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Hamiltonian Model of Lorentz Invariant Particle Interactions*

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A model is constructed of interaction in a quantum mechanical system of two spinless particles. The interaction is shown to produce nontrivial scattering. Lorentz symmetry of the model is established by the construction of generators of a unitary representation of the inhomogeneous Lorentz group. The total momentum and angular momentum operators are the same as for a system of two free particles. This ensures familiar transformation properties under space translations and space rotations. The Hamiltonian satisfies the asymptotic condition relative to the Hamiltonian for a system of free particles. The use of the asymptotic condition is shown to be Lorentz invariant. The scattering amplitude is a manifestly invariant function of the particle momentum variables, and can be made to have a variety of analyticity properties by a suitable choice of the arbitrary form factors which occur in the model.

I. INTRODUCTION

THIS paper was motivated by a recent study of special relativistic invariance in Hamiltonian particle dynamics.[1,2] This study has emphasized two distinct aspects of relativistic invariance. The first of these is the symmetry of the theory under the inhomogeneous Lorentz group, reflecting the principle of special relativity that the laws of physics should be invariant under transformations of reference frames. This symmetry is guaranteed by the existence of ten infinitesimal generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$, for time translations, space translations, space rotations, and rotation-free Lorentz transformations, respectively, satisfying the Lie (Poisson or commutator) bracket equations characteristic of the inhomogeneous Lorentz group.[1,3]

The second aspect involves the manifest invariance or the explicit transformation properties of specific quantities. The philosophy of recent work is to describe particle interactions by an S matrix or scattering amplitude. Relativistic invariance is taken to mean that the scattering amplitude is a manifestly invariant function of the particle momentum variables.

In this paper we construct a model of interaction in a quantum mechanical system of two spinless particles. The interaction is shown to produce nontrivial scattering. Lorentz symmetry of the model is established by the construction of ten Hermitian operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ which generate a unitary representation of the inhomogeneous Lorentz group. The generators $\mathbf{P}$ and $\mathbf{J}$ are just the usual total momentum and angular momentum operators for a system of two free particles. This ensures that all quantities, for example the particle position and momentum variables, will transform in the familiar manner under space translations and space rotations. The Hamiltonian operator H satisfies the asymptotic condition relative to the Hamiltonian operator H_0 of a system of two free particles. We show that our use of the asymptotic condition is a Lorentz invariant procedure. The scattering amplitude is a manifestly invariant function of the particle momentum variables.

Theories of interaction in a quantum mechanical system of two particles have been constructed previously by Bakamjian and Thomas[4] and by Foldy.[5] These theories exhibit Lorentz symmetry by the existence of generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ of a representation of the inhomogeneous Lorentz group. But the interaction in these theories is described only by the Hamiltonian. Our model goes further by providing a solution of the scattering problem in which the asymptotic condition is satisfied and in which the scattering amplitude is manifestly invariant.

With a suitable choice of the arbitrary form factors that occur in our model, the scattering amplitude can be made to have a variety of analyticity properties as a function of energy or angular momentum. In particular, causality conditions can be satisfied and Regge pole behavior can be produced. Our constructions can be used also to make a model field theory with nontrivial scattering which satisfies all of the usual field theory axioms except that it transforms nonlocally.

To build our model, we define two unitary operators Ω_+ and Ω_-. Then we construct the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ of the representation of the inhomogeneous Lorentz group by using Ω_+ to make a unitary transformation of the generators H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ of the representation which is characteristic of a system of two free particles. The scattering amplitude is easily found because Ω_+ and Ω_- are carefully chosen so that they turn out to be the wave operators for the scattering problem defined by H and H_0. Our constructions depend in a fundamental way on our use of variables in terms of which H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ have the form of a reduction into irreducible representations of the direct product of the two single-particle representations of the inhomogeneous Lorentz group.[6] These variables are introduced in

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[1] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).
[2] D. G. Currie, J. Math. Phys. 4, 1470 (1963); thesis, University of Rochester, 1962 (unpublished).
[3] P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).
[4] L. H. Thomas, Phys. Rev. 85, 868 (1952); B. Bakamjian and L. H. Thomas, *ibid.* 92, 1300 (1953); B. Bakamjian, *ibid.* 121, 1849 (1961).
[5] L. L. Foldy, Phys. Rev. 122, 275 (1961).
[6] A. J. Macfarlane, J. Math. Phys. 4, 490 (1963). We follow the

Sec. II. In Sec. III we define Ω_+ and Ω_- and construct the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$. Section IV contains the solution for the scattering. In Sec. V we explain the Lorentz invariance of our use of the asymptotic condition. In Sec. VI and VII we conclude with remarks on analyticity properties and model nonlocal field theories. All of the actual work involved in the construction of our model is contained in three appendices.

In classical mechanics one can describe the motion of particles by the time dependence of their positions in space. It is natural to postulate, as part of the requirement of relativistic invariance, that the time-dependent values of the particle position variables transform in the familiar manner of space-time events under space translations, space rotations, and Lorentz transformations. This postulate, together with the postulate that there exists generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ establishing symmetry under the inhomogeneous Lorentz group, has been used to prove theorems that there can be no interaction in a classical mechanical system of two or three spinless particles.[1,2,7]

The transformation properties of particle positions under space translations and space rotations are as well established in quantum mechanics as in classical mechanics, but the role of Lorentz transformations of particle positions in quantum mechanics is not so clear. In quantum mechanics one cannot provide a direct physical interpretation for the equations which are the analogs of those characteristic of Lorentz transformations of particle positions in classical mechanics.[8] In this paper we adopt the attitude that one may ignore the Lorentz transformation properties of the particle positions and require only that the scattering amplitude be Lorentz invariant in a quantum-mechanical theory of particle dynamics.

