Dynamical Symmetries and Symmetry Algebras

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We consider the interaction of three multiplets of particles under the assumption that the members of each one of these multiplets have the same mass and the same spin. The vertex self-consistency conditions lead to an algebraic structure involving the coupling matrices. This structure, referred to as a symmetry algebra and denoted by the symbol (N,n,ν), is characterized essentially by the numbers of particles N, n, and ν belonging to each one of the three multiplets and is independent of the particular underlying dynamics. A question of particular interest is whether dynamical self-consistency implies the existence of a symmetry group that leaves the interaction invariant. We analyze this problem in detail for three particularly simple but instructive symmetry algebras. It is shown that the algebra $(N, n, \nu=Nn)$ corresponds to the case of maximal symmetry, the interaction being invariant under the group $U(N)\times U(n)$. The algebra $(N, n, \nu=Nn-1)$ is shown to have a solution only if $n=N$, in which case it corresponds to symmetry under the group $U(N)$. Finally we consider the particularly instructive algebra $(N=3, n=3, \nu=3)$ which is shown to admit of three physically distinct solutions, which correspond to invariance of the interaction under a three-parameter Abelian group, the orthogonal group in three dimensions, and the 24-element permutation group S_4, respectively.

I. INTRODUCTION

SYMMETRY groups have come to play an increasing role in elementary-particle physics. To a large extent this has been a direct consequence of the fact that strongly interacting particles apparently organize themselves into multiplets. A natural association of these multiplets with symmetry groups suggests itself, with the multiplets furnishing (irreducible unitary) representations of the relevant group. The simplest assumption about the interaction is that it is invariant under the group, though this assumption may be too restrictive, especially in the case of broken symmetries. This orthodox approach to symmetry groups, however, makes no attempt to answer the following questions: First, which groups arise and what irreducible representations of these groups are relevant? Second, what, if any, is the relation between dynamics of the interacting multiplets and the resultant (approximate) symmetry group?

In this area significant conceptual advances have been made in the last few years. The key concept is the notion of dynamical self-consistency. The idea of consistency constraints in dynamics possessing symmetries is of course not unfamiliar: Symmetries, in general, lead to conservation laws which are constraints on the dynamics. What is new in the present context is the idea of determining the symmetry group, multiplet structure, and interaction structure in terms of a coupled self-consistency requirement. We will return to this question in the final section of this paper.

The plan of the paper is as follows: Section II transcribes the self-consistency requirements into algebraic nonlinear equations which constitute a symmetry algebra. The following section shows how such a scheme implements the existence of a symmetry group in a particularly simple case. Section IV deals with the derivation of unitary symmetry in the present framework. In Sec. V we present the detailed analysis of the structure of a more complicated symmetry algebra and exhibit the three inequivalent self-consistent coupling schemes associated with this algebra. In the next section we show how each of these coupling schemes is related to a symmetry group of the interaction. Certain general questions are reviewed in the concluding section.

II. SELF-CONSISTENCY AND SYMMETRY

The self-consistency condition for interacting multiplets can be formulated in terms of the propagators[1] or vertex functions[2] or scattering amplitudes.[3] The self-consistency relations for these *functions* can be reduced to a system of nonlinear algebraic relations for a finite set of coupling constants provided the dependence on space-time variables can be factorized out. By far the most direct method of achieving factorizability is to assume that the various members of a definite multiplet have (approximately) the same mass; this assumption may not be necessary (see below).

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[1] E. C. G. Sudarshan, L. S. O'Raifeartaigh, and T. S. Santhanam, Phys. Rev. 136, B1092 (1964); E. C. G. Sudarshan, in *Symposia on Theoretical Physics*, edited by A. Ramakrishnan (Plenum Press, Inc., New York, 1966), Vol. 2.

[2] R. E. Cutkosky, Ann. Phys. (N. Y.) 23, 415 (1963); in *Proceedings of the Seminar on Unified Theories of Elementary Particles*, edited by D. Lurie and N. Mukunda (University of Rochester, Rochester, New York, 1963).

[3] R. E. Cutkosky, Phys. Rev. 131, 1888 (1963); E. C. G. Sudarshan, Phys. Letters 9, 286 (1964); J. C. Polkinghorne, Ann. Phys. (N. Y.) 34, 153 (1965). See also H. Fröhlich, W. Heitler, and N. Kemmer, Proc. Roy. Soc. (London) 166, 154 (1938).

