall redo with greater care CGS's analysis. They had concluded that $[A_\alpha,A_\beta]=0$ from the fact that the scattering amplitude of the form

$$\begin{aligned}t_{\alpha\beta}{}^{ij}(w)=&-(g^2/w)([A_\alpha,A_\beta])^{ij}\\&+w^{-2}([A_\alpha,[\gamma,A_\beta]])^{ij}+O(1/g^2)\end{aligned}$$

[8] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters 15, 35 (1965).

had to be bounded in the strong-coupling limit. However, we can expand the coupling matrices A_α in inverse powers of the coupling strength g^2. This is a consequence of the fact that A_α is finite in the strong-coupling limit and that with pure Yukawa coupling, the sign of the coupling can be changed at will by an obvious canonical transformation. Thus the expansion is in powers of $1/g^2$ rather than $1/g$:

$$A_\alpha\left(\frac{1}{g^2}\right)=A_\alpha{}^{(0)}+\frac{1}{g^2}A_\alpha{}^{(1)}+\frac{1}{g^4}A_\alpha{}^{(2)}+\cdots.$$

Thus

$$\begin{aligned}t_{\alpha\beta}{}^{ij}(w)=&-(g^2/w)[A_\alpha{}^{(0)},A_\beta{}^{(0)}]^{ij}\\&-(1/w)([A_\alpha{}^{(1)},A_\beta{}^{(0)}]+[A_\alpha{}^{(0)},A_\beta{}^{(1)}])^{ij}\\&+(1/w^2)[A_\alpha{}^{(0)},[\gamma,A_\beta{}^{(0)}]]^{ij}+0(1/g^2).\end{aligned}$$

The boundedness of the scattering amplitude implies only the weaker condition

$$[A_\alpha{}^{(0)},A_\beta{}^{(0)}]=0.$$

We notice that in the terms independent of g in the above expansion of $t_{\alpha\beta}{}^{ij}(w)$, the coefficient of $1/w$ is antisymmetric in α and β, while the coefficient of $1/w^2$ is symmetric, consistent with the requirements of crossing symmetry. The CGS assumption of $[A_\alpha,A_\beta]=0$ implies that the scattering amplitude is symmetric in α and β, which is at variance with experiment.

An explicit dynamical model does not seem to lead to a group structure very easily. In this connection, we note that the intermediate-coupling models[8,9] are highly speculative. Thus, it seems as if the derivation of a symmetry on the basis of a model has very small chances of success. On the other hand, our way of looking at the subject is decidedly advantageous. The contention is merely that the noninvariance groups and the consequent dynamical implications describe satisfactorily the state of affairs of strongly interacting particles. Specific dynamical models like the pole models or the one-particle-exchange models, while explaining isolated phenomena, give an inconsistent over-all description. This may be because the assumptions involved are far too restrictive in nature. We shall describe how one can obtain quite a few results by making a few innocent-looking assumptions in order to obtain a NIG. In the following, we shall keep in mind the analysis of the hydrogen atom, where the dynamics control the structure of the NIG. We take the converse of the above statement that the structure of the NIG will govern the dynamics, though a satisfactory explanation of the exact interrelationship is not known at present. We know what isobar states exist, and this corresponds, in the case of the hydrogen atom, to a knowledge of the energy spectrum. The dynamical postulate we made was to identify the meson-coupling matrices with the noninvariant generators. With this identification of the noninvariant generators, we are able to obtain the various NIG's.

For the hydrogen atom, on the other hand, we were aware of the existence of the various energy levels labeled by n, each being described by a symmetric tensor representation of $O(4)$ of rank $(n-1)$. In order to make explicit this extra information, we sought and found a NIG [$O(5)$ or $O(4,1)$] which could accommodate as many of the levels as we desired in one single representation. As we emphasized earlier, the NIG was a secondary construct, and the noninvariant generators were expressible in terms of the primitive dynamical variables. If we were to proceed in an analogous fashion in the case of systems of strongly interacting particles, we would make an educated guess as to what the NIG could be from a knowledge of the isobar spectrum. This would, no doubt, help us in classifying these states, but very little information can be gleaned until such time as the noninvariant generators are identified. We do not deal with such things as dynamical variables, and therefore this identification of the noninvariant generators is of crucial importance.

We proceed to consider the NIG for the psuedo-scalar-symmetric theory. The invariance group of the Hamiltonian is $SU(2)^I\otimes SU(2)^J$ and the commutation relations (CR's) are the following:

$$[I_i,I_j]=i\epsilon_{ijk}I_k,$$
$$[J_\alpha,J_B]=i\epsilon_{\alpha\beta\gamma}J_\gamma,$$
$$[I_i,J_\alpha]=0.$$

Baryon multiplets, which furnish a UIR of this group, are distinguished by their quantum numbers (I,I_z,J,J_z). The meson coupling matrices $Q_{i\alpha}$, whose elements are the coupling constants of the p-wave pion to the isobar states, transform as vectors with respect to the isospin and spin rotation groups. That is,

$$[Q_{i\alpha},I_j]=i\epsilon_{ijk}Q_{k\alpha},$$
$$[Q_{i\alpha},J_\beta]=i\epsilon_{\alpha\beta\gamma}Q_{i\gamma}.$$

We express the dynamical postulate which picks out the appropriate NIG as

$$[Q_{i\alpha},Q_{j\beta}]=\theta(i\epsilon_{ijk}\delta_{\alpha\beta}I_k+i\epsilon_{\alpha\beta\gamma}\delta_{ij}J_\gamma).$$

When $\theta=0$, the NIG is $T_9\times(SU(2)^I\otimes SU(2)^J)$, the noncompact strong-coupling group that CGS obtained. A detailed study of this case has been undertaken by Singh and by Bose.[10]

$\theta=+1$ corresponds to the choice of $SU(4)$ as a NIG, and $\theta=-1$ to $SL(4,R)$ as the NIG. Our investigations in connection with the method of master analytic

[9] The relations between NIG models and bootstrap conditions have been considered in a very elegant paper by D. B. Fairlie, Phys. Rev. **155**, 1694 (1967).

[10] V. Singh, Phys. Rev. **144**, 1275 (1966); S. K. Bose, *ibid.* **145**, 1247 (1966).

representations (MAR)[11] have established the fact that these two groups are closely related, so we shall investigate the case of compact $SU(4)$—with considerable hindsight—and make comments relevant to $SL(4,R)$ by a judicious use of the method of MAR.

We seek the UIR of $SU(4)$ with $SU(2)^I$ and $SU(2)^J$ diagonal. For a special class of representations of $SU(4)$, those with $I=J=\lambda$, the various independent reduced matrix elements of Q, the noninvariant generators, are the following:

$$\langle\lambda+1\|Q\|\lambda\rangle=[(2\lambda+1)/(2\lambda+3)]^{1/2}[r^2-16(\lambda+1)^2]^{1/2},$$
$$\langle\lambda\|Q\|\lambda\rangle=r.$$

The value of r will determine the isospin-spin multiplets that occur in each $SU(4)$ representation. If $r=10$, the only states that occur are $I=J=\frac{1}{2}$. If $r=14$, $I=J=\frac{1}{2}$ and $\frac{3}{2}$ are included, and so on.