II. LORENTZ SYMMETRY FOR FREE PARTICLES

We consider a system of two particles with zero spins and positive masses m_1 and m_2. We work with operators defined on wave functions of the momentum variables $\mathbf{p}^1$ and $\mathbf{p}^2$ for the two particles, the inner product of two wave functions f and g being defined by

$$(f,g)=\int f(\mathbf{p}^1,\mathbf{p}^2)^*g(\mathbf{p}^1,\mathbf{p}^2)(1/2W_1)(1/2W_2)d^3p^1d^3p^2\,, \tag{2.1}$$

where $(W_1)^2=(\mathbf{p}^1)^2+(m_1)^2$ and $(W_2)^2=(\mathbf{p}^2)^2+(m_2)^2$.

For two free particles we have infinitesimal generators H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ for a unitary representation of the inhomogeneous Lorentz group in the canonical forms[6]

$$H_0=W_1+W_2\,, \tag{2.2}$$

$$\mathbf{P}_0=\mathbf{p}^1+\mathbf{p}^2\,, \tag{2.3}$$

$$\mathbf{J}_0=\mathbf{q}^1\times\mathbf{p}^1+\mathbf{q}^2\times\mathbf{p}^2\,, \tag{2.4}$$

$$\mathbf{N}_0=W_1\mathbf{q}^1+W_2\mathbf{q}^2\,, \tag{2.5}$$

where $\mathbf{q}^1$ and $\mathbf{q}^2$ are the coordinate operators canonical to $\mathbf{p}^1$ and $\mathbf{p}^2$ that are defined by the relations

$$(q_j^nf)(\mathbf{p}^1,\mathbf{p}^2)=i(\partial/\partial p_j^n)f(\mathbf{p}^1,\mathbf{p}^2) \tag{2.6}$$

for $n=1,2$; $j=1,2,3$ and for any momentum space wave function f. (We choose units for which $\hbar=c=1$.) One can check that the operators (2.2)–(2.5) are Hermitian with the inner product (2.1) and that they satisfy the commutation relations[9] characteristic of the inhomogeneous Lorentz group.

It will be convenient to make a change of variables from $\mathbf{p}^1$ and $\mathbf{p}^2$ to M, $\mathbf{K}$, and $\mathbf{e}$, with M the total mass of the two-particle system, $\mathbf{K}$ the total momentum, and $\mathbf{e}$ a unit vector introduced[10] to describe the relative motion of the particles. (Since $\mathbf{e}$ is a unit vector, only its spherical polar angles are variables and altogether there are six variables as before.) These variables are related to $\mathbf{p}^1$ and $\mathbf{p}^2$ by the equations[6]

$$\begin{aligned}
&\mathbf{K}=\mathbf{p}^1+\mathbf{p}^2\,,\\
&M^2=(W_1+W_2)^2-\mathbf{K}^2\,,\\
&\mathbf{e}=\mathbf{q}-(M+W_1+W_2)^{-1}q_0\mathbf{K}\,,\\
&\mathbf{q}=M\lambda^{-1/2}(M)\{\mathbf{p}^1-\mathbf{p}^2-[(m_1^2-m_2^2)/M^2]\mathbf{K}\}\,,\\
&q_0=M\lambda^{-1/2}(M)\{W_1-W_2-[(m_1^2-m_2^2)/M^2]\\
&\qquad\times(W_1+W_2)\}\,,\\
&\lambda(M)=[M^2-(m_1+m_2)^2][M^2-(m_1-m_2)^2]\,.
\end{aligned} \tag{2.7}$$

We can write the momentum space wave functions as functions of $\mathbf{K}$, M, and $\mathbf{e}$. Let us, at the same time, make a spherical harmonic decomposition in the angles of $\mathbf{e}$ and write

$$f(\mathbf{p}^1,\mathbf{p}^2)=2M^{1/2}\lambda^{-1/4}(M)\sum_{lm}f_{lm}(M,\mathbf{K})Y_{lm}(\mathbf{e})\,, \tag{2.8}$$

[where it is to be understood that the summation is from $l=0$ to infinity, $m=-l$ to l, and that $Y_{lm}(\mathbf{e})$ is the spherical harmonic function of the spherical polar angles of the unit vector $\mathbf{e}$]. Each state vector f can be represented either by the wave function $f(\mathbf{p}^1,\mathbf{p}^2)$ or by the sequence of functions $f_{lm}(M,\mathbf{K})$. For functions $f(\mathbf{p}^1,\mathbf{p}^2)$ we have the inner product (2.1) which for functions $f_{lm}(M,\mathbf{K})$ takes the form[6]

$$(f,g)=\int_{m_1+m_2}^{\infty}\int\frac{1}{2}(M^2+\mathbf{K}^2)^{-1/2}\times d^3K\sum_{lm}f_{lm}(M,\mathbf{K})^*g_{lm}(M,\mathbf{K})\,. \tag{2.9}$$

notation of this paper. In it the reader will find an extensive list of further references.

[7] J. T. Cannon and T. F. Jordan, University of Rochester Report NYO-10263 (to be published).

[8] See Ref. 1, especially the last part of Sec. III. However, these equations are not without content in quantum mechanics. Their consequences for a single free particle with spin have been investigated by T. F. Jordan and N. Mukunda, University of Rochester Report NYO-10270 (to be published).

[9] L. L. Foldy, Phys. Rev. 102, 568 (1956).

[10] A. J. Macfarlane, Rev. Mod. Phys. 34, 41 (1962).