We briefly review the often discussed notion of vertex self-consistency.[2,4] Consider three multiplets ϕ_A, ϕ_a, and ϕ_α with N, n, and ν members, respectively.

$$1\leq A\leq N;\quad 1\leq a\leq n;\quad 1\leq\alpha\leq\nu.$$

Let us denote the coupling for the (virtual) transition

$$\phi_A\rightarrow\phi_a+\phi_\alpha$$

by the vertex function

$$\Gamma_\alpha{}^A{}_a(X,x,\xi)\,,$$

where X, x, and ξ denote the space-time variables of the three fields, including the spin indices if these fields carry spin. The requirement of factorizability is then equivalent to asserting

$$\Gamma_\alpha{}^A{}_a(X,x,\xi)=g_\alpha{}^A{}_a\Gamma(X,x,\xi)\,.$$

The postulate of dynamical self-consistency requires that the lowest-order modified vertex function be the same as the original vertex function. In general the lowest-order vertex modification arises in fifth order. There is no third-order contribution to the vertex unless one of the following two special conditions is met.[5]

S_1 The multiplets ϕ_A and ϕ_a are identical.

S_2 The multiplet ϕ_α is self-adjoint.

In the general case, dynamical self-consistency leads to the fifth-order relation

$$g_\alpha g_\beta{}^\dagger g_\gamma g_\alpha{}^\dagger g_\beta=\lambda_3 g_\gamma\,, \tag{2.1}$$

where we have employed a matrix notation, $g_\alpha{}^A{}_a$ being the (A,a) matrix element of the matrix g_α (which is, in general, rectangular) and summation over repeated indices is understood. Higher-order vertex modifications lead to higher-order self-consistency relations which will be discussed later.

On the other hand, if one of the two special conditions S_1 or S_2 applies, dynamical self-consistency gives rise to the third-order relations

$$S_1:\quad g_\alpha g_\beta g_\alpha{}^\dagger=\lambda_2 g_\beta\,,\qquad g_\alpha{}^\dagger g_\beta g_\alpha=\lambda_2' g_\beta\,,$$

and

$$S_2:\quad g_\alpha g_\beta{}^\dagger g_\alpha=\lambda_2'' g_\beta\,.$$

Either by viewing a suitable limit of the vertex function $\Gamma_\alpha{}^A{}_a(X,x,\xi)$ as the wave function of any one of the particles considered as a bound state and requiring orthonormality of the wave functions[2] or by requiring the equivalence of propagators within a multiplet,[1] we get the second-order inhomogeneous equations

$$g_\alpha g_\alpha{}^\dagger=(\lambda_1/N)\mathbf{1}_{(N)}\,,\qquad g_\alpha{}^\dagger g_\alpha=(\lambda_1/n)\mathbf{1}_{(n)}\,, \tag{2.2}$$

and

$$\mathrm{tr}\{g_\alpha g_\beta{}^\dagger\}=(\lambda_1/\nu)\delta_{\alpha\beta}\,.$$

In these equations, the characteristic constants λ_1, λ_2, and λ_3 depend on the masses of the particles, the strength of coupling, the details of cutoff (to simulate higher-order effects), etc.

Equations (2.1) and (2.2) and their higher-order generalizations provide a characteristic mathematical structure. We refer to such a structure as a symmetry algebra. The derivation of a symmetry group from a symmetry algebra is, then, the purpose of attempts at the derivation of dynamical symmetries. It is not known at present whether all symmetry algebras can be associated with Clebsch-Gordan coefficients of a suitable (finite or compact) group. This question deserves further study.

The derivation of a symmetry algebra can also be based on self-consistency requirements for the scattering amplitude within a suitable scheme of approximations. On the other hand, if one makes use of the requirement that the propagator remains degenerate within a multiplet[1,6,7] when it is in interaction, one arrives at the relations[1,8]

$$g_\alpha g_\beta{}^\dagger g_\gamma g_\alpha{}^\dagger g_\beta g_\gamma{}^\dagger=(\mu_3/N)\mathbf{1}_{(N)}\,,\qquad g_\alpha{}^\dagger g_\beta g_\gamma{}^\dagger g_\alpha g_\beta{}^\dagger g_\gamma=(\mu_3/n)\mathbf{1}_{(n)}\,,$$

and

$$\mathrm{tr}\{g_\alpha g_\beta{}^\dagger g_\gamma g_\alpha{}^\dagger g_\beta g_\delta{}^\dagger\}=(\mu_3/\nu)\delta_{\gamma\delta}\,.$$

Similarly, one finds relations of the fourth order in the case of the special algebras of the type S_1 or S_2. These equations are weaker than the fifth- (third-) degree relations derived above (i.e., they are implied by them but do not imply them).

