Integral Representations of Two-Point Functions*

S. Deser, *Physics Department, Brandeis University, Waltham, Massachusetts*

AND

W. Gilbert and E. C. G. Sudarshan,† *Lyman Laboratory, Harvard University, Cambridge, Massachusetts*

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An integral representation for a general matrix element of two field operators between eigenstates labelled by an arbitrary number of momenta is presented. This representation is conveniently parametrized over a set of invariants related to the total momentum and momentum transfer, and explicitly incorporates spectral conditions. Physically interesting dispersion relations for off-forward scattering cannot be obtained from this representation alone.

1. INTRODUCTION

IN previous papers[1] we have suggested a simple representation for the vertex function and forward scattering amplitude, consistent with the usual requirements. This representation was of interest due to the particularly transparent form of the parametrization, and to the ease with which the mass restrictions could be incorporated. A natural question arises therefore, as to the possibility of reaching with the same method similar forms in the cases of the off-forward amplitude and the most general two-point function with arbitrary momenta labelling the initial and final eigenstates. Indeed, it is easily shown that a similar representation can be written for the off-forward case, which becomes especially simple in terms of the total momentum vector and a quantity related to the momentum transfer, and involves one more parametric dependence. Similarly, it will be shown that the general case involves only one further space-like vector and corresponding parameter. Thus the kinetical preliminaries and the incorporation of causality requirements involve no particular difficulties, and the introduction of information combined in the mass spectrum can then be carried through. While the existence of three parameters greatly complicates the algebra involved, there is no difficulty of principle in incorporating these restrictions by methods similar to those of I and II. Just as in I and II, the representations so obtained satisfy all the usual requirements but are not dictated by them.

While these representations yield information concerning the analytic behavior of the corresponding generalized Green's functions, their specializations to the energy shell in the case of off-forward scattering for situations of physical interest do not yield analytic information needed to derive dispersion relations. In view of their simple parametric structure and of the information displayed by the representations, they may prove, however, to be useful in studying virtual processes.

[1] Deser, Gilbert, and Sudarshan, Phys. Rev. **115**, 731 (1959), and preceding paper [Phys. Rev. **117**, 266 (1960)]. These will be referred to as I and II, respectively.

2. INTEGRAL REPRESENTATIONS FOR THE MATRIX ELEMENTS OF THE COMMUTATOR

We wish to derive a representation for the momentum state matrix elements of the commutator of two local field operators:

$$h(x,p,p') = \langle p | [j_1(x/2), j_2(-x/2)] | p' \rangle, \quad (1)$$

under the restrictions

(a) $[j_1(x/2), j_2(-x/2)] = 0$ for $x^2 < 0$,

(b) $\langle p | j_1 | n \rangle = 0$, $n^2 < n_1^2$
$\langle p | j_2 | n \rangle = 0$, $n^2 < n_2^2$

and (c) Lorentz invariance.

By the methods of paper I, one can obtain a representation for the Fourier transform, $f(k)$, of $h(x)$. Before this is done, it is convenient to define the time-like state momentum vector P and a spacelike unit vector Q orthogonal to P as follows:

$$\tfrac{1}{2}(p+p') = P, \quad (p-p')^\lambda - (p-p') \cdot P P^\lambda / P^2 = \bar{Q}^\lambda,$$
$$Q^\lambda = \bar{Q}^\lambda / (\bar{Q}^2)^{\frac{1}{2}}. \quad (2)$$

In terms of these, $f(k,P,Q)$ has the representation

$$f(k,P,Q) = \int d\mu \, d\beta \, d\gamma \, H(\mu,\beta,\gamma) \, \epsilon(\beta P^2 + P \cdot k)$$
$$\times \delta(k^2 + 2P \cdot k\beta + 2Q \cdot k\gamma - \mu), \quad (3)$$

where the causality condition is embodied in the lower limit of the μ integration: $\mu \geqslant \mu_0 = -\beta^2 P^2 - \gamma^2 Q^2 = \gamma^2 - \beta^2 P^2$. The integrations over γ and β run out to infinity, and γ is, of course, the new parameter corresponding to the new momentum Q. The weight function $H(\mu,\beta,\gamma)$ depends implicitly on P and Q.

In order to proceed with the imposition of the mass-spectrum conditions in Sec. II, the assumption must be made that the weight function H vanishes sufficiently rapidly for large μ and β. Alternatively, we could have proceeded as in I, and derived this form from a Jost-Lehmann representation by using a modified Dyson identity to rotate one of the parameters from a spacelike to a timelike direction; this would have yielded a weight function nonvanishing only in the region

$$\beta^2 P^2 + \gamma^2 \leqslant P^2; \quad \mu \geqslant \mu_0. \quad (4)$$

This procedure would also have necessitated the additional assumption that the weight function was a Schwartzian distribution.