In terms of the new variables $\mathbf{K}$, M, and $\mathbf{e}$, the operators H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ are [6]

$$(H_0f)_{lm}(M,\mathbf{K})=(M^2+\mathbf{K}^2)^{1/2}f_{lm}(M,\mathbf{K})\,, \tag{2.10}$$

$$(\mathbf{P}_0f)_{lm}(M,\mathbf{K})=\mathbf{K}f_{lm}(M,\mathbf{K})\,, \tag{2.11}$$

$$(\mathbf{J}_0f)_{lm}(M,\mathbf{K})=-i\mathbf{K}\times\boldsymbol{\nabla}f_{lm}(M,\mathbf{K})+(\mathbf{I}f)_{lm}(M,\mathbf{K})\,, \tag{2.12}$$

$$\begin{aligned}(\mathbf{N}_0f)_{lm}(M,\mathbf{K})=\,&i(M^2+\mathbf{K}^2)^{1/2}\boldsymbol{\nabla}f_{lm}(M,\mathbf{K})\\&+[M+(M^2+\mathbf{K}^2)^{1/2}]^{-1}\\&\qquad\mathbf{K}\times(\mathbf{I}f)_{lm}(M,\mathbf{K})\,,\end{aligned} \tag{2.13}$$

where $\boldsymbol{\nabla}$ is the gradient operator with respect to the $\mathbf{K}$ variables and

$$(I_rf)_{lm}(M,\mathbf{K})=[l(l+1)]^{1/2}\sum_n C(l1nrm)f_{ln}(M,\mathbf{K}) \tag{2.14}$$

with $C(l1nrm)$ a Clebsch-Gordan coefficient and with $r(=1,0,-1)$ referring to a spherical component of $\mathbf{I}$. Here, the total angular momentum $\mathbf{J}_0$ of the two-particle system appears as the sum of a term which represents the orbital angular momentum arising from the motion of the total system and the term $\mathbf{I}$ which represents the intrinsic angular momentum arising from the relative motion of the particles.

The reader who wants to understand why we choose the variables $\mathbf{K}$, M, and $\mathbf{e}$, make the spherical harmonic decomposition (2.8), and put the operators H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ in the forms (2.10)–(2.13), needs to recognize that this is just what is needed to effect a decomposition into irreducible representations of the representation of the inhomogeneous Lorentz group generated by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$.[6] In the forms (2.2)–(2.5) these operators exhibit the structure of a direct product of two irreducible representations with zero spins and positive masses m_1 and m_2. From the forms (2.10)–(2.13) we see that for each fixed l and M the functions $f_{lm}(M,\mathbf{K})$ form a space which is invariant under H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$. On this space these operators generate an irreducible representation of the inhomogeneous Lorentz group with mass M and spin l.[6] Equations (2.9)–(2.13) state how these irreducible representations are combined to form a representation appropriate for two free spinless particles.

III. LORENTZ SYMMETRY FOR INTERACTING PARTICLES

At the heart of our model are two operators Ω_+ and Ω_- which are defined by

$$(\Omega_\pm f)_{lm}(M,\mathbf{K})=f_{lm}(M,\mathbf{K})+\int dM'\left(\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right)^{1/4}\times\frac{G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)}f_{lm}(M',\mathbf{K}) \tag{3.1}$$

for any sequence of wave functions f_{lm}, with

$$B_{l\pm}(M)=B_l(M\pm i\epsilon)\,, \tag{3.2}$$

$$B_l(z)=1-\int\frac{G_l(M)^2}{z-M}dM\,. \tag{3.3}$$

[It is to be understood that integrations over M and M' variables as in Eqs. (3.1) and (3.3), are to be from m_1+m_2 to infinity. Quantities such as in Eqs. (3.1) and (3.2) which contain an ϵ are to be taken in the limit as the positive number ϵ goes to zero.] Our choice of the functions G_l is limited only by the requirements that they are real and that all the equations in which they occur are meaningful. Otherwise the G_l are arbitrary.

In Appendix A it is established that the operators Ω_+ and Ω_- satisfy the equations

$$\Omega_+{}^+\Omega_+=\Omega_-{}^+\Omega_-=1\,, \tag{3.4}$$

$$\Omega_+\Omega_+{}^+=\Omega_-\Omega_-{}^+=1\,, \tag{3.5}$$

and are therefore unitary operators. [Here, as always, we use the inner product (2.1) or (2.9).]

The generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ of the representation of the inhomogeneous Lorentz group are defined in our model by

$$\begin{aligned}H&=\Omega_+H_0\Omega_+{}^+\,,\\ \mathbf{P}&=\Omega_+\mathbf{P}_0\Omega_+{}^+\,,\\ \mathbf{J}&=\Omega_+\mathbf{J}_0\Omega_+{}^+\,,\\ \mathbf{N}&=\Omega_+\mathbf{N}_0\Omega_+{}^+\,,\end{aligned} \tag{3.6}$$

where H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ are the generators (2.2)–(2.5) or (2.10)–(2.13) for a system of two free particles. Since H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ are Hermitian and Ω_+ is unitary, H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ are Hermitian [in the inner product (2.1) or (2.9)]. The unitary property of Ω_+ also ensures that the operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ satisfy the same commutation relations characteristic of the inhomogeneous Lorentz group as are satisfied by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$. The Lorentz symmetry of our model is established by the unitary representation of the inhomogeneous Lorentz group generated by H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$.[1]

In Appendix B it is shown that

$$\mathbf{P}=\mathbf{P}_0\,, \tag{3.7}$$

$$\mathbf{J}=\mathbf{J}_0\,, \tag{3.8}$$

$$\Omega_-H_0\Omega_-{}^+=H\,, \tag{3.9}$$

$$\begin{aligned}(Hf)_{lm}(M,\mathbf{K})=\,&(H_0f)_{lm}(M,\mathbf{K})\\&+\int dM'\left(\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right)^{1/4}G_l(M)G_l(M')\\&\times[1+F_l(\mathbf{K}^2,M,M')]f_{lm}(M',\mathbf{K})\,,\end{aligned} \tag{3.10}$$

where $F_l(\mathbf{K}^2,M,M')$ is a function which depends on the function $G_l(M)$. The term containing $F_l(\mathbf{K}^2,M,M')$ represents a "relativistic correction" to the Hamiltonian H which otherwise appears as the sum of the free-particle Hamiltonian H_0 and a separable potential operator.