All the self-consistency relations discussed so far have the property that they are invariant under a group of transformations[1,8] on the matrices g_α. Consider for example the correspondence

$$g_\alpha\rightarrow W_{\alpha\beta}g_\beta\,. \tag{2.3}$$

If the $\nu\times\nu$ matrix $W_{\alpha\beta}$ is unitary the algebraic relations (2.1) and (2.2) are unaltered. Similarly, if U and V are $N\times N$ and $n\times n$ unitary matrices, the correspondence

$$g_\alpha\rightarrow Ug_\alpha V \tag{2.4}$$

also leaves the system of equations unaltered. The existence of these transformations implies that the solution of the self-consistency relations cannot be a unique set of matrices. These linear transformations have a

[4] R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963); J. S. Dowker and J. E. Paton, Nuovo Cimento **30**, 450 (1963); Hong-Mo Chan, P. DeCelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963); Nuovo Cimento **33**, 70 (1964); E. C. G. Sudarshan, Current Sci. (India) **34**, 202 (1965).

[5] As the three multiplets enter the basic vertex on equal footing, we can always arrange the notation in such a fashion as to single out ϕ_α in the statement of the conditions S_1 and S_2.

[6] J. J. Sakurai, Phys. Rev. Letters **10**, 446 (1963); J. S. Dowker, Nuovo Cimento **34**, 773 (1964); J. S. Dowker and P. A. Cook, *ibid.* **37**, 335 (1965).

[7] O. Fleishman, R. Musto, L. O'Raifeartaigh, and P. S. Rao, Phys. Rev. (to be published).

[8] E. C. G. Sudarshan, Current Sci. (India) **34**, 202 (1965).

simple meaning in terms of the particles of a multiplet: They correspond to the choice of a new basis for the particles within a multiplet. If two sets of matrices are related by such transformations, they are physically equivalent and the solutions of the self-consistency relations may be put into equivalence classes of physically indistinguishable solutions. In the sequel we shall investigate these questions for a special case.

The special symmetry algebras of the type S_1 or S_2 are characterized by a restricted group of equivalence transformations. The third-order self-consistency relations are invariant under (2.3) and (2.4) only if

$$S_1:\quad V=U^\dagger;\quad S_2:\quad W_{\alpha\beta}=W_{\alpha\beta}^*.$$

The restriction S_1 guarantees that the transformations on the multiplets ϕ_A and ϕ_a coincide if these particles are identical and S_2 implies that the field operator associated with a self-adjoint particle remains Hermitian under the transformation $W_{\alpha\beta}$.

III. MAXIMAL SYMMETRY

A symmetry algebra is essentially characterized by the three multiplicities N, n, and ν. In the absence of special conditions of the type S_1 or S_2, the three multiplets enter the self-consistency relations on equal footing. The distinguished appearance of the index α is due to the use of sets of matrices (for want of a more appropriate mathematical entity) to display the coupling structure. Therefore, without loss of generality, we can assume that the three multiplicities be ordered

$$N\leq n\leq \nu,$$

and we denote the corresponding symmetry algebra by the symbol (N,n,ν).

It is clear that given the two multiplicities N, n the third, ν, cannot be arbitrarily large. This is because the quadratic relations (2.2) imply that the ν matrices g_α are linearly independent and the number of linearly independent $N\times n$ matrices cannot exceed Nn. Therefore, for fixed N, n, $N\leq n$, the range of the third multiplicity ν is given by $n\leq\nu\leq Nn$.

As an illustration of the method of deducing a symmetry group from a suitable symmetry algebra, we consider in this section the particularly simple case $\nu=Nn$. For this algebra the coupling matrices g_α form a complete set of $N\times n$ matrices. This immediately implies that this algebra admits of a symmetry group. In fact, consider the matrix $Ug_\alpha V$. Due to the completeness of the set g_α, this quantity can be expanded according to

$$Ug_\alpha V=W_{\alpha\beta}(U,V)g_\beta, \tag{3.1}$$

where the $W_{\alpha\beta}(U,V)$ are a set of coefficients depending upon U and V. The self-consistency relation $\mathrm{tr}\{g_\alpha g_\beta^\dagger\}=(\lambda_1/Nn)\delta_{\alpha\beta}$ implies that the matrix $W_{\alpha\beta}(U,V)$ is unitary provided U and V are. Equation (3.1) shows that the interaction is invariant under the simultaneous transformations

$$\phi_A\rightarrow U_{AB}\phi_B,\quad \phi_a\rightarrow V^\dagger{}_{ab}\phi_b, \tag{3.2}$$

and

$$\phi_\alpha\rightarrow W_{\alpha\beta}(U,V)\phi_\beta.$$

As the transformations U and V are independent, the symmetry group associated with the algebra (N,n,Nn) is $U(N)\otimes U(n)$. This is clearly the largest possible symmetry group compatible with any symmetry algebra (N,n,ν), because symmetry transformations are always equivalence transformations and hence the symmetry group must always be a subgroup of $U(N)\otimes U(n)$.