3. SPECTRAL CONDITIONS

By a decomposition with respect to a complete set of intermediate states, one can write (apart from numerical factors)

$$f(k)=\sum \delta(k+P-n_1)\langle p|j_1|n_1\rangle\langle n_1|j_2|p'\rangle$$
$$-\sum \delta(k-P-n_2)\langle p|j_2|n_2\rangle\langle n_2|j_1|p'\rangle,\quad (5)$$

which implies that

$$f(k)=f(k^2,2P\cdot k,2Q\cdot k)=0$$
$$\text{for}\quad (P+k)^2\leqslant n_1^2,\quad (P-k)^2\leqslant n_2^2,\quad (6)$$

provided one is really in the physical region of the invariants defined by

$$4k^2\leqslant\{(2P\cdot k)^2/P^2\}-(2Q\cdot k)^2.\quad (7)$$

The integral representation contains a δ function which restricts the integration of the weight function $H(\mu,\beta,\gamma)$ to the plane

$$k^2+2P\cdot k\beta+2Q\cdot k\gamma=\mu.\quad (8)$$

The mass conditions assert that in the physical domain of the invariants, i.e., on one side of the saddle

$$4k^2=\{(2P\cdot k)^2/P^2\}-(2Q\cdot k)^2,\quad (9)$$

in the wedge $\Re$ included by the planes

$$2P\cdot k+k^2=n_1^2-P^2,\quad -2P\cdot k+k^2=n_2^2-P^2,\quad (10)$$

(the proper wedge being chosen by including the points for k^2 large and negative), the function vanishes.

To make explicit use of this property, one has to overcome the complication of the $\epsilon(P\cdot k+\beta P^2)$ factor. We already know that $H(\mu,\beta,\gamma)$ vanishes on one side of the saddle.

$$\mu\leqslant\mu_0=\gamma^2-\beta^2P^2,\quad (11)$$

in the (μ,β,γ) space. By an explicit calculation one can show that for all points in the physical region, the expression $\epsilon(P\cdot k+\beta P^2)$ changes sign in the finite segment intercepted by the saddle; and thus may be replaced by $\epsilon(P\cdot k)$ for physical values of k. We thus have the result that $\int d\mu d\beta d\gamma\, H(\mu,\beta,\gamma)\delta(k^2+2P\cdot k\beta +2Q\cdot k\gamma-\mu)$ vanishes for points lying in the physical part of the wedge $\Re(10)$.

Two possible physical situations can arise according as this wedge does not or does have a common domain with the unphysical region of the saddle (9). We shall call these case A and case B, respectively. In case A the physical region where the function is known to vanish is a wedge with no straight line boundaries, truncated by a curved surface, but in case B part of the wedge is not bounded by the saddle and the region does have a straight edge of finite length.

A point $(k^2,2P\cdot k,2Q\cdot k)$ determines a plane in the (μ,β,γ) space and the vanishing of $f(k)$ in the (truncated) wedge corresponds to the integrals along a dense set of planes in the (μ,β,γ) space vanishing identically. Provided the function H is of sufficiently fast decrease, these conditions are adequate to impose support conditions on H. To this we now turn.

4. SUPPORT CONDITIONS

Case A. $n_1+n_2>2P$

In this and the following sections we assume that the weight function $H(\mu,\beta,\gamma)$ is a distribution of sufficiently fast decrease so that the condition for the vanishing of the integrals along a set of planes dense in a certain region $\mathcal{S}'$, in general unbounded, is that the weight function itself should vanish in $\mathcal{S}'$.

We now observe that the mass-spectrum limits discussed in the previous section do indeed provide such a dense set of planes, corresponding to the (dense) set of points of the wedge $\Re$. The domain $\mathcal{S}'$ in the (μ,β,γ) plane is bounded by the envelope of the planes generated by the boundaries of the domain $\Re$. The boundary of $\Re$ consists of the intersections

$$\mathcal{G}_1:\begin{cases} k^2+2P\cdot k=n_1^2-P^2,\\ 4k^2+(2Q\cdot k)^2-(2P\cdot k)^2/P^2=0,\end{cases}$$

and

$$\quad (12)$$

$$\mathcal{G}_2:\begin{cases} k^2-2P\cdot k=n_2^2-P^2,\\ 4k^2+(2Q\cdot k)^2-(2P\cdot k)^2/P^2=0.\end{cases}$$