By applying the unitary transformation Ω_+ to the free-particle Hamiltonian H_0, we have constructed a Hamiltonian H which contains an interaction term. We maintain the Lorentz symmetry of the theory by applying the same unitary transformation to $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ to produce a complete set of generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$ for a unitary representation of the inhomogeneous Lorentz group. The fact that $\mathbf{P}$ and $\mathbf{J}$ are the same as $\mathbf{P}_0$ and $\mathbf{J}_0$ means that all quantities, for example the particle momentum operators $\mathbf{p}^1$ and $\mathbf{p}^2$, will transform under space translations and rotations just as they do in a theory of free particles. Of course $\mathbf{N}$ is not the same as $\mathbf{N}_0$.

IV. SCATTERING

We now show that the interaction introduced in the preceding section produces nontrivial scattering in the two-particle system and that the scattering amplitude is a manifestly invariant function.

We have already remarked that the Hamiltonian operator H has the form of the free-particle Hamiltonian H_0 plus a separable potential operator. Our construction of this model has been guided by knowledge of solutions of scattering problems with a separable potential.[11] In fact the solution has been built into the model. For $\Omega_\pm$ are the wave operators for the scattering problem defined by H and H_0.[12] This follows from three facts that are proved in the appendices. The first, which we have already noted in Eqs. (3.4) and (3.5), is that $\Omega_\pm$ are unitary operators. The second is a combination of the unitarity of $\Omega_\pm$, the definition (3.6) of H, and Eq. (3.9), which we state as

$$H\Omega_\pm=\Omega_\pm H_0, \tag{4.1}$$

The third, which is proved in Appendix C, is that

$$\lim_{t\to\mp\infty} e^{iH_0t}\Omega_\pm e^{-iH_0t}=1. \tag{4.2}$$

These can be taken as the defining properties of the wave operators for scattering by the Hamiltonian H relative to the free Hamiltonian H_0.[12]

The asymptotic condition is satisfied by our model; from the three properties of $\Omega_\pm$ stated above, one can show that[12]

$$\Omega_\pm=\lim_{t\to\mp\infty} e^{iHt}e^{-iH_0t}. \tag{4.3}$$

It also follows that the Hamiltonian operator H has no bound states. For the ranges of the operators (4.3) are known to be contained in the subspace of continuum "eigenstates" of H[12]. But we know that every state is contained in the range of each of the operators $\Omega_\pm$ because $\Omega_\pm$ are unitary.

The scattering operator S is defined by[12]

$$S=\Omega_-^{+}\Omega_+, \tag{4.4}$$

By using Eq. (4.1) and the adjoint of Eq. (4.3) we find that

$$S=\lim_{t\to\infty} e^{iH_0t}e^{-iHt}\Omega_+$$

$$=\lim_{t\to\infty} e^{iH_0t}\Omega_+e^{-iH_0t}. \tag{4.5}$$

One can use either Eq. (4.4) or Eq. (4.5) to evaluate the S operator. In Appendix C it is shown that

$$(Sf)_{lm}(M,\mathbf{K})=\frac{B_{l-}(M)}{B_{l+}(M)}f_{lm}(M,\mathbf{K}). \tag{4.6}$$

Since $B_{l+}(M)$ is the complex conjugate of $B_{l-}(M)$, we may write

$$(Sf)_{lm}(M,\mathbf{K})=e^{2i\delta_l(M)}f_{lm}(M,\mathbf{K}), \tag{4.7}$$

where $\delta_l(M)=\arg[B_{l-}(M)]$.

Two things should be noted. First we see that, in general, there will be nontrivial scattering. By choosing various functions $G_l(M)$ we can produce various phase shifts $\delta_l(M)$. This can be done independently for each l. For example, by choosing G_l to be nonzero only for certain values of l, we can produce scattering in those partial-wave channels with no scattering in the others.

We also see that the scattering amplitude is a manifestly invariant function. It depends only on the variables l and M. That these are invariant functions of the particle momentum variables $\mathbf{p}^1$ and $\mathbf{p}^2$ is evident from an inspection of the formulas (2.10)–(2.14) for H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$. The change of variables (2.7) which was used to put H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$ in the forms (2.10)–(2.14) is a fundamental part of our construction of an invariant scattering amplitude. The manifest invariance of the scattering amplitude is invariance under the representation of the inhomogeneous Lorentz group generated by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$, not under the representation generated by H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$. This is in accord with the picture of scattering which supports the asymptotic condition: scattering is between initial and final states in which the particles are free and the scattering amplitude is a function of the variables describing the initial and final free-particle motion.

V. LORENTZ INVARIANCE OF THE WAVE OPERATORS AND ASYMPTOTIC CONDITION

We now show that our use of Eq. (4.1), the asymptotic condition (4.2), and the wave operators (4.3) is a

[11] E. C. G. Sudarshan, *1961 Brandeis Summer Institute Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1962).

[12] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 7; J. M. Jauch, Helv. Phys. Acta **31**, 127 and 661 (1958); T. F. Jordan, J. Math. Phys. **3**, 414 and 429 (1962).

Lorentz invariant procedure. Specifically, we show that we get the same wave operators $\Omega_\pm$ if we take the limits (4.3) in any Lorentz transformed reference frame, and that operators $\Omega_\pm$ satisfy the asymptotic condition (4.2) in any Lorentz transformed reference frame if they satisfy it in one frame. We also show that $\Omega_\pm$ satisfy Eq. (4.2) in any Lorentz transformed reference frame if they satisfy it in one frame.

Scattering involves a comparison of the dynamics of a system of interacting particles with that of a system of free particles.[12] In the interacting system the representation of the inhomogeneous Lorentz group is generated by H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{N}$. In the free system the representation of the inhomogeneous Lorentz group is generated by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{N}_0$.