IV. UNITARY SYMMETRY

In this section, we consider the next simplest algebra, $(N,n,Nn-1)$. In this case the $Nn-1$ linearly independent matrices g_α do not form a complete set. Therefore, in general, $Ug_\alpha V$ cannot be expanded the g_α's and the interaction does not admit, the full group $U(N)\otimes U(n)$. The problem is to find the restrictions on U and V such that $Ug_\alpha V$ still lies in the subspace spanned by the matrices g_α. To investigate these restrictions, let us first introduce the matrix g_0 that completes the set g_α:

$$\mathrm{tr}\{g_0g_\alpha^\dagger\}=0;\quad \mathrm{tr}\{g_0g_0^\dagger\}=\lambda_1/(Nn-1). \tag{4.1}$$

As an immediate consequence of the self-consistency relations

$$g_\alpha g_\alpha^\dagger=(\lambda_1/N)\mathbf{1}_{(N)},\quad\text{and}\quad g_\alpha^\dagger g_\alpha=(\lambda_1/n)\mathbf{1}_{(n)}, \tag{4.2}$$

and the relations

$$g_\alpha g_\alpha^\dagger+g_0g_0^\dagger=[\lambda_1 n/(Nn-1)]\mathbf{1}_{(N)},\quad\text{and}\quad g_\alpha^\dagger g_\alpha+g_0^\dagger g_0=[\lambda_1 N/(Nn-1)]\mathbf{1}_{(n)}, \tag{4.3}$$

which follow from the completeness of the set g_0, g_α, we have

$$g_0g_0^\dagger=[\lambda_1/N(Nn-1)]\mathbf{1}_{(N)},\quad\text{and}\quad g_0^\dagger g_0=[\lambda_1/n(Nn-1)]\mathbf{1}_{(n)}. \tag{4.4}$$

These two relations state that the N vectors $g_0{}^{A}{}_{a}$ $(A=1,\cdots,N)$ are orthogonal and complete. Hence, $N=n$ and the $n\times n$ matrix g_0 is unitary up to a factor. This shows that the symmetry algebra $(N,n,Nn-1)$ has a nontrivial self-consistent solution only if $N=n$ and we assume that this is indeed the case.

Let us now proceed to investigate the symmetry group of this algebra. In terms of the matrix g_0, it is easy to see what the restrictions on U and V are in order that $Ug_\alpha V$ lies in the subspace spanned by the N^2-1 matrices g_α: The one-dimensional orthogonal subspace must be left invariant:

$$Ug_0V=g_0.$$

TABLE I. Standard solutions for the algebra (3,3,3).

	g_1	g_2	g_3
α	$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$	$\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
β	$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0&0&i\\0&0&-i\\-i&i&0\end{pmatrix}$
γ	$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&-1\\1&-1&0\end{pmatrix}$

Therefore, the matrix V is determined by U:

$$V=g_0^{-1}U^\dagger g_0. \tag{4.5}$$

Because of (4.4), V is unitary for unitary U. This shows that the symmetry group of the algebra (N, N, N^2-1) is $U(N)$.

To make contact with more familiar descriptions of unitary symmetry, we note that the equivalence transformation

$$\phi_A\to\hat{\phi}_A=\left(\frac{\lambda_1}{N(N^2-1)}\right)^{1/2}g_0{}^{\dagger A}{}_B\phi_B,$$

and

$$g_\alpha\to\hat{g}_\alpha=\left(\frac{\lambda_1}{N(N^2-1)}\right)^{1/2}g_0{}^\dagger g_\alpha,$$

implies

$$\mathrm{tr}\hat{g}_\alpha=0.$$

Therefore, the matrices $\hat{g}_\alpha$ can be represented as linear combinations of the N^2-1 traceless Hermitian generators G_α of the group $SU(N)$:

$$\hat{g}_\alpha=\lambda_1{}^{1/2}W_{\alpha\beta}G_\beta,$$

where the normalization of the generators G_α is given by

$$\mathrm{tr}\{G_\alpha G_\beta{}^\dagger\}=[1/(N^2-1)]\delta_{\alpha\beta}.$$

This implies that the transformation $W_{\alpha\beta}$ is unitary and this accomplishes the proof that up to equivalence transformations the coupling matrices g_α coincide with the generators of the group $SU(N)$.