The method of MAR yielded the expression for $\tilde{Q}$, the noninvariant generators of $SL(4,R)$,

$$\langle\lambda+1\|\tilde{Q}\|\lambda\rangle=[(2\lambda+1)/(2\lambda+3)]^{1/2}[R^2+16(\lambda+1)^2]^{1/2},$$
$$\langle\lambda\|\tilde{Q}\|\lambda\rangle=R.$$

In this group, we have an arbitrary parameter R which enters into the expressions for the reduced matrix elements.

To obtain the Euclidean-type group $T_9\times(SU(2)^I\otimes SU(2)^J$, we consider the transformation

$$Q\to Q'=Q/r \quad (\text{or } Q/R),$$

and formally let $r\to\infty$ (or $R\to\infty$) in the expressions for the reduced matrix elements. We thus obtain

$$\langle\lambda+1\|Q'\|\lambda\rangle=[(2\lambda+1)/(2\lambda+3)]^{1/2}a,$$
$$\langle\lambda\|Q'\|\lambda\rangle=a.$$

The Q'a have been identified with meson coupling matrices, and the matrix elements of Q between baryon states give the baryon-baryon-meson coupling constants.

In $SU(4)$ we find that

$$\frac{|\langle N^{*+}|\pi^0|p\rangle|^2}{|\langle p|\pi^0|p\rangle|^2}=2\left(1-\frac{36}{r^2}\right),$$

where the particle symbols have been utilized to label the isobar states (with $J_z=\frac{1}{2}$) and the noninvariant generators. From the above ratio, it is possible to deduce the width of N^{*+} as a function of the parameter r.

If $r=10$, only $J=I=\frac{1}{2}$, $\frac{3}{2}$ states occur, and the above ratio becomes 1/28, which corresponds to a N^* width of 80 MeV. For $r=14$, only $I=J=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ states are included, and the width is 103 MeV, while in the strong-coupling limit the ratio becomes 2 and the width is 125 MeV.

For the NIG $SL(4,R)$, we have the relation

$$\frac{|\langle N^{*+}|\pi^0|p\rangle|^2}{|\langle p|\pi^0|p\rangle|^2}=2\left(1+\frac{36}{R^2}\right)\geq 2,$$

and thus the N^* width is always greater than 125 MeV. R is here an arbitrary real parameter and is determined by fixing the above ratio. However, this would reduce the predictive power of the theory, from small to almost nothing. So the group $SL(4R)$ is not what we want. To obtain the strong-coupling group, we may just set $R\to\infty$ in the above relation.

Another relation that we obtain, this time in all three NIG's, is

$$\frac{|\langle N^{*+}|\pi^0|N^{*+}\rangle|^2}{|\langle p|\pi^0|p\rangle|^2}=\tfrac{1}{5}.$$

Thus, $(N^{*+}N^{*+}\pi^0)$ coupling seems to be the feeblest coupling of the three, and there is some justification for ignoring such terms in calculations such as reciprocal bootstrap.

The method of MAR depends crucially on the fact that the same analytic function serves as matrix elements for all the NIG. This uniqueness of the analytic function trivially explains the similarity in results for the diagonal matrix elements in all the three NIG's.

A. Magnetic Moments

If we assume that the isovector part of the magnetic-moment operator transforms as a π^0 current and the isoscalar part as J, we will be able to express the magnetic moments of the isobars in terms of two parameters, which we choose to be the magnetic moments of p and n.

For the $J_3=j$ components, we obtain the relations

$$\mu(N^{*+})=(9/5)\mu(p)+(6/5)\mu(n),$$
$$\mu(N^{*0})=(6/5)\mu(p)+(9/5)\mu(n).$$

The best we can do at present is to compare the above with the predictions of static $SU(6)$[12] theory after setting

$$\mu(p)/\mu(n)=-\tfrac{3}{2},$$

(which is equivalent to assuming the correct D/F

[11] The method of MAR is a prescription for obtaining the unitary (and other linear) representations of certain noncompact groups by considering a "related" compact group. It consists of two steps: (a) Multiply the operators of the compact group by suitable factors of i (dictated by Weyl's unitary trick) to obtain the algebra of the related noncompact group. Now perform this operation on the matrix elements of the generators of the compact group and identify the resulting expressions with the matrix elements of the generators of the related noncompact group. (b) Analytically continue these matrix elements (following Dirac) into the region in which the corresponding generators are Hermitian. This process may include possible analytic continuation of the parameters of the representation. For a detailed version of MAR with many many applications, see J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, Institute for Advanced Study Report (to be published).

[12] For a summary of $SU(6)$ results see the excellent review article by A. Pais, Rev. Mod. Phys. 38, 215 (1966).

ratio), and then we have

$$\mu(N^{*++})=\mu(p)\,,$$
$$\mu(N^{*0})=0\,,$$

both of which, not surprisingly, agree with the usual $SU(6)$ results.

Some comments on the difference between the conventional $SU(4)$ theory and the $SU(4)$ NIG theory are in order. Firstly, as the name suggests, we do not have $SU(4)$ as an invariance group, but have a smaller invariance group $SU(2)^I\otimes SU(2)^J$. In the conventional $SU(4)$ theory, one cannot write a $\bar{B}BM$ vertex (with p-wave pions) that is $SU(4)$-invariant. That is, in the strict $SU(4)$-invariant limit, processes such as $N^*\to N\pi$ are forbidden. In our theory, on the other hand, we get around this difficulty because of a smaller invariance group. Also, only isobar states are classified into UIR's of $SU(4)$, and not mesons, so there are only nine mesons, corresponding to the nine noninvariant generators, and not 15 as in the conventional theory. As yet we have made no comment on the classification of meson-baryon states.

B. Meson-Baryon Scattering

The scattering amplitude at energy w for a process such as

$$M_{i\alpha}+B\to \bar{M}_{j\beta}+B'\,,$$

written as a matrix with respect to B and B', is given by

$$T_{i\alpha,j\beta}(w)\,,$$

where we have used two indices to describe a meson state, one for isospin and the other for spin.[13] The amplitude matrix for $M_{j\beta}+B\to\bar{M}_{i\alpha}+B'$ is given by

$$T_{j\beta,i\alpha}(w)\,.$$

Both of these amplitude matrices transform in the same way with respect to the invariance group K. The matrix

$$[T_{i\alpha,j\beta}(w)-T_{j\beta,i\alpha}(w)]\,,$$

which is antisymmetric in the meson indices, is assumed to be an energy-dependent multiple of the commutator of the matrices $Q_{i\alpha}$ and $Q_{j\beta}$. That is,

$$[T_{i\alpha,j\beta}(w)-T_{j\beta,i\alpha}(w)]=f(w)[Q_{i\alpha},Q_{j\beta}]\,.$$

This is a matrix equation, where the baryon indices have been suppressed. Our earlier consideration of the symmetry of the driving term in the Chew-Low equation would have led us to this relation, and CGS's assumption of $T_a\times(SU(2)^I\otimes SU(2)^J)$ would have led them to conclude that

$$T_{i\alpha,j\beta}(w)=T_{j\beta,i\alpha}(w)\,.$$

[13] J. G. Kuriyan and E. C. G. Sudarshan, Phys. Rev. Letters 16, 825 (1966).

We emphasize that this postulate, even though motivated by the study of the driving term, is not meant to imply that the symmetry of the driving term is the symmetry of the scattering amplitude. The assumption we make has to be elevated to the level of another postulate in the theory.