Consider a transformation to a reference frame moving in the x direction with respect to a given frame with a velocity $v=\tanh a$. We may use the "Heisenberg picture" to represent this transformation[1]; operators are transformed but state vectors are unchanged. For the interacting system

$$H \to H \cosh a - P_1 \sinh a \tag{5.1}$$

and for the free system

$$H_0 \to H_0 \cosh a - P_{01} \sinh a\,. \tag{5.2}$$

Under this transformation the wave operators (4.3) go into

$$\lim_{t\to\mp\infty} e^{iHt\cosh a}e^{-iP_1t\sinh a}e^{iP_{01}t\sinh a}e^{-iH_0t\cosh a}. \tag{5.3}$$

(Here we use the fact that H and $\mathbf{P}$ commute with each other, as do H_0 and $\mathbf{P}_0$.) Due to the fact established in Appendix B that

$$\mathbf{P}=\mathbf{P}_0 \tag{5.4}$$

the operators (5.3) are equal to

$$\lim_{t\to\mp\infty} e^{iHt\cosh a}e^{-iH_0t\cosh a},$$

which are the same as the operators (4.3). The same result holds, of course, for frames moving in the y or z direction. Under transformations to frames that are rotated in space or translated in space or time with respect to the given frame there is no change in H or H_0 because H commutes with H, $\mathbf{P}$, $\mathbf{J}$, and H_0 commutes with H_0, $\mathbf{P}_0$, $\mathbf{J}_0$. Hence, there is no change in the wave operators (4.3). The wave operators (4.3) are thus the same in all reference frames which are related by transformations of the inhomogeneous Lorentz group. This result depends only on the fact that Eq. (5.4) is satisfied in our model.

Consider again the Lorentz transformation under which H_0 is transformed according to the relation (5.2). The left-hand sides of Eqs. (4.2) go into

$$\lim_{t\to\mp\infty} e^{iH_0t\cosh a}e^{-iP_{01}t\sinh a}\Omega_\pm e^{iP_{01}t\sinh a}e^{-iH_0t\cosh a}. \tag{5.5}$$

Due to the fact established in Appendix B that

$$\Omega_\pm P_0 = P_0\Omega_\pm \tag{5.6}$$

the operators (5.5) are equal to

$$\lim_{t\to\mp\infty} e^{iH_0t\cosh a}\Omega_\pm e^{-iH_0t\cosh a},$$

which are the same as the left-hand sides of Eqs. (4.2). As before, the same result holds for frames moving in the y or z direction and holds trivially for frames that are rotated in space or translated in space or time. Therefore, wave operators $\Omega_\pm$ satisfying the asymptotic condition (4.2) in one reference frame satisfy it in any frame that can be reached by a transformation of the inhomogeneous Lorentz group. This result depends only on the fact that Eq. (5.6) is satisfied in our model.

Consider once more the Lorentz transformation under which H and H_0 are transformed according to the relations (5.1) and (5.2). Equation (4.1) goes into

$$(H\cosh a - P_1\sinh a)\Omega_\pm = \Omega_\pm(H_0\cosh a - P_{01}\sinh a)$$

which is valid because of Eq. (4.1) and the equation

$$P\Omega_\pm = \Omega_\pm P_0\,, \tag{5.7}$$

which is also established in Appendix B. Once more, the same result holds for frames moving in the y or z direction and holds trivially for frames that are rotated in space or translated in space or time. Hence wave operators $\Omega_\pm$ satisfying Eq. (4.1) in one reference frame satisfy it in any frame that can be reached by a transformation of the inhomogeneous Lorentz group. This result depends only on the fact that Eq. (5.7) is satisfied in our model.

We emphasize that these results depend only on the three conditions (5.4), (5.6), and (5.7). These have the property that any two of them implies the third. In particular, under the condition (5.7) for invariance of the equality (4.1), Eqs. (5.4) and (5.6) are equivalent conditions for the invariance of our use of the asymptotic condition (4.2) or the wave operators (4.3). These conditions ensure that the scattering found by comparison of the interacting and free systems is independent of the frame in which the comparison is made.

VI. ANALYTICITY PROPERTIES

From the definitions (3.2) and (3.3) of $B_{l\pm}$ there follows the relation

$$B_{l+}(M)-B_{l-}(M)=i2\pi G_l(M)^2. \tag{6.1}$$

Using this, we can write the S matrix (4.6) in the form[10,13]

$$\begin{aligned}(Sf)_{lm}(M,\mathbf{K})=&f_{lm}(M,\mathbf{K})\\ &+2i\lambda^{1/2}(M)M^{-2}A_l(M)f_{lm}(M,\mathbf{K})\end{aligned} \tag{6.2}$$

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[13] For the case of equal mass particles $m_1=m_2$, this agrees with G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), Eq. (9–5).

and find that the partial-wave scattering amplitude $A_l(M)$ is

$$A_l(M) = -\lambda^{-1/2}(M) M^2 \pi G_l(M)^2 / B_{l+}(M). \quad (6.3)$$

The purpose of this section is to observe that the scattering amplitude can be made to have various analyticity properties as a function of complex variables l and M, depending the choice of $G_l(M)$. In general, $A_l(M)$ is, of course, not analytic in either l or M. But if we choose $G_l(M)$ to be the boundary value, evaluated at integral non-negative l and real $M > m_1 + m_2$, of an analytic function of two complex variables l and M, then the partial-wave scattering amplitude $A_l(M)$ is also the boundary value of an analytic function of these complex variables.

For scattering in the center-of-mass frame, in other words for a state in which $\mathbf{p}^1 = -\mathbf{p}^2$, the variable M is just the total energy of the two free particles. (M^2 is commonly denoted by s.) A "causality" condition can be formulated in terms of the analyticity of the scattering amplitude in the upper half M plane.[14] Such a condition can be satisfied by a suitable choice of $G_l(M)$. This is not surprising since causality conditions are also satisfied by a variety of static nonrelativistic potentials.

In the $M^2 = s$ plane the partial-wave scattering amplitude $A_l(M)$ has an invariable branch cut from $(m_1+m_2)^2$ to infinity. Any additional left-hand cut can be obtained by including it in the analyticity properties of $G_l(M)$.