It is worth pointing out that in the above derivation no assumption has been made about the existence of other constraints like invariance under a subgroup or even the existence of a conserved additive quantum number. Once the $SU(N)$ invariance is established, the invariance under its subgroups is of course automatic.

To conclude this section we note that the symmetry algebra $(N,N,N^2-1)_{S_1}$ (the multiplets ϕ_A and ϕ_a coincide) has also been shown to lead to the symmetry group $U(N)$. The demonstration is more involved and use has to be made of the third-order self-consistency relation. On the other hand, the present analysis can be extended to the algebra $(N,N,N^2)_{S_1}$ in a trivial fashion. One finds again the group $U(N)$.

V. STRUCTURE OF THE SYMMETRY ALGEBRA (3,3,3)

It does not seem possible to deal with the remaining types of symmetry algebras for $n\leq\nu\leq Nn-2$ in a general fashion. In order to see what has to be expected, we have investigated in detail the particular symmetry algebra $(3,3,3)_{S_1\cap S_2}$ involving the coupling of a self-adjoint triplet ϕ_α to a triplet $\phi_A=\phi_a$. Because of these special conditions, the coupling matrices g_1, g_2, and g_3 are Hermitian and the symmetry algebra under consideration is characterized by the relations

$$g_\alpha{}^\dagger=g_\alpha, \tag{5.1}$$

$$g_\alpha g_\alpha=\mathbf{1}, \tag{5.2}$$

$$\mathrm{tr}\{g_\alpha g_\beta\}=\delta_{\alpha\beta}, \tag{5.3}$$

$$g_\beta g_\alpha g_\beta=\lambda_2 g_\alpha, \tag{5.4}$$

$$g_\beta g_\gamma g_\alpha g_\beta g_\gamma=\lambda_3 g_\alpha. \tag{5.5}$$

The set of these relations could be enlarged by adding generalizations of the last equation to obtain equations of degree $(2n+1)$ involving n indices which occur twice each in any order, summed over independently. The normalization of the matrices g_α has been chosen such that $\lambda_1=3$. All these equations are invariant under the transcription

$$g_\alpha\to W_{\alpha\beta}Ug_\beta U^\dagger, \tag{5.6}$$

provided the matrices $W_{\alpha\beta}$ and U are orthogonal and unitary, respectively. The main result of our analysis is the assertion that by means of a transformation of the type (5.6) every solution of Eqs. (5.1) to (5.5) can be brought to one of the three standard solutions displayed in Table I. In other words, the manifold of solutions of the relations (5.1) to (5.5) consists of three equivalence classes, each class being obtained from the associated standard solution α, β, or γ, by means of all possible transformations of the type (5.6).

The method of the proof is simple. One first shows that by means of a W transformation of the type (5.6), it is always possible to obtain

$$\mathrm{tr}g_1=\det g_1=0. \tag{5.7}$$

Making use of the U transformations in (5.6), one then diagonalizes g_1. The conditions (5.7), together with the normalization (5.3), imply

$$g_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}. \tag{5.8}$$

One then inserts this form for g_1 in the self-consistency relations (5.1) to (5.5) and obtains explicit nonlinear equations for the determination of the 18 unknown matrix elements of g_2 and g_3. These equations turn out to be manageable but the analysis is too lengthly to be reproduced here in full. One finds the three standard solutions displayed in Table I.

The three classes of solutions are characterized by three different algebraic properties.

$$\begin{aligned} \alpha:&\quad g_\alpha g_\beta - g_\beta g_\alpha = 0;\\ \beta:&\quad g_\alpha g_\beta - g_\beta g_\alpha = (-i/\sqrt{2})\epsilon_{\alpha\beta\gamma} g_\gamma;\\ \gamma:&\quad g_\alpha g_\beta + g_\beta g_\alpha = (1/\sqrt{2}) g_\gamma, \quad \text{(cycl.)}. \end{aligned} \tag{5.9}$$

These properties are invariant with respect to the transformations (5.6) with the exception of the algebraic characterization of class γ, which is not invariant with respect to W transformations. This, together with the fact that the standard solution γ does not satisfy the algebraic properties characteristic of the classes α or β, implies immediately that the three classes are disjoint in the sense that there is no transformation of the type (5.6) by means of which any two of the three standard solutions can be transformed into each other.

We conclude this structure analysis by a demonstration of the algebraic dependence of the higher-degree relations on the quadratic, cubic, and quintic relations (5.1) to (5.5). A consequence of such a demonstration is that these lower-degree relations completely characterize the symmetry algebra. On the other hand, the quintic relations (5.5) are necessary. There are solutions of the relations (5.1) to (5.4) which do not satisfy (5.5).