Notice that the planes generated by points on the plane $k^2+2P\cdot k=n_1^2-P^2$ all pass through the point $\mu=n_1^2-P^2$, $\beta=1$, $\gamma=0$ and similarly all the planes generated by points on $k^2-2P\cdot k=n_2^2-P^2$ pass through $\mu=n_2^2-P^2$, $\beta=-1$, $\gamma=0$. Further any point on the saddle $k^2+(Q\cdot k)^2-(P\cdot k)^2/P^2=0$ generates a plane tangent to the saddle $\mu+\beta^2P^2-\gamma^2=0$. Thus $\mathcal{G}_1$ generates a set of tangent planes to the (μ,β,γ) saddle passing through $(n_1^2-P^2, 1, 0)$ and $\mathcal{G}_2$ generates a set through $(n_2^2-P^2, -1, 0)$. The problem is thus reduced to finding the region $\mathcal{S}'$ swept out by these planes and bounded in part by the envelopes of these two sets of planes.

To find these envelopes, let us first parametrise the set of planes generated by $\mathcal{G}_1$. Let any definite plane of this set touch the saddle $\mu=\gamma^2-\beta^2P^2$ at the point (μ',β',γ'); then these quantities satisfy
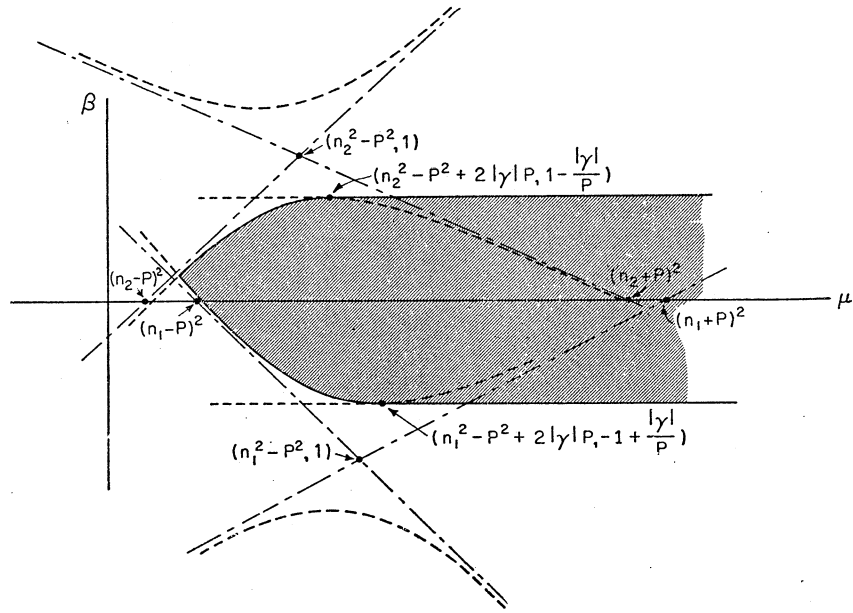
$$\mu'=\gamma'^2-\beta'^2P^2;\quad \mu'=P^2-n_1^2-2\beta'P^2.$$

The latter condition follows since the tangent at the point to the saddle is

$$\tfrac{1}{2}(\mu+\mu')=\gamma\gamma'-\beta\beta'P^2,$$

and any plane of the set generated by $\mathcal{G}_1$ passes through the point $(n_1^2-P^2, 1, 0)$. Consequently, the set may be parametrised by their points of tangency $\mu'=P^2-n_1^2+2P^2\lambda$, $\beta'=\lambda$, $\gamma'=\pm[(1-\lambda)^2P^2-n_1^2]^{\frac{1}{2}}$, so that the

FIG. 1. The support of the weight function $H(\mu,\beta,\gamma)$ in a (μ,β) plane, case A.

planes form the family:

$$\mu=n_1^2-P^2+2P^2(1+\beta)\lambda\pm2\gamma[(1-\lambda)^2P^2-n_1^2]^{\frac{1}{2}}. \quad (13)$$

The envelope of this family is

$$\mu=n_1^2+(1+2\beta)P^2-2n_1[(1+\beta)^2P^2-\gamma^2]^{\frac{1}{2}}; \quad |\beta|\leqslant1. \quad (14)$$

Similarly, the envelope of the family generated by $\mathcal{I}_2$ is

$$\mu=n_2^2+(1-2\beta)P^2-2n_2[(1-\beta)^2P^2-\gamma^2]^{\frac{1}{2}}; \quad |\beta|\leqslant1. \quad (15)$$

These envelopes are segments of hyperboloids with their centers of symmetry at the points $(n_1^2-P^2, 1, 0)$ and $(n_2^2-P^2, -1, 0)$ and are ruled surfaces with the one-parameter family of tangent planes touching them along a line. For $\gamma=0$ either hyperboloid degenerates into a pair of lines; for one of these it is

$$\begin{cases} 2(n_1P-P^2)\beta+\mu-(n_1-P)^2=0, \\ -2(n_1P+P^2)\beta+\mu-(n_2+P)^2=0, \end{cases} \quad (16)$$

which incidentally constitute the asymptotes for the envelope for all values of γ.