It is also possible for $A_l(M)$ to have interesting properties as a function of the complex variable l. Specific choices of $G_l(M)$ which lead to Regge pole behavior have been considered by Acharya.[15]

Interest in the possibility of describing a relativistic quantum-mechanical system of interacting particles in terms of just particle variables has been stimulated by recent developments in the theory of strong interactions. Most of this work is aimed at the construction of an invariant scattering amplitude satisfying the principles of unitarity, analyticity, and crossing symmetry.[16] Despite numerous calculations, no scattering amplitude satisfying these conditions *exactly* is known.

The model constructed in the preceding sections shows that the requirement of relativistic symmetry is not incompatible in a particle theory with an interaction which gives nontrivial scattering described by a manifestly invariant scattering amplitude. We also see that the scattering amplitude can have a variety of familiar analyticity properties. Further, our S matrix is unitary, which provides a simple illustration of the fact that unitarity alone puts no essential constraint on the analyticity of the scattering amplitude in either the energy or the angular momentum. Finally, while the relativistic invariance and unitarity requirements are satisfied exactly in our model, it exhibits no crossing symmetry. In fact crossing symmetry seems to present a nontrivial mathematical problem: to date no nontrivial *exact* solution is known for any multichannel theory which satisfies crossing symmetry and in which channels mix on crossing.

VII. MODEL FIELD THEORIES

In the preceding sections we constructed a model of a system of two interacting particles by making a unitary transformation of a description of two free particles. By a simple extension of this technique we can construct a model field theory which gives nontrivial two-particle scattering with a manifestly invariant scattering amplitude and which satisfies all of the axioms of relativistic quantum field theory except that it does not transform locally. One of the authors has outlined this construction in some detail.[17] We will just summarize the recipe:

Take a free neutral scalar relativistic quantum field theory in the Fock representation. Construct the projection operator to the subspace of two-particle states. Make a unitary transformation which is equal to the unitary transformation Ω_+ of the preceding sections on the two-particle subspace and which is equal to the identity transformation on the orthogonal complement of the two-particle subspace.

The field theory so obtained has a standard particle interpretation and gives scattering only in two-particle channels. It contains a unitary representation of the inhomogeneous Lorentz group, a unique invariant vacuum state, and local commutation relations. It fails to satisfy the usual axioms of relativistic quantum field theory only in that the transformation of the fields is not a local or point transformation. It is interesting to note that such a violation of the point transformation property occurs in the radiation gauge formulation of quantum electrodynamics.[18]

APPENDIX A

In this Appendix we prove that the operators $\Omega_\pm$ are unitary. Our first task is to establish a formula for the adjoint operators $\Omega_\pm^+$. By definition we have that

$$(g, \Omega_\pm^+ f) = (\Omega_\pm g, f).$$

[14] J. S. Toll, Phys. Rev. **104**, 1760 (1956).
[15] R. Acharya (to be published); thesis, University of Rochester, 1962 (unpublished).
[16] See Ref. 13.
[17] E. C. G. Sudarshan, J. Math. Phys. **4**, 1029 (1963).
[18] E. C. G. Sudarshan, Phys. Rev. **123**, 2183 (1961), Appendix.

Using the inner product (2.9) and the definition (3.1) of $\Omega_\pm$, recognizing that B_{l+} and B_{l-} are complex conjugates of each other, and making some changes of variables and order of integration, we develop the above relation as

$$\int dM \int (1/2)(M^2+\mathbf{K}^2)^{-1/2}d^3K \sum_{lm} g_{lm}(M,K)^*(\Omega_\pm{}^+f)_{lm}(M,\mathbf{K}) = \int dM \int (1/2)(M^2+\mathbf{K}^2)^{-1/2}d^3K \sum_{lm} f_{lm}(M,\mathbf{K})$$
$$\times\left\{g_{lm}(M,\mathbf{K})^* + \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\mp}(M')(M'-M\mp i\epsilon)} g_{lm}(M',\mathbf{K})^*\right\} = \int dM \int (1/2)(M^2+\mathbf{K}^2)^{-1/2}$$
$$\times d^3K \sum_{lm} g_{lm}(M,\mathbf{K})^* \left\{f_{lm}(M,\mathbf{K}) - \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\mp}(M)(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K})\right\},$$

from which we deduce that

$$(\Omega_\pm{}^+f)_{lm}(M,\mathbf{K}) = f_{lm}(M,\mathbf{K}) - \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\mp}(M)(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}). \tag{A1}$$

Next we establish Eq. (3.4). To do this we just use the formulas (3.1) and (A1) to write

$$(\Omega_\pm{}^+\Omega_\pm f)_{lm}(M,\mathbf{K}) = f_{lm}(M,\mathbf{K}) + \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}) - \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}$$
$$\times\frac{G_l(M)G_l(M')}{B_{l\mp}(M)(M'-M\pm i\epsilon)} f_{lm}(M'\mathbf{K}) - \int dM''\left[\frac{M^2+\mathbf{K}^2}{M''^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M'')}{B_{l\mp}(M)(M''-M\pm i\epsilon)}$$
$$\times\int dM'\left[\frac{M''^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M'')G_l(M')}{B_{l\pm}(M')(M'-M''\pm i\epsilon)} f_{lm}(M',\mathbf{K}). \tag{A2}$$

Then we use the identity

$$[(M''-M\pm i\epsilon)(M'-M''\pm i\epsilon)]^{-1} = [(M'-M\pm i\epsilon)(M''-M\pm i\epsilon)]^{-1} + [(M'-M\pm i\epsilon)(M'-M''\pm i\epsilon)]^{-1} \tag{A3}$$

and the definitions (3.2) and (3.3) of $B_{l\pm}$ to develop the double integral term of Eq. (2) as

$$-\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\mp}(M)B_{l\pm}(M')(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}) \times \int dM'' G_l(M'')^2\left[\frac{1}{(M''-M\pm i\epsilon)} - \frac{1}{(M''-M'\mp i\epsilon)}\right]$$
$$= \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\mp}(M)B_{l\pm}(M')(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}) \times [B_{l\pm}(M') - B_{l\mp}(M)].$$

From this we see that the double integral term of Eq. (A2) cancels out the two single integral terms leaving us with

$$(\Omega_\pm{}^+\Omega_\pm f)_{lm}(M,\mathbf{K}) = f_{lm}(M,\mathbf{K}),$$

which is just Eq. (3.4).