A typical higher-degree relation would have the form

$$\sum_{\beta_1\cdots\beta_n} g_{\beta_1}\cdots g_{\beta_n} g_\alpha g_{\gamma_1}\cdots g_{\gamma_n} = \lambda_{n+1} g_\alpha, \tag{5.10}$$

where the indices $\gamma_1, \cdots, \gamma_n$ are identical with $\beta_1, \cdots, \beta_n$ in some order. (The constant λ_{n+1} may depend on the permutation of $\gamma_1, \cdots, \gamma_n$, which gives $\beta_1, \cdots, \beta_n$). The above relation is obviously satisfied in the case of the algebra α where the matrices g_α commute. For the algebra β the matrices g_α are the generators of the rotation group and the left-hand side of (5.10) is a tensor operator of rank one. Any such tensor operator which acts exclusively on the representation space to map it onto itself must be a constant multiple of the generators. Therefore, all the higher-degree relations are automatically satisfied.

For the algebra γ, an entirely analogous argument can be given, if use is made of the symmetry group associated with this algebra. This symmetry group is discussed in the next section and we shall come back to the problem of verifying the higher-order self-consistency relations in the case of algebra γ at the end of that section.

VI. SYMMETRY GROUPS ASSOCIATED WITH THE ALGEBRAS α, β, AND γ

This section is devoted to the question of the possible existence of a symmetry group that can be associated with the three symmetry algebras α, β, and γ. For the algebras α and β the question is trivially answered: The algebra α is associated with a three-parameter Abelian group and the algebra β with the orthogonal group $O(3)$ in three dimensions. In these cases, the coupling matrices constitute the Lie algebra of the generators of the relevant continuous group. The problem is, however, not trivial for the algebra γ. In order to show that this coupling scheme is invariant under a symmetry group, we have to show that there is a group G—this group does not have to be a continuous group—that admits of two three-dimensional representations D and R such that R is contained in the reduction of the direct product $D\otimes D^*$ and the coupling matrices are the Clebsch-Gordan coefficients for this reduction:

$$D g_\alpha D^\dagger = R_{\beta\alpha} g_\beta. \tag{6.1}$$

We find that this is indeed the case: The algebra γ corresponds to the coupling of two triplets invariant under the symmetric group S_4 on four variables with the triplets transforming as three-dimensional irreducible representations. A direct proof of this result is given below; at the same time the proof shows that S_4 is the largest symmetry group compatible with this algebra.

We first note that the nine matrices g_1, g_2, g_3, g_2g_3, g_3g_1, g_1g_2, $(g_1)^2$, $(g_2)^2$, $(g_3)^2$ are all linearly independent. If we insert (6.1) into the anticommutation relations,

$$g_\alpha g_\beta + g_\beta g_\alpha = (1/\sqrt{2}) g_\gamma, \quad \text{(cycl.)}$$

characteristic of the algebra γ we find that the matrix $R_{\alpha\beta}$ must have one of the following forms:

$$\begin{matrix} \begin{pmatrix} x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}, & \begin{pmatrix} x & 0 & 0\\ 0 & 0 & y\\ 0 & z & 0\end{pmatrix}, & \begin{pmatrix} 0 & x & 0\\ 0 & 0 & y\\ z & 0 & 0\end{pmatrix},\\ \begin{pmatrix} 0 & x & 0\\ y & 0 & 0\\ 0 & 0 & z\end{pmatrix}, & \begin{pmatrix} 0 & 0 & x\\ y & 0 & 0\\ 0 & z & 0\end{pmatrix}, & \begin{pmatrix} 0 & 0 & x\\ 0 & y & 0\\ z & 0 & 0\end{pmatrix}, \end{matrix} \tag{6.2}$$

with the real variables x, y, and z satisfying

$$yz = x; \quad zx = y; \quad xy = z. \tag{6.3}$$

There are no conditions on the matrix D as D leaves algebraic relations invariant. For (nontrivial) solutions of (6.3), either all three variables x, y, and z equal unity or two of them are -1 and the third equals $+1$. This makes altogether 24 possible choices for the matrix $R_{\alpha\beta}$, compatible with the anticommutation relations. These 24 matrices can be generated by taking all possible matrix products of the three matrices

$$A = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

Under matrix multiplication, the 24 matrices characterized above form a group isomorphic to the symmetric group S_4 on four variables.