We can now study the consequences of such a postulate. The meson-baryon scattering amplitude in terms of the elastic (non-spin-flip) and spin-flip amplitudes is given by

$$T=f+g(\mathbf{n}\cdot\boldsymbol{\sigma})\,.$$

In the forward direction $\mathbf{n}=0$ and $T=f$.

We define the quantity

$$X(B_1M_1,B_2M_2)=f(B_1M_1\to B_2M_2)-f(B_1\bar{M}_2\to B_2\bar{M}_1)\,.$$

For example,

$$X(p\pi^+,p\pi^+)=f(p\pi^+\to p\pi^+)-f(p\pi^-\to p\pi^-)\,,$$

and we obtain a nontrivial relation

$$X(p\pi^+,p\pi^+)=+\tfrac{1}{2}\sqrt{2}X(p\pi^-,n\pi^0)\,.$$

For the spin-flip amplitude we define

$$Y(B_1M_1\to B_2M_2)=g(B_1M_1\to B_2M_2)+g(B_1\bar{M}_2\to B_2\bar{M}_1)\,.$$

We observe that $Y(B_1M_1\to B_2M_2)$ is proportional to the matrix element of the commutator of two noninvariant generators between baryon states. Our NIG postulate ensures us that this commutator is a linear combination of the invariance generators

$$[Q_{i\alpha},Q_{j\beta}]=(i\epsilon_{ijk}\delta_{\alpha\beta}I_k+i\epsilon_{\alpha\beta\gamma}\delta_{ij}J_\gamma)\,.$$

Since we are considering only the spin-flip part, the relevant contribution can only come from the matrix elements of J_γ. However, J_γ is an isospin singlet; therefore, only if $B_1=B_2$ does J_γ have nonzero matrix elements. That is,

$$Y(B_1M_1\to B_2M_2)=0\quad\text{only if}\quad B_1\neq B_2\,.$$

One example of such a relation is

$$Y(p\pi^-,n\pi^0)=0\,,$$

which yields a trivial identity. We notice that Y is defined as the sum of amplitudes, and the reason for that is as follows: The matrix $T_{\alpha\beta}$ in spin-isospin space is expressed as

$$T_{\alpha\beta}=M^{(+)}\delta_{\alpha\beta}+M^{(-)}[\tau_\alpha,\tau_\beta]/2\,,$$

where $M^{(\pm)}=A^{(\pm)}+B^{(\pm)}(\boldsymbol{\sigma}\cdot\mathbf{n})$. Since we consider linear combinations of amplitudes which are antisymmetric in meson indices, for the non-spin-flip part we can safely identify f with $A^{(-)}$. For the spin-flip part, greater care must be exercised. If the antisymmetrization involved only the isospin of the meson, then $B^{(-)}$ would contribute. However, the differences of the amplitudes $(T_{\alpha\beta}-T_{\beta\alpha})$ imply the exchange of both isospin and spin labels. Thus the spin-flip part (of the differences) of the amplitudes g must be identified with $B^{(+)}$.

For spin-$\frac{1}{2}$ (isospin-$\frac{1}{2}$) objects, the only symmetric term we can construct is $\delta_{\alpha\beta}$ (δ_{jk}), and the only antisymmetric term is $\epsilon_{\alpha\beta\gamma}\sigma_\gamma$ $(\epsilon_{jkl}\tau_l)$. So we obtain nothing more than the consequences of spin (isospin) invariance.

In a forthcoming paper, Deshpande[14] has made the assumption that

$$t_{\alpha\beta}(\omega)=f(\omega)[A_\alpha,A_\beta]+g(\omega)[A_\alpha[\gamma,A_\beta]].$$

This is more restrictive than our assumption on the differences of amplitudes, and he is able to obtain scattering-length relations which are in good agreement with experiment. It is interesting to note that Balachandran *et al.*[15] are able to compute each scattering length using current algebra and they obtain a theoretical prediction of $a_{33}=0.08$, as opposed to the Woolcock value $a_{33}=0.215\pm0.005$. Thus, for p waves Balachandran *et al.* predict the sum rule $a_{33}-a_{31}=a_{13}-a_{11}$.

C. Meson+Baryon → Meson+Baryon Resonance

For the case of baryon-resonance production, we observe that the initial state is a $(\frac{1}{2}\,\frac{1}{2})$ object describing a nucleon, while the final state is a $(\frac{3}{2}\,\frac{3}{2})$ object describing a baryon resonance [we use the symbols (IJ) to describe an isobar]. We are considering differences in scattering amplitudes,

$$T_{i\alpha,j\beta}(w)-T_{j\beta,i\alpha}(w),$$

and by our postulate this is proportional to the matrix element of the commutators of the of the two invariant generators that describe the meson current sources taken between a $(\frac{1}{2}\,\frac{1}{2})$ and a $(\frac{3}{2}\,\frac{3}{2})$ state. Our assumption of the NIG implies that the commutator yields a linear combination of the invariance generators I and J. But neither I nor J, being invariance generators, can connect states belonging to distinct UIR of the invariance group. So we have

$$T(B+M_\alpha\to B^*+M_\beta)=T(B+\bar{M}_\beta\to B^*+\bar{M}_\alpha),$$

in particular

$$T(\pi^+p\to\pi^+N^{*+})=T(\pi^-p\to\pi^-N^{*+}),$$
$$T(\pi^+p\to\pi^0N^{*++})=T(\pi^0p\to\pi^-N^{*++}).$$

Let us consider the first of these relations in terms of the isospin-$\frac{1}{2}$ channel A_1 and the isospin-$\frac{3}{2}$ channel A_3. We obtain the equation

$$A_1=(10)^{1/2}A_3.$$

This prediction was first made in connection with the $SU(6)_W$ group and was compared with experiment by Olsson.[16] Olsson's analysis (using the Olsson-Yodh technique) of the data led him to predict that

$$A_1=3.34A_3,$$

which is extremely close to our result.

Secondly, from the data of Daronian *et al.*[17] on the above process in the form of scattering cross sections (at 1.6 BeV/c);

$$\sigma(\pi^+p\to\pi^+N^{*+})=0.9\text{ mb},$$
$$\sigma(\pi^-p\to\pi^-N^{*+})=1.0\text{ mb}.$$

We point out that both the $B+M\to B+M$ and the $B+M\to B^*+M$ results depend only on the commutation relation (CR) of the NIG and they are therefore true for every choice of representation of the NIG (i.e., they are independent of γ). By the principle of MAR, it follows that these relations hold for $SU(4)$, SL$(4,R)$, and $T(9)\times[SU(2)\otimes SU(2)]$. The relations test the basic NIG identifications of the currents, and their agreement with experiments verifies the choice made for the meson sources.

III. UNITARY-SYMMETRIC PSEUDOSCALAR THEORY

In order to include strange particles in our discussions, we consider $SU(3)\otimes SU(2)^J$ as our symmetry group K of the Hamiltonian. The low-lying isobar states which the Hamiltonian describes fall into multiplets which furnish UIR of the symmetry group K. Since the internal symmetry group is $SU(3)$, the meson currents transform as octets with respect to this group and vectors with respect to the $SU(2)^J$ group. We have thus a p-wave octet of pseudoscalar mesons.