Finally we establish Eq. (3.5). We need to use the fact that

$$B_{l+}(M) - B_{l-}(M) = 2\pi i G_l(M)^2, \tag{A4}$$

which follows from the definitions (3.2) and (3.3) of $B_{l\pm}$. We use the formulas (3.1) and (A1) again to write

$$(\Omega_\pm\Omega_\pm{}^+f)_{lm}(M,\mathbf{K}) = f_{lm}(M,\mathbf{K}) + \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}) - \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}$$
$$\frac{G_l(M)G_l(M')}{B_{l\mp}(M)(M'-M\pm i\epsilon)} f_{lm}(M',\mathbf{K}) - \int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)}$$
$$\times\int dM''\left[\frac{M'^2+\mathbf{K}^2}{M''^2+\mathbf{K}^2}\right]^{1/4} \frac{G_l(M')G_l(M'')}{B_{l\mp}(M')(M''-M'\pm i\epsilon)} f_{lm}(M'',\mathbf{K}). \tag{A5}$$

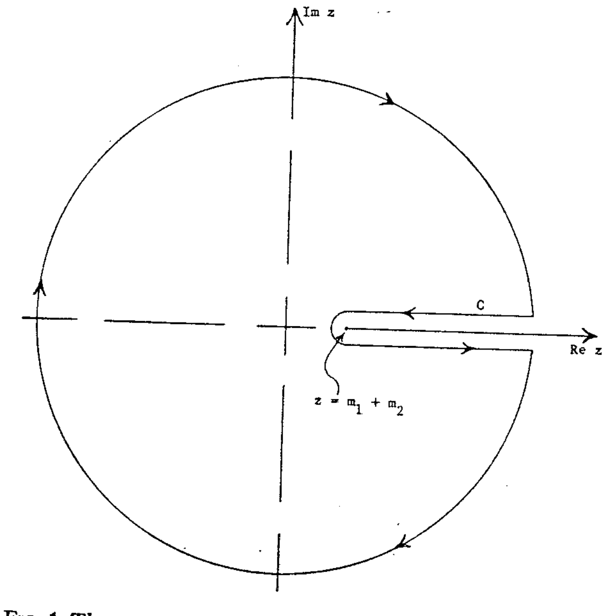

FIG. 1. The contour C for the integrals (A7) and (B3) is closed by adding to it a circle of large radius.

Using Eq. (A4), we develop the double integral term of Eq. (A5) as

$$-\int dM''\left[\frac{M^2+\mathbf{K}^2}{M''^2+\mathbf{K}^2}\right]^{1/4} G_l(M)G_l(M'')f_{lm}(M'',\mathbf{K})I\,, \tag{A6}$$

where

$$I=\int dM'\frac{G_l(M')^2}{B_{l+}(M')B_{l-}(M')(M'-M\pm i\epsilon)(M''-M'\pm i\epsilon)}$$

$$=\frac{1}{2\pi i}\int dM'\frac{1}{(M'-M\pm i\epsilon)(M''-M'\pm i\epsilon)}\left[\frac{1}{B_{l-}(M')}-\frac{1}{B_{l+}(M')}\right].$$

We can write I as the contour integral

$$I=\frac{1}{2\pi i}\int_C dz\frac{1}{B_l(z)(z-M\pm i\epsilon)(M''-z\pm i\epsilon)}\,,$$

where C is the contour shown in Fig. 1 which circumscribes the part of the real axis between m_1+m_2 and infinity. The part of the contour below the real axis gives the term with B_{l-} and the part above the real axis gives the term with B_{l+}. Since $B_l(z)$ approaches 1 as z becomes infinite, the integrand of I is of the order z^{-2} for large z. Hence we can close the contour C by adding to it a circle of large radius (Fig. 1) which contributes nothing to I. Now $B_l(z)$ has no zeros within the region enclosed by C, so the only contributions to I are from the poles at $M\mp i\epsilon$ and $M''\pm i\epsilon$. These give

$$I=[B_{l\pm}(M'')(M''-M\pm i\epsilon)]^{-1}-[B_{l\mp}(M)(M''-M\pm i\epsilon)]^{-1}.$$

Substituting this into the term (A6), we see that the double integral term of Eq. (A5) cancels the single integral terms leaving us with

$$(\Omega^{\pm}\Omega_{\pm}{}^{+}f)_{lm}(M,\mathbf{K})=f_{lm}(M,\mathbf{K})\,,$$

which is just Eq. (3.5).

APPENDIX B

In this Appendix we derive the explicit forms of H, $\mathbf{P}$, $\mathbf{J}$ and establish Eqs. (3.7)–(3.10).