We now wish to show that matrices D with the property (6.1) exist. The representative $D(A)$ of the matrix A must satisfy the relations

$$D(A)g_1D(A)^\dagger = g_1,$$
$$D(A)g_2D(A)^\dagger = -g_2,$$

and

$$D(A)g_3D(A)^\dagger = -g_3.$$

Such a matrix is given by

$$D(A)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Similarly, one finds

$$D(B)=\tfrac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{bmatrix},$$

and

$$D(C)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is straightforward to extend this set of representatives to a representation of the full group S_4 and to verify that

$$D(s)g_\alpha D(s)^\dagger = R_{\beta\alpha}(s)g_\beta, \quad s\in S_4. \tag{6.4}$$

The orthogonal representation $R_{\beta\alpha}(s)$ is obtained by taking products of the matrices A, B, and C, whereas the unitary representation $D(s)$ arises from the corresponding products of the matrices $D(A)$, $D(B)$, and $D(C)$. Note that the representations $R_{\beta\alpha}(s)$ and $D(s)$ are inequivalent.

We are now in a position to verify that the self-consistency relations (5.10) are satisfied to all orders for the algebra γ as well. For this purpose, let us denote the left-hand side of (5.10) by d_α. Then, under the transformation $g_\alpha \to D(s)g_\alpha D(s)^\dagger$, this quantity is transformed into $D(s)d_\alpha D(s)^\dagger$ on account of the unitarity of the representation $D(s)$. On the other hand, according to (6.4), the above unitary transformation of the matrices g_α is equivalent to the transformation $g_\alpha \to R_{\beta\alpha}(s)g_\beta$ and, due to the orthogonality of the matrices $R_{\beta\alpha}(s)$, this transformation sends d_α into $R_{\beta\alpha}(s)d_\beta$. Thus we have

$$D(s)d_\alpha D(s)^\dagger = R_{\beta\alpha}(s)d_\beta,$$

and it is easy to verify that this implies that d_α is proportional to g_α. This completes the verification of the higher-order self-consistency relations (5.10) for the algebra γ.

Having demonstrated the relation of the algebras α, β, and γ to the three respective groups, we can ask for conservation laws. In the first two cases the symmetry group is continuous and we have additive conservation laws, the additively conserved dynamical quantities being associated with the elements of the Lie algebra. There is, however, a distinction: For algebra α, the additive quantum numbers are all mutually commutative. There are thus three distinct additive conservation laws. If we so choose, we can identify a suitable linear combination of them with the electric charge. However, in this case the field is strictly neutral, i.e., the quanta associated with this field do not carry nonzero values of any of these quantum numbers.

The situation is quite different for algebra β, where we have the familiar isospin-invariant couplings. In this case, although there are three linearly independent additive quantum numbers, they do not commute; only one of them can be diagonalized. Apart from trivial modifications, this quantity can be identified with the electric charge. In this case the quanta of the field are charged and the charge varies linearly within the multiplet.

We have thus a curious situation; "electric" charge conservation does not by itself choose between α and β but the requirement of charge conservation with charges so *assigned* that the quanta of the field ϕ_α carry charge does select out β as the only acceptable interaction.

In the case of the algebra γ, we have only multiplicative quantum numbers. No additive conservation law exists; the requirement of charge conservation with assigned charges together with the symmetry algebra structure selects out β as the only acceptable solution. We have thus "derived" charge independence for this coupling of triplets.

VII. CONCLUDING REMARKS

The discussion of the last section showed how, in some special cases, a symmetry algebra could be associated with a suitable symmetry group. In the general case, however, this does not follow. The linear transformations on the matrices preserving their symmetry algebra structure are present in all cases and one could define equivalence classes of such coupling matrices.

If the coupling matrices are further constrained by additional requirements, it may become possible to deduce a Lie-algebra structure. A well-known example is the totally antisymmetric coupling of a multiplet with itself which is appropriate for describing the self-coupling of vector mesons.[3,7] In this case, we could show that the coupling matrices constitute the regular representation of a (semisimple) Lie group.

Often, on physical grounds, it may be appropriate to postulate invariance of the coupling scheme under some well-defined group of transformations, discrete or continuous; the Lie algebra structure that may be deduced in such cases would involve any such continuous group as a proper subgroup. One example of this is the derivation[4] of $SU(3)$ symmetry for the coupling of two octet pseudoscalar mesons and an octet vector meson: In this case, the invariance under a $SU(2)\times U(1)$ subgroup would correspond to the conservation of isotopic spin and electric charge (or, equally, hypercharge) and it would be natural to impose it. One could further impose invariance of the coupling under charge conjugation. These constraints are sufficient[4] to deduce $SU(3)$ symmetry for the interaction, and a Lie algebra structure for the coupling matrices.