In analogy to the charge-symmetric pseudoscalar theory that we have considered, we will generate new NIG, since the dynamical postulate involving the identification of the noninvariant generator and the invariance group has been made. The NIG in this case are the noncompact

$$T_{24}\times\{(SU(3)\otimes SU(2)^J\}$$

and the compact $SU(6)$.

Let F_i be the generators of $SU(3)$, J_α that of $SU(2)^J$, and $Q_{i\alpha}$ the noninvariant generators. We can express the fact that the invariance group K is $SU(3)\otimes SU(2)^J$ by the following set of CR's:

$$[F_i,F_j]=if_{ijk}F_k,$$
$$[J_\alpha,J_\beta]=i\epsilon_{\alpha\beta\gamma}J_\gamma,$$
$$[F_i,J_\alpha]=0.$$

[14] N. G. Deshpande (to be published). We would like to thank Deshpande for correcting some errors in our work and for sending us his paper prior to publication.

[15] A. P. Balachandran, M. Gundzik, and F. Nicodemi, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1965); Nuovo Cimento **44**, 1257 (1966).

[16] M. G. Olsson, Phys. Rev. Letters **15**, 710 (1965); **15**, 768(E) (1965).

[17] G. Yodh (private communication).

In addition, $Q_{i\alpha}$ are the components of a p-wave octet operator:

$$[Q_{i\alpha},F_j]=if_{ijk}Q_{k\alpha}\,,$$
$$[Q_{i\alpha},J_\beta]=i\epsilon_{\alpha\beta\gamma}Q_{i\gamma}\,.$$

The dynamical postulate involving the commutator of two noninvariant generators is that

$$[Q_{i\alpha},Q_{j\beta}]=\theta(i\delta_{ij}\epsilon_{\alpha\beta\gamma}J_\gamma+if_{ijk}\delta_{\alpha\beta}F_k+id_{ijk}\epsilon_{\alpha\beta\gamma}Q_{k\gamma})\,.$$

Here $\theta=0$ corresponds to the choice of the noncompact group

$$T_{24}\times\{SU(3)\otimes SU(2)^J\}\,,$$

as our NIG, and $\theta=1$ corresponds to the choice of the compact $SU(6)$.

We shall restrict ourselves to the study of the compact noninvariance group $SU(6)$, since this seems to be the most promising one. Unlike the previous case of $SU(4)$, to obtain the most general expression for the various matrix elements of these groups is an arduous task, and perhaps an unfruitful one. So we content ourselves with a particular choice, the representation of $SU(6)$ with a $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet of baryons. This is the familiar 56-dimensional representation of baryons.

The mesonic currents cause transitions between the octet and the decuplet, and it is possible to relate the various independent reduced matrix elements to one another.

The independent reduced matrix elements which occur are the following:

$$\langle 10\tfrac{3}{2}\|Q\|10\tfrac{3}{2}\rangle=\alpha\,,$$
$$\langle 8\tfrac{1}{2}\|Q\|10\tfrac{3}{2}\rangle=\beta\,,$$
$$\{\langle 8\tfrac{1}{2}\|Q\|8\tfrac{1}{2}\rangle\}_1=\gamma\,,$$
$$\{\langle 8\tfrac{1}{2}\|Q\|8\tfrac{1}{2}\rangle\}_2=\delta\,,$$

with

$$\langle 10\tfrac{3}{2}\|Q\|8\tfrac{1}{2}\rangle=\beta'$$

related to β through the Hermiticity relation. [The magnetic quantum numbers are necessarily absent, and in the state $|Nj\rangle$, N is the dimension of $SU(3)$ representation and j is the spin. δ and γ are the antisymmetric and symmetric coupling of $8\otimes 8$, respectively.]

We use the canonical method of evaluating these reduced matrix elements to obtain

$$\alpha=\sqrt{2}\gamma=-\beta\,,$$
$$\delta=\tfrac{2}{5}\gamma\sqrt{5}\,,$$
$$\beta'=(\sqrt{\tfrac{2}{5}})\beta\,.$$

The consistent set of solutions relating α, β, β', γ, and δ implies that there exists a representation of the algebra with only the $8\,\frac{1}{2}$ and $10\,\frac{3}{2}$ states. In this connection we point out the equivalence of this approach to that used by Lee[18] in a derivation of $SU(6)$ results with the help of current algebra. He defined currents which obey the algebra of $SU(6)$ but with other Lorentz transformations also specified. He then found that by using an intermediate set of states of $8\,\frac{1}{2}$ and $10\,\frac{3}{2}$, he could obtain a consistent set of solutions, and concluded, incorrectly, that this method is quite distinct from the group-theoretical formulation. The equivalence was first pointed out by Sudarshan and Okubo.[18] The relation between β, α, and β' is already inherent in our $SU(4)$ calculation, while the γ-to-δ ratio, which *is* related to the D/F ratio, is the only new information contained in the above reduced matrix elements.

We find that

$$|\langle p|\pi^0|N^{*+}\rangle|^2/|\langle p|\pi^0|p\rangle|^2=1.28\,,$$

which is identical to the $SU(4)$ result. This is no surprise, since the above ratio involves only nonstrange particles, and so we are restricting ourselves to the $SU(4)$ subgroup of $SU(6)$—and the $SU(6)$ representation is chosen so that the $SU(4)$ subrepresentation involving nucleon isobars is identical to the one we considered earlier.

By identifying the strangeness-conserving Fermi component of the weak interactions with the corresponding component of the isotopic-spin generator, and identifying the Gamow-Teller component with the corresponding component of the pion coupling (apart from the change in coupling constant), we can immediately write sum rules for the weak interactions. For the strangeness-changing weak interactions we use, correspondingly, couplings proportional to the strangeness-violating "conserved" generator and the kaon coupling, respectively. We shall not undergo the tedium of displaying the predictions.

A. Magnetic Moments

To derive the electromagnetic properties of baryons, we assume that the magnetic-moment operator transforms as $(\sqrt{3}\pi^0+\eta)$, where the particle symbols have been utilized to specify the quantum numbers. These are p-wave pseudoscalar mesons. They carry a spin $J=1$ and are the $J_z=0$ components. We can thus use the Wigner-Eckart theorem and obtain the following set of relations:

$$\mu(\Sigma^-)=-\mu(\Sigma^0)=\mu(\Lambda)=\tfrac{1}{2}\mu(n)=\mu(\Xi^-)=\tfrac{1}{2}\mu(\Xi^0)$$
$$=-\tfrac{1}{3}\mu(p)=-\tfrac{1}{3}\mu(\Sigma^+)=-\tfrac{1}{3}\mu_T(\Sigma^0\to\Lambda)\,,$$
$$\mu(p)=\tfrac{1}{2}\mu(N^{*++})=-\mu(N^{*+})=-\mu(N^{*-})=\mu(Y^{*+})$$
$$=-\mu(Y^{*-})=-\mu(\Xi^{*-})=-\mu(\Omega^-)\,.$$

We can use the same operator to compute radiative

[18] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965); **14**, 850 (E) (1965); E. C. G. Sudarshan, *ibid.*, **14**, 1083 (1965); S. Okubo, Phys. Letters **17**, 172 (1965).