First we find H and prove Eqs. (3.9) and (3.10). Using formulas (2.10), (3.1), and (A1), we find that

$$\begin{aligned}(\Omega_\pm H_0\Omega_\pm^{+}f)_{lm}(M,\mathbf{K})={}&(H_0f)_{lm}(M,\mathbf{K})+\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}\frac{(M'^2+\mathbf{K}^2)^{1/2}G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)}f_{lm}(M',\mathbf{K})\\ &-\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}\frac{(M^2+\mathbf{K}^2)^{1/2}G_l(M)G_l(M')}{B_{l\mp}(M)(M'-M\pm i\epsilon)}f_{lm}(M',\mathbf{K})-\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}\\ &\times\frac{(M'^2+\mathbf{K}^2)^{1/2}G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)}\times\int dM''\left[\frac{M'^2+\mathbf{K}^2}{M''^2+\mathbf{K}^2}\right]^{1/4}\frac{G_l(M')G_l(M'')}{B_{l\mp}(M')(M''-M'\pm i\epsilon)}f_{lm}(M'',\mathbf{K}).\end{aligned}\tag{B1}$$

Using Eq. (A4) and proceeding just as we did with the double integral term of Eq. (A5), we develop the double integral term of Eq. (B1) as

$$-\int dM''\left[\frac{M^2+\mathbf{K}^2}{M''^2+\mathbf{K}^2}\right]^{1/4}G_l(M)G_l(M'')f_{lm}(M'',\mathbf{K})I\,,\tag{B2}$$

where

$$I=\frac{1}{2\pi i}\int_C dz\frac{(z^2+\mathbf{K}^2)^{1/2}}{B_l(z)(z-M\pm i\epsilon)(M''-z\pm i\epsilon)},\tag{B3}$$

with C the part of the contour shown in Fig. 1 which circumscribes the part of the real axis between m_1+m_2 and infinity. As in Appendix A, we close the contour C by adding to it a circle of large radius (Fig. 1). But now the integrand of I approaches $-z^{-1}$ as z becomes infinite. Hence the integral around the large circle does not vanish but contributes a term -1 to I. When substituted into the term (B2), the contributions to I from the poles at $M\mp i\epsilon$ and $M''\pm i\epsilon$ give terms in the double integral of Eq. (B1) which cancel the single integral terms, as in Appendix A. The only other contribution to I comes from the branch cut for the function $(z^2+\mathbf{K}^2)^{1/2}$. Since we can not evaluate this for arbitrary G_l, we simply name it $-F_l(\mathbf{K}^2,M,M'')$. Adding this contribution to the contribution -1 from the large circle, substituting in the double integral term (B2) of Eq. (B1), and remembering that the pole contributions have cancelled the single integral terms, we see that Eq. (B1) becomes identical to Eqs. (3.9) and (3.10) with the definition (3.6) of H.

We could prove Eq. (3.7) by a calculation of the above kind. But it is easier just to see, by inspecting the formulas (2.11) and (3.1), that

$$(\Omega_\pm\mathbf{P}_0f)_{lm}(M,\mathbf{K})=\mathbf{K}(\Omega_\pm f)_{lm}(M,\mathbf{K})=(P_0\Omega_\pm f)_{lm}(M,\mathbf{K})\,.\tag{B4}$$

By a similar inspection of the formulas (2.12), (2.14), and (3.1), we can see that

$$(\Omega_\pm\mathbf{J}_0f)_{lm}(M,\mathbf{K})=(\mathbf{J}_0\Omega_\pm f)_{lm}(M,\mathbf{K})\,.$$

For it is clear that the second term $\mathbf{I}$ of $\mathbf{J}_0$ commutes with $\Omega_\pm$ and the fact that

$$-i\mathbf{K}\times\nabla\mathbf{K}^2=0$$

makes it evident that the first term does also.

APPENDIX C

In this Appendix we prove Eq. (4.2) and (4.6). From the formulas (2.10) and (3.1) we have that

$$\begin{aligned}(e^{iH_0t}\Omega_\pm e^{-iH_0t}f)_{lm}(M,\mathbf{K})={}&f_{lm}(M,\mathbf{K})+\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}e^{it[(M^2+\mathbf{K}^2)^{1/2}-(M'^2+\mathbf{K}^2)^{1/2}]}\\ &\times\frac{G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M\pm i\epsilon)}f_{lm}(M',\mathbf{K})\,.\end{aligned}\tag{C1}$$

We use the identity

$$\frac{1}{M'-M\pm i\epsilon}=\frac{(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}}{M'-M}\times(\mp i)\int_{0,-\infty}^{\infty,0}e^{iy[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]}$$

to write the last term of Eq. (C1) as

$$\mp i\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}\int_{0,-\infty}^{\infty,0} dy\, e^{i(y-t)\,[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]}$$

$$\times\frac{[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M)}f_{lm}(M',\mathbf{K})=\mp i\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}$$

$$\times\int_{-t,-\infty}^{\infty,-t} dx\, e^{ix\,[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]}\times\frac{[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]G_l(M)G_l(M')}{B_{l\pm}(M')(M'-M)}f_{lm}(M',\mathbf{K}). \quad \text{(C2)}$$

In the limit as t approaches $\mp\infty$ this term vanishes, leaving us with Eq. (4.2).

If, for the case of Ω_+, we take the limit as t approaches $+\infty$ of the term (C2), we get

$$-2\pi i\int dM'\left[\frac{M^2+\mathbf{K}^2}{M'^2+\mathbf{K}^2}\right]^{1/4}\delta[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]\times\frac{[(M'^2+\mathbf{K}^2)^{1/2}-(M^2+\mathbf{K}^2)^{1/2}]G_l(M)G_l(M')}{B_{l+}(M')(M'-M)}f_{lm}(M',\mathbf{K})$$

$$=-2\pi i\frac{G_l(M)^2}{B_{l+}(M)}f_{lm}(M,\mathbf{K}),$$

which is equal, by the identity (A4), to

$$\frac{B_{l-}(M)-B_{l+}(M)}{B_{l+}(M)}f_{lm}(M,\mathbf{K})=\frac{B_{l-}(M)}{B_l^{+}(M)}f_{lm}(M,\mathbf{K})-f_{lm}(M,\mathbf{K}).$$

Substituting this for the last term of Eq. (C1), we have that

$$\left(\lim_{t\to\infty}e^{iH_0t}\Omega_+e^{-iH_0t}f\right)_{lm}(M,\mathbf{K})=\frac{B_{l-}(M)}{B_{l+}(M)}f_{lm}(M,\mathbf{K}),$$

which is just Eq. (4.6) with S evaluated according to Eq. (4.5). One can obtain Eq. (4.6) also by using the formulas (3.1) and (A1) with the definition (4.4) of S.