In the general case, we should not expect a Lie algebra structure even if we have a continuous symmetry group for the interaction. This is more or less obvious for the cases when no two of the numbers N, n, ν are equal. In this case, it is impossible to define the commutator of two matrices but the symmetry algebra structure may hold. In particular, if we have an interaction invariant under some group G, with these multiplets furnishing irreducible representations of G, then the coupling coefficients are proportional to the (generalized) Clebsch-Gordan coefficients. If there is only one invariant of G that can be made up of the three multiplets, then the Clebsch-Gordan coefficients are unique. In this case (simple reducibility), since any modified vertex is also an invariant of G, it follows that the symmetry-algebra structure is obtained. No restriction need be made, provided the appropriate symmetry-algebra structure is assumed. In this connection it is worthwhile to point out again that the symmetry-algebra structure treats all the three indices A, a, and α on an equivalent footing, corresponding to the equivalent treatment of the three multiplets; the distinguished appearance of the index α is due to the use of sets of matrices (for want of a more appropriate mathematical entity) to display the coupling structure.

The case of nonsimple reducibility, i.e., the existence of more than one linearly independent invariants constructed out of the three multiplets is of particular interest in relation to the Yukawa coupling between octets of baryons and mesons. In this case, there are two types of $SU(3)$-invariant couplings, usually called the D- and F-type couplings, and there are therefore two coupling constants: The D/F ratio is not dictated by $SU(3)$ invariance. The imposition of a symmetry-algebra structure for this coupling would restrict the allowed D/F values. Thus, in some cases, the symmetry-algebra structure may contain information not contained in the postulate of invariance under a group.

In the derivation of unitary symmetry in the previous sections, or in the other cases referred to above, the full structure of the symmetry algebra was not made use of explicitly; it was sufficient to use the lower-order relations. Thus, in the derivation of unitary symmetry from the symmetry algebra $(N, N, N^2-1)_{S_1}$, we make use of the quadratic and quartic relations only; having deduced $SU(N)$ invariance, since the $N\times N\times(N^2-1)$ coupling has only one invariant, it follows that all the higher-order equations are automatically satisfied. This phenomenon appears in other cases also. In these cases the higher-order relations are not algebraically independent of the lower-order relations. Though some exploratory investigations of these inter-relationships have been made,[9] more detailed study is warranted.

In the discussions so far, we have considered only the interaction of three multiplets of particles, the effect of other particles being totally ignored. This is certainly a drastic approximation. We have also ignored the question of the sign of the interaction (potential); this sign is important in making it plausible that the particles in fact bind themselves to form other particles. In our discussion, however, this was not directly relevant, since the sign of the interaction depends also on the space-time coupling scheme which we have left unspecified.

Once the interactions are such that other particles are also strongly coupled, they must be taken into account.[10] If there is a hierarchy of interactions in which some are very strong and some the next strongest, etc., we could first consider only the very strong interactions in an effort to "derive" a symmetric-interaction structure. If these strongly interacting particles interact with others by the next strongest interactions, in suitable cases we could, by invoking self-consistency, derive[3] for their interactions symmetry under the same group or a smaller group. This latter case is of particular interest in connection with the phenomenological descriptions of a hierarchy of strong interactions with smaller invariance groups. This line of investigation deserves attention.

There is another aspect that also opens up an interesting possibility: We have considered interactions that are strong enough to produce tightly bound states for which we could apply self-consistency notions. Suppose now the interaction is still stronger, so that a variety of isobars are formed which tend to become degenerate. In this limit we must not ignore their presence but they must be treated on the same footing; the symmetry group of the theory enlarges itself. If the number of isobars to be taken account of is finite, then we expect an enlarged compact group; otherwise a suitable noncompact group. The authors hope to return to this problem at a future date.

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[9] J. S. Dowker (unpublished).

[10] A beautiful theory of higher symmetries of strongly-coupled systems has recently been presented by T. Cook, C. J. Goebel, and B. Sakita [Phys. Rev. Letters **15**, 35 (1965)]. Their idea is followed up by V. Singh, Phys. Rev. **144**, 1275 (1966); S. K. Bose, *ibid.* **145**, 1247 (1966); J. G. Kuriyan and E. C. G. Sudarshan, Phys. Letters **21**, 106 (1966); Phys. Rev. Letters **16**, 825 (1966). See also the contributions by C. J. Goebel and N. Mukunda, in *Proceedings of the Conference on Non-Compact Groups in Particle Physics*, edited by Y. Chow (W. A. Benjamin Inc., New York, 1966).