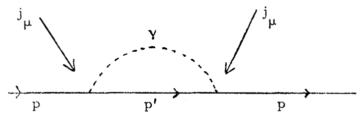

FIG. 1. Feynman diagram for EM mass differences.

decays of baryon resonances. We get

$$\begin{aligned}
&M(N^{*+}\to p+\gamma)\\
&\quad=-M(Y^{*+}\to\Sigma^{+}+\gamma)=M(N^{*0}\to n+\gamma)\\
&\quad=-2M(Y^{*0}\to\Sigma^{0}+\gamma)=-M(\Xi^{*0}\to\Xi^{0}+\gamma)\\
&\quad=2M(Y^{*0}\to\Lambda+\gamma)/\sqrt{3}\,,\\
&M(Y^{*-}\to\Sigma^{-}+\gamma)=0=M(\Xi^{*-}\to\Xi^{-}+\gamma)\,,\\
&M(N^{*+}\to p+\gamma)=\tfrac{2}{3}\sqrt{2}\mu(p)\,.
\end{aligned}$$

These agree with the usual $SU(3)$ and $SU(6)$ results except for phases. We follow the definition of particle states and the Clebsch-Gordan (CG) coefficients of $SU(3)$ as given in the tables of Chilton and McNamee.[19] The $(Y^{*+}\to\Sigma^{+})/(Y^{*0}\to\Sigma^{0})$ and the $(N^{*+}\to p)/(N^{*0}\to n)$ are consequences of charge independence alone.[20]

One of the most remarkable predictions of this theory is that

$$\mu(n)/\mu(p)=-\tfrac{2}{3}\,,$$

which is in very good agreement with the experimental ratio of -0.68. Data on other relations are quite sketchy.

B. Electromagnetic Mass Differences

In a perturbation-theoretic calculation of electromagnetic (EM) mass differences, one considers a Feynman diagram of the kind given in Fig. 1. The mass difference Δm due to this electromagnetic effect is, by the usual Feynman rules, proportional to

$$\langle p|(j_\mu j_\mu)_+|p\rangle.$$

Thus to obtain the EM mass difference, we must use a second-order operator.

We first obtain the contribution to EM mass differences from the *charge part* of the interaction. We *identify* the charge operator with the invariant generator of $SU(3)$ in the combination $(I_z+\frac{1}{2}Y)$. This identification enables us to make the following set of predictions:

$$\begin{aligned}
&M(N^{*+})-M(N^{*0})\\
&\quad=M(Y_1^{*+})-M(Y_1^{*0})=M(p)-M(n)\\
&\qquad=M(\Sigma^{+})-M(\Sigma^{0})\,, \qquad \text{(I.1)}\\
&M(\Xi^{*-})-M(\Xi^{*0})\\
&\quad=M(N^{*-})-M(N^{*0})=M(Y_1^{*-})-M(Y_1^{*0})\\
&\quad=M(\Xi^{-})-M(\Xi^{0})=M(\Sigma^{-})-M(\Sigma^{0})\,, \qquad \text{(I.2)}\\
&M(N^{*++})-M(N^{*-})=3[M(N^{*+})-M(N^{*0})]\,, \qquad \text{(I.3)}
\end{aligned}$$

which are consistent with the static $SU(6)$ predictions.

We then compute the contribution to the EM differences due to the *magnetic moment operator*, already identified as the noninvariant generator in the combination $(\sqrt{3}\pi^0+\eta)$.[21]

Thus we have the product of two magnetic moment operators, and in the intermediate set of states we are justified in restricting ourselves to the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet baryon states. This is only because we have *identified* the operator as a component of the noninvariant generator. We obtain the following:

$$\begin{aligned}
&M(N^{*0})-M(N^{*+})\\
&\quad=M(Y^{*0})-M(Y^{*+})=\tfrac{1}{5}[M(n)-M(p)]\,, \qquad \text{(II.1)}\\
&M(N^{*-})-M(N^{*0})\\
&\quad=M(Y^{*-})-M(Y^{*0})=M(\Xi^{*-})-M(\Xi^{*0})\\
&\qquad=(7/5)[M(n)-M(p)]\,, \qquad \text{(II.2)}\\
&\sqrt{3}M_T(\Sigma^0\to\Lambda)\\
&\quad=[M(\Sigma^0)-M(\Sigma^+)-M(n)+M(p)]\,, \qquad \text{(II.3)}\\
&M(\Sigma^+)-M(\Sigma^-)\\
&\quad=M(\Xi^0)-M(\Xi^-)+M(p)-M(n)\,, \qquad \text{(II.4)}\\
&M(\Sigma^-)-M(\Sigma^+)=(16/11)[M(\Xi^-)-M(\Xi^0)]\,, \qquad \text{(II.5)}\\
&M(\Sigma^-)-M(\Sigma^0)=(5/11)[M(\Xi^-)-M(\Xi^0)]\,, \qquad \text{(II.6)}\\
&M(\Sigma^0)-M(\Sigma^+)=(11/5)[M(n)-M(p)]\,, \qquad \text{(II.7)}
\end{aligned}$$

as well as the relation that follows from charge independence alone,[22]

$$M(N^{*++})-M(N^{*-})=3M(N^{*+})-3M(N^{*0})\,.$$

If we consider the contribution to the EM mass difference due to the charge and magnetic-moment operator to several orders, we find that there are two parameters in the problem and we can thus express all the EM mass differences in terms of the EM mass differences of the nucleons and the Σ^0 and Σ^-, i.e.,

$$\Delta N=M(p)-M(n)$$

and

$$\Delta\Sigma=M(\Sigma^0)-M(\Sigma^-)\,.$$

Thus we obtain

$$\begin{aligned}
&M(\Sigma^+)-M(\Sigma^0)=\tfrac{1}{5}(8\Delta N+3\Delta\Sigma)\,, \qquad \text{(III.1)}\\
&M(\Sigma^+)-M(\Sigma^-)=(11/10)(\Delta N+\Delta\Sigma)\,, \qquad \text{(III.2)}\\
&M(\Xi^0)-M(\Xi^-)=\tfrac{1}{5}(3\Delta N+8\Delta\Sigma)\,, \qquad \text{(III.3)}\\
&M(N^{*+})-M(N^{*0})\\
&\quad=M(Y^{*+})-M(Y^{*0})=\tfrac{1}{5}(3\Delta N-2\Delta\Sigma)\,, \qquad \text{(III.4)}\\
&M(N^{*0})-M(N^{*-})=M(Y^{*0})-M(Y^{*-})=M(\Xi^{*0})\\
&\qquad-M(\Xi^{*-})=(6\Delta\Sigma+\Delta N)/5. \qquad \text{(III.5)}
\end{aligned}$$

[19] P. McNamee and F. Chilton, Rev. Mod. Phys. **36**, 1005 (1964).

[20] R. E. Marshak, S. Okubo, and E. C. G. Sudarshan, Phys. Rev. **106**, 599 (1957).

[21] Strictly speaking, we can only take something which transforms in this fashion, and then we are forced to take all possible intermediate states. Thus the results we obtain are dependent on our assumption.

[22] S. Weinberg and S. B. Treiman, Phys. Rev. **116**, 465 (1959).

The comparison of set I with experiment is already in the literature; the excellent review of $SU(6)$ by Pais[12] provides all the necessary information. We do not have sufficient experimental data to check Eqs. (II.1) and (II.2) of set II.

Equation (II.3) is the Macfarlane-Sudarshan $\Sigma-\Lambda$ transition[23] mass relation obtained from $SU(3)$ and analyzed in detail by Dalitz and Von Hippel,[24] who estimated $M_T(\Sigma^0-\Lambda)$ and found it to be in good agreement with the above prediction.