We may now define the domain S'. For this purpose notice that the limiting planes which are tangent to the saddle at infinity are tangents to the envelopes for the largest permissible value of μ and these are parallel to the μ axis. Thus the region S, the support of $H(\mu,\beta,\gamma)$ and the complement of S', is bounded by the envelopes and their horizontal tangents:

$$S: \begin{cases} \mu=\max\{n_1^2+(1+2\beta)P^2-2n_1[(1+\beta)^2P^2-\gamma^2]^{\frac{1}{2}} \\ \quad +n_2^2+(1-2\beta)P^2-2n_2[(1-\beta)^2P^2-\gamma^2]^{\frac{1}{2}}\}, \\ \quad -1+|\gamma|/P\leqslant\beta\leqslant1-|\gamma|/P. \end{cases} \quad (17)$$

This region is illustrated in Fig. 1 where S is shaded.

One observes that the cross sections in the (β,γ) plane are bounded for all μ; the maximum allowable

value for $|\gamma|$ is P at which point the two tangent lines coincide. The region in the (μ,β) plane is widest for $\gamma=0$ in which case the hyperbolic boundaries degenerate into their asymptotes, the parallel strip is bounded by the lines $\beta=\pm1$; and we have a configuration familiar from the representation of the forward scattering amplitude of II; we notice also that the mass parameter κ occurring there is here replaced by the quantity P.
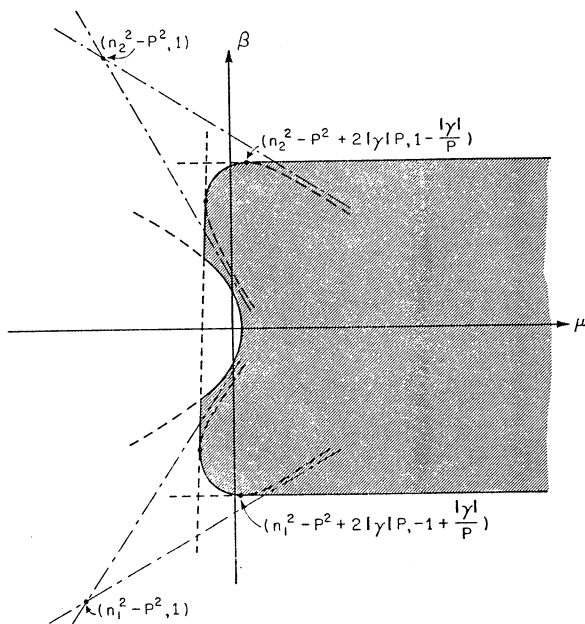
Case B. $n_1+n_2\leqslant2P$

In this case the associated geometric problem is very similar, except for the fact that for very small values of the variable $2Q\cdot k$ the wedge $\mathcal{R}$ does not extend up to the bounding curves $\mathcal{I}_1$ and $\mathcal{I}_2$ but is terminated by the line

$$l: \begin{cases} k^2+2P\cdot k=n_1^2-P^2, \\ k^2-2P\cdot k=n_2^2-P^2, \end{cases} \quad (18)$$

up to a definite value of the variable $2Q\cdot k$. There are two such values (for either sign of $2Q\cdot k$) and at the corresponding points the curves $\mathcal{I}_1$, $\mathcal{I}_2$ and the line l intersect. These points are

$$k^2=\tfrac{1}{2}(n_1^2+n_2^2)-P^2,$$

$$2P\cdot k=\tfrac{1}{2}(n_1^2-n_2^2),$$

$$2Q\cdot k=\pm\left(\frac{[(n_1+n_2)^2-4P^2][(n_1-n_2)^2-4P^2]}{4P^2}\right)^{\frac{1}{2}} \quad (19)$$

$$=\pm2\Gamma.$$

The line l generates a set of planes passing through the two points $(n_1^2-P^2, 1, 0)$ and $(n_2^2-P^2, -1, 0)$. The domain S is bounded by the subsets of planes generated by the segments of the curves $\mathcal{I}_1$, $\mathcal{I}_2$ and of the line

FIG. 2. The support of H for case B.