Equation (II.4) is the well-known Coleman-Glashow[25] relation, which agrees well with experiment.

Comparing Eq. (II.5) with experiment, we obtain

$$M(\Sigma^-)-M(\Sigma^+)=7.8 \text{ MeV},$$
$$(16/11)[M(\Xi^-)-M(\Xi^0)]=9.5 \text{ MeV}.$$

The error in the determination of the mass of the Ξ is large enough to make the above prediction quite a reasonable one.

As for Eq. (II.6),

$$M(\Sigma^-)-M(\Sigma^0)=5.1 \text{ MeV},$$
$$(5/11)[M(\Xi^-)-M(\Xi^0)]=3 \text{ MeV}.$$

We find it to be in poor agreement with experiment. In addition, one can use this equation in conjunction with Eqs. (II.4) and (II.5) to obtain equally poor predictions. So this is an unacceptable prediction.

In Eq. II.7

$$M(\Sigma^0)-M(\Sigma^+)=2.86 \text{ MeV},$$
$$(11/5)[M(n)-M(p)]=2.86 \text{ MeV}.$$

The agreement is remarkably good.

Comparing Eq. (III.1) with experiment, we find

$$M(\Sigma^+)-M(\Sigma^0)=-2.7 \text{ MeV},$$
$$\tfrac{1}{5}[8\Delta N+3\Delta\Sigma]=-4.95 \text{ MeV}. \quad \text{(III.1}')$$

Similarly, we find

$$M(\Sigma^+)-M(\Sigma^-)=-7.81 \text{ MeV}.$$
$$(11/10)[\Delta N+\Delta\Sigma]=-6.75 \text{ MeV}, \quad \text{(III.2}')$$

and

$$M(\Xi^0)-M(\Xi^-)=-6.5 \text{ MeV},$$
$$\tfrac{1}{5}[3\Delta N+8\Delta\Sigma]=-8.53 \text{ MeV}. \quad \text{(III.3}')$$

We should, however, take these predictions as merely an indication of the fact that one can make such calculations of static quantities in this theory. We would not like to consider these static results as the most compelling reasons for considering this theory a success. The static results should be judged in the light of the original assumptions regarding the identification of the magnetic moment operators.

[23] A. J. Macfarlane and E. C. G. Sudarshan, Nuovo Cimento **31**, 1176 (1964).

[24] R. H. Dalitz and F. von Hippel, Phys. Letters **10**, 155 (1964).

[25] S. Coleman and S. Glashow, Phys. Rev. Letters **6**, 423 (1961).

C. Meson-Baryon Scattering

We use the same notation as that employed in the discussion of $SU(4)$ as a NIG. We expressed the scattering amplitude as $T=f+g(\mathbf{n}\cdot\boldsymbol{\sigma})$, and defined

$$X(B_1M_1,B_2M_2)=f(B_1M_1\to B_2M_2)-j(B_1\bar{M}_2\to B_2\bar{M}_1),$$
$$Y(B_1M_1,B_2M_2)=g(B_1M_1\to B_2M_2)+g(B_1\bar{M}_2\to B_2\bar{M}_1),$$

where f corresponds to the non-spin-flip part and g to the spin-flip part of the amplitudes.

We recall that when the invariance group was $SU(4)$ we postulated that $t_{i\alpha,j\beta}(w)-t_{j\beta,i\alpha}(w)$ was proportional to $[Q_{i\alpha},Q_{i\beta}]$, considered as matrices in the baryon space. The proportionality factor was an energy-dependent quantity. The algebra of $SU(4)$ assured us that the commutator yielded an invariance generator according to

$$[Q_{i\alpha},Q_{j\beta}]=i\epsilon_{ijk}\delta_{\alpha\beta}I_k+i\epsilon_{\alpha\beta\gamma}\delta_{ij}J_\gamma.$$

For the case at hand, which is an $SU(6)$ NIG, if we made a similar postulate for the differences between scattering amplitudes, we would notice the important fact that the commutator yields invariant generators, as well as noninvariant generators, according to

$$[Q_{i\alpha},Q_{j\beta}]=if_{ijk}\delta_{\alpha\beta}F_k+i\epsilon_{\alpha\beta\gamma}\delta_{ij}J_\gamma+i\epsilon_{\alpha\beta\gamma}d_{ijk}Q_k.$$

We consider $X(B_1M_1,B_2M_2)$, defined earlier, which is the matrix element between baryon states B_1 and B_2 of the non-spin-flip part of the above CR. Since F_k, the unitary spin generator, is the only spin singlet which occurs on the right-hand side of the commutator of the two Q's, we need to consider only the matrix elements of F_k between baryon states which have definite $SU(3)$ [and some $SU(2)$] quantum numbers. Thus the non-spin-flip part of all processes will be expressible in terms of one parameter.

The results are the following:

$$\begin{aligned}
&X(nK^+,nK^+)\\
&=X(p\pi^+,\Sigma^+K^+)=\sqrt{2}X(n\pi^+,\Sigma^0K^+)=X(p\pi^+,p\pi^+)\\
&=-\sqrt{2}X(p\pi^-,\Sigma^0K^0)=-X(n\pi^+,n\pi^+)\\
&\qquad\qquad\qquad=-\tfrac{1}{2}\sqrt{2}X(p\pi^-,n\pi^0)\\
&=X(pK^-,n\bar{K}^0)=\tfrac{1}{2}X(pk^+,pk^+)=(-\sqrt{\tfrac{2}{3}})X(p\pi^-,\Lambda k^0),\\
&X(p\bar{K}^0,\Xi^0K^+)=X(p\pi^-,\Sigma^-K^+)=X(n\bar{K}^0,\Xi^-k^+)=0.
\end{aligned}$$

The set of relations

$$\begin{aligned}\tfrac{1}{2}X(pK^+,pK^+)&=X(nK^+,nK^+)\\&=X(p\pi^+,p\pi^+)\end{aligned}$$

are the Johnson-Treiman (JT)[26] relations derived from $SU(6)$ theory as relationships involving total cross sections. We have, however, rederived and extended their validity to the entire non-spin-flip amplitude. In the forward direction the spin-flip term vanishes identically, and hence, of course, the forward scattering amplitude is entirely a non-spin-flip amplitude.

[26] K. Johnson and S. B. Treiman, Phys. Rev. Letters **14**, 189 (1965).

It is important to notice that we have utilized only the $SU(3)$ property of the baryons, as we have considered matrix elements of the generators of $SU(3)$ between baryon states. The $SU(6)$ properties such as the D/F ratio have not been used and, consequently, *no matter what representation of SU(6) the* $\frac{1}{2}^+$ octet states are assigned (**56**, **70**, **700**, $\cdots$), *the above equations, including the JT relations, are valid.*

To be more explicit, if we had assigned the $\frac{1}{2}^+$ octet baryons to the 70- (rather than the 56-) dimensional representation of $SU(6)$, then we would have had, instead of α, β, γ, and δ, many more independent reduced matrix elements of the noninvariant generators, corresponding to the transitions between the various $SU(3)$ representations in the 70-plet. The *sole* difference between the assignment of the baryon to the 56-plet and the 70-plet would manifest itself in the change in the values of the reduced matrix elements of the noninvariant generator. However, in the above scattering relations, we consider *only* the invariant generators, which are the generators of $SU(3)$, and the only nonvanishing matrix elements are the diagonal ones. Thus we have a representation-independent statement of the JT relation for the non-spin-flip part of the amplitude.

On comparing with experimental data, we find that $X(pK^+,pK^+)\simeq 2X(nK^+,nK^+)$, but the second equality is in disagreement.[27] The reason for this is not too difficult to find, because the processes involving π^+p and K^+p are so different that the comparison must be made with due account taken of kinematic corrections, i.e., we remember that the proportionality factors involved energy-dependent multiples.

One can combine the two relations and write

$$\sigma(K^+p)-\sigma(K^-p)=\sigma(K^+n)-\sigma(K^-n)+\sigma(\pi^+p)-\sigma(\pi^-p),$$

a relation derived by Volkov and Ruegg and by Rühl.[28] This agrees remarkably well with experiment.

Many people have derived the JT relations, but this is the first derivation where a *representation-independent* relation is obtained for the entire non-spin-flip amplitude (and not only for the forward scattering amplitude).

For the spin-flip amplitude, we define

$$Y(B_iM_\alpha,B_jM_\beta)=g(M_\alpha B_i\to M_\beta B_j)+g(\bar{M}_\beta B_i\to \bar{M}_\alpha B_j).$$

The contributions to $Y(B_iM_\alpha,B_jM_\beta)$ must again come from the matrix element of the commutator of two of the Q's between baryon states. The commutator under consideration was given earlier.

F_k, the generator of $SU(3)$, is a scalar in spin space and cannot possibly contribute to the spin-flip amplitude. We choose our mesons judiciously so that the d_{ijk} coupling is zero. An example of this is the process, involving a π meson and nucleons, considered in the $SU(4)$ theory. This means that we can effectively ignore the noninvariant generator (Q) term. J_γ is a scalar in unitary-spin space and will have nonvanishing matrix elements only when the initial and final states have the same unitary-spin quantum numbers.

The predictions are

$$Y(p\pi^-,n\pi^0)=Y(p\pi^-,\Sigma^-K^+)=Y(pK^0,\Xi^0K^+)=0.$$

D. Baryon Resonance Production

The initial baryon state belongs to a $\frac{1}{2}^+$ octet and the final to a $\frac{3}{2}^+$ decuplet. Following the reasoning in the previous section, we note that neither F_k nor J_γ, being invariance generators, can contribute to such processes. In order to avoid bringing in the $Q_{k\gamma}$, we consider only such processes involving mesons which have zero d_{ijk} coupling. For this we make the prediction

$$T(BM_\alpha\to BM_\beta)=T(B\bar{M}_\beta\to B^*\bar{M}_\alpha).$$

One such is the relation we obtained from the $SU(4)$ theory,

$$T(\pi^+p\to\pi^+N^{*+})=T\pi^-(p\to\pi^+N^{*+})$$

and the consequent isospin amplitude relation $A_1=(\sqrt{10})A_3$. We discussed the experimental agreement of the above earlier.

For Kaons,

$$T(K^-p\to K^0\Xi^{*0})-T(\bar{K}^0p\to K^+\Xi^{*0})=0,$$

which also gives an isospin amplitude relation $A_0=3A_2$.

However, the isospin triangle relation for Ξ^* production gives the following result:

$$T(\bar{K}^0p\to K^+\Xi^{*0})+T(K^-p\to K^0\Xi^{*0})=T(K^-p\to K^+\Xi^{*-}).$$

Thus our prediction implies that

$$T(K^-p\to K^+\Xi^{*-})=2T(K^-p\to K^0\Xi^{*0}),$$

or

$$R=\sigma(K^-p\to K^+\Xi^{*-})/\sigma K^-p\to K^0\Xi^{*0})=4.$$

Experimentally one obtains

at $P_{K^-}=2.24$ BeV/c, $R\simeq 0.4$ (London);

at $p_{K^-}=3$ BeV/c, $R=0.25$ (Badier *et al.*).

This prediction is the same as that obtained from the $SU(6)_W$ theory.[29]

IV. CONCLUSION

We can now indulge in some speculative reasoning, as follows: Consider incident pions of very low energy in the Chew-Low expression for the πp scattering amplitude. The Born term would be of the form

$$T_{\alpha\beta}{}^{ij}(w)=\frac{-g^2}{w}\sum_k\frac{(A_\alpha A_\beta)^{ij}}{M_k-M_i-w}+\frac{(A_\beta A_\alpha)^{ij}}{M_k-M_j+w},$$

[27] W. Galbraith *et al.*, Phys. Rev. **138**, 913 (1965).

[28] Volkov and Ruegg (unpublished); W. Ruhl, in *Proceedings of the Seminar on High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Agency, Vienna, 1965).

[29] J. Carter *et al.*, Phys. Rev. Letters **15**, 373 (1965).

since the pions are of low energy, $M_k - M_i \simeq 0$, and

$$T_{\alpha\beta}{}^{ij}(w) = (-g^2/w)[A_\alpha, A_\beta]^{ij}.$$

We thus see that $A(\pi^+ p \to \pi^+ p) = -A(\pi^- p \to \pi^- p)$ in this approximation.

Consequently, we obtain the isospin amplitude relation $2A_3 + A_1 = 0$, which is extremely well satisfied by s-wave pion scattering lengths and has been recently derived by Tomozawa, by Balachandran, *et al.*, by Raman and Sudarshan, by Weinberg, and by Hamprecht.[30]

It is of interest to note that the group-theoretic framework discussed above can be related to dynamical assumptions about the high-energy behavior of the meson-baryon scattering amplitude. We know that the Born approximation with a single intermediate state behaves like $1/w$. If we require that the complete amplitude fall off sufficiently for large w, we can deduce that the integral of the imaginary part of the scattering amplitude should obey some sum rules.[31] If we approximate the intermediate states by resonant baryons in the direct and crossed channels, we have

$$t_{\beta\alpha}{}^{ji}(w) = -g^2 \sum_k \left\{ \frac{(A_\beta{}^{jk} A_\alpha{}^{ki})}{M_k - M_i - w} + \frac{A_\alpha{}^{jk} A_\beta{}^{ki}}{M_k - M_j + w} \right\}.$$

If $t(w)$ goes down faster than w^{-1} for large values of w, we can deduce the relation

$$A_\beta{}^{jk} A_\alpha{}^{ki} = A_\alpha{}^{jk} A_\beta{}^{ki}.$$

which is the same as[32]

$$[A_\alpha, A_\beta]^{ji} = 0.$$

More generally, if we knew that the leading term in the amplitude in the asymptotic region was a "Regge-pole" contribution of the type

$$(1/w) f_{\beta\alpha}{}^{ji},$$

then we could obtain

$$-g^2[A_\alpha, A_\beta] = f_{\beta\alpha}.$$

It is now up to us to make definite the nature of the coupling $f_{\beta\alpha}$. If we choose it so as to assure ourselves that the Lie algebra of the invariance group K and the extra generators A_α close under commutation, we can obtain one or another form of the NIG. We have thus "deduced" the NIG from the asymptotic properties of the meson-baryon scattering amplitude.