which form boundaries of $\mathfrak{R}$. The family of planes generated by l is

$$\mu = \tfrac{1}{2}(n_1{}^2 + n_2{}^2) - P^2 + \tfrac{1}{2}(n_1{}^2 - n_2{}^2)\beta - 2\gamma\gamma'. \quad (20)$$

This family has two planes common to the families generated by $\mathcal{G}_1$ and $\mathcal{G}_2$, namely

$$\mu = \tfrac{1}{2}(n_1{}^2 + n_2{}^2) - P^2 + \tfrac{1}{2}(n_1{}^2 - n_2{}^2)\beta \pm 2\Gamma\gamma. \quad (21)$$

The domain $\mathcal{S}$ is, then, bounded in part by these envelopes and these two planes. In order to see precisely how the domain is bounded it is necessary to examine the segment of the envelope actually generated by the allowed tangent planes. For this purpose notice that if the one-parameter family of tangent planes is labeled by the value of γ' for the point of contact of this plane with the saddle, the representative points (lines) along the envelope curve (surface) in the (μ,β) plane $[(\mu,\beta,\gamma)$ space] form a monotonic sequence. For $\gamma' = 0$ these are points at infinity $\beta = \pm\infty$ while for $\gamma' \to \infty$ these approach the points $\beta = \mp(1 - |\gamma|/P)$. Hence in the present case (case B) where $|\gamma'|$ cannot be less than a certain value, it follows that only a finite segment of the asymptote is a true boundary. The exact points can be obtained analytically.

But a more elegant method is to notice that in this case the planes (21) are also tangent to these envelope surfaces and furthermore, they are *common* tangents to these envelopes. It can also be seen that the points (lines) of contact of these tangents are the required limiting points (lines) on the envelopes. Consequently, the domain $\mathcal{S}$ is bounded by this plane, the saddle, the segments of the asymptotes concave towards the μ axis and the tangents to the envelopes at their limiting

points parallel to the μ axis. The typical configuration is illustrated in Fig. 2, where $\mathcal{S}$ is shaded.

For both very small and very large values of γ the line (21) intersects the parabola $\mu = \gamma^2 - \beta^2 P^2$, but for suitable parameters there may exist a range of values of γ for which the line does not intersect the parabola, in which case the parabola is not a boundary of the configuration $\mathcal{S}$ in the (μ,β) plane for these values of γ.

Again, for $\gamma = 0$, the hyperbolas degenerate into their asymptotes and the pattern is similar to the one obtained in the study of the forward-scattering amplitude, with P replacing κ. To obtain the forward-scattering limit, it is necessary to reinstate the unit of length for Q, which has been implicit in the dimensional parameter γ. In this limit, the γ dependence drops out of the δ function in Eq. (3), and the new weight function, $H(\mu,\beta)$, is the integral $\int_{-\infty}^{\infty} d\gamma\, H(\mu,\beta,\gamma')$, where the limits on the dimensionless γ' have become $(-\infty, \infty)$, independent of β. The range of μ and β is just that obtaining in the $\gamma = 0$ plane.

5. GENERALIZED TWO-POINT GREEN'S FUNCTION

From the considerations of the previous section we have obtained a structure for the matrix element of a local commutator between two states labeled only by their four-momenta. This enables us to write down the retarded Green's function for the corresponding cases by the familiar replacement of

$$\delta(k^2 + 2P \cdot k\beta + 2Q \cdot k\gamma - \mu)$$

by the expression

$$\{k^2 + 2P \cdot k\beta + 2Q \cdot k\gamma - \mu + i(P \cdot k + \beta P^2)\epsilon\}^{-1},$$

with the same weight functions $H(\mu,\beta,\gamma)$. In case A we can separate the contributions coming from the two orders of the commutator in the physical region; and write the corresponding time-ordered Green's function by dropping the sign factor in the imaginary infinitesimal part:

$$G_+(k) = \int_{\mathcal{S}} d\mu\, d\beta\, d\gamma \, \frac{H(\mu,\beta,\gamma)}{k^2 + 2P \cdot k\beta + 2Q \cdot k\gamma - \mu + i\epsilon}. \quad (22)$$

In case B we cannot define the time-ordered Green's function for all physical momenta since there is no way of separating out the contributions from the two orderings of the commutator for those values of the invariants in the physical region of overlap. In this case the $i\epsilon$ trick generates a Green's function which coincides with the true time-ordered Green's function over most physical momenta, but is not necessarily identical to it for a restricted range of the real momentum k.

It is to be stressed at this point that since the classification into case A or case B is dependent on $n_1 + n_2 - 2P$, for sufficiently