In conclusion, we would like to touch upon a few things which have not been discussed yet. We notice that there was no necessity to introduce vector mesons in this theory. We could either take the point of view that they are not basic entities, and therefore deserve to be ignored, or consider the scalar coupling coefficients that occur on the right-hand side when we consider the matrix elements of the commutator of two noninvariant generators as representing the conserved coupling of vector mesons. Also, we could deviate from our static-model assumptions and define our particle states to have baryon-number-zero objects as well; then we could assign the mesons to a 35-dimensional representation *à la* static $SU(6)$ theory. The important difference here is that the roles of the vector and pseudoscalar mesons will be interchanged, because pseudoscalar mesons are p-wave entities. So the singlet that occurs in this representation is the X^0 and not the ϕ. This seems to be the conclusion of the bootstrap approach to $SU(6)$ theory.[33]

The second point to ponder is the fact that only p-wave pseudoscalar mesons have been considered, and it seems as if we cannot do anything for s-wave pseudoscalar mesons.

The third point concerns the relationship of this theory to static models. We considered an intermediate-coupling model to obtain the compact NIG: Is it possible to obtain the latter in some dynamical model? At every point, it seems in retrospect, we were motivated by the static-model results, and yet our assumptions have not yet been justified in any serious fashion. The only justification (though perhaps of the most significant kind) comes from the agreement of our predictions with experimental results.[34] On the positive side, we have succeeded in obtaining many of the static results of the conventional $SU(6)$ theory without having the serious defect of the usual $SU(6)$ approach—that an $SU(6)$ invariant BBM vertex for p-wave pseudoscalar mesons does not exist. Previously, one was forced either to consider groups such as $SU(6) \otimes O(3)$[35] or to turn a deaf ear to these objections.

In the scattering predictions, we seem to be able to obtain some good results: The JT relationship, first derived for the forward amplitude from $SU(6)$ theory, is now a representation-independent statement for the entire non-spin-flip amplitude. The N^* production relation in πp scattering, which was first derived in $SU(6)_W$, is now true for p-wave pions and agrees with experi-

[30] Y. Tomozawa (unpublished); A. P. Balachandran *et al.*, (Ref. 15); B. Hamprecht (unpublished); K. Raman and E. C. G. Sudarshan, Phys. Letters **21**, 450 (1966); S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[31] A partial list of early papers using the superconvergence relations includes M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. **99**, 986 (1955); A. P. Balachandran, P. G. O. Freund, and C. R. Schumacher, Phys. Rev. Letters **12**, 209 (1964); A. P. Balachandran, Ann. Phys. (N. Y.) **30**, 476 (1964); Phys. Rev. **137**, B177 (1965); Nuovo Cimento **42**, 804 (1966); V. de Alfaro, S. Fubini, G. Rosetti, and G. Furlan, Phys. Letters, **21**, 576 (1966); L. D. Soloviev, JINR—Joint Institute for Nuclear Research (Dubna), JINR Report (unpublished).

[32] See, L. K. Pande, International Center for Theoretical Physics, ICTP Report (unpublished), where the isobar mass spectrum which is a consequence of the above group structure, is obtained by explicit calculation.

[33] R. H. Capps, Phys. Rev. Letters **14**, 31 (1965); J. G. Belinfante and R. E. Cutkosky, *ibid.* **14**, 33 (1965).

[34] D. B. Fairlie [Phys. Rev. **155**, 1694 (1967)] has considered the derivation of the intermediate coupling model from a bootstrap hypothesis. N. Mukunda, E. C. G. Sudarshan, and A. Bohm [Phys. Letters **24**, B301 (1967)] have shown how to construct a relativistic NIG which give the spin isospin sequence $I = J = \frac{1}{2}, \frac{3}{2}, \cdots$, by considering a 14-parameter Lie algebra.

[35] K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Letters **14**, 163 (1965).

ment. This relation is an angle- and energy-independent statement. In the case of Ξ^* production we obtained a result, not in agreement with experiment, such as that derived from $SU(6)_W$. There are many more results which we have not been able to analyze because of a lack of experimental information.

If group theory is to compete in any serious way with dispersion or field theories then one must be able to make comments on the momentum dependence of form factors (or in our language, matrix elements). In our approach, this is equivalent to the statement that the invariance group must include the Poincaré group, and the noninvariant generators (belonging to a suitable NIG) must mix states of the different representations of the invariance group (hence, different momenta). If we identify the noninvariant generators as, say, the isovector current, then the analytic expressions of the matrix elements of this operator will be our isovector form factors, and their momentum dependence will be known. Even though the problem can be formulated so easily, the task of solving it is a difficult one.

ACKNOWLEDGMENTS

We would like to thank A. P. Balachandran, A. Gleeson, N. Mukunda, M. Olsson, and S. Pakvasa, and particularly N. Mukunda, for many illuminating conversations. One of us (JGK) would like to express his gratitude to the late Professor J. R. Oppenheimer and to Professor Carl Kaysen for the hospitality extended to him at the Institute for Advanced Study.

PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Relativistic Harmonic Oscillator and Superconvergent Amplitudes*

G. Cocho
University of Mexico, Mexico D. F., Mexico

AND

C. Fronsdal,† I. T. Grodsky,† and R. White
University of California, Los Angeles, California
(Received 1 May 1967)

Current commutation relations and Fubini sum rules are saturated in a model that is a relativistic version of the three-dimensional harmonic oscillator. The model is essentially determined by the requirement that a local nonderivative relativistic coupling of the oscillator to an external electromagnetic field give rise to form factors that reduce to the usual result in the nonrelativistic limit. The relativistic form factors decrease as a power of the invariant momentum transfer, although they fall off exponentially in the limit $c\rightarrow\infty$. Vertex functions and scattering amplitudes are investigated, and it is found that (i) the Compton scattering amplitudes for current-particle interactions satisfy Fubini sum rules. (ii) All strong-interaction amplitudes are superconvergent in the Born approximation, in which an infinite/equal-mass multiplet is either exchanged or forms a set of intermediary states. (iii) Scattering amplitudes can be arranged in a hierarchy of increasing convergence (e.g., no spin flip, single spin flip, double spin flip), as suggested by de Alfaro *et al.* Finally, the problem of introducing the mass spectrum in the one-particle propagator is discussed.

I. INTRODUCTION

Recent work on current algebras and superconvergence relations has been directed towards the construction of more or less realistic models, in which the attempt is made to saturate these relations with a number of idealized states.[1] In a nonrelativistic framework this is certainly possible; the success of the bootstrap in the static model may be cited as an example of saturation of a superconvergence relation with a small number of single-particle states. On the other hand, it has been discovered that saturation of superconvergence relations in a relativistic theory requires an infinite number of states.[1] As has been emphasized repeatedly, the technical complications that arise from an infinite number of single-particle states are compensated by unexpected dividends. In fact, it has been shown that what is originally introduced as a discrete set of one-particle states sometimes turn out to represent a continuum of two-particle states, in addition to a number (finite or infinite) of physical one-particle states.[2]

The three-dimensional harmonic oscillator seems to provide a suitable starting point for a relativistic model because the energy spectrum is not unlike that which might be expected for elementary particles. The absence

* Supported in part by the National Science Foundation and by the Comision Nacional de Energia Nuclear de Mexico.
† Present address: International Centre for Theoretical Physics, Trieste, Italy.
[1] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Institute di Fisica Dell' Universita, Torino Report, December, 1966 (unpublished).
[2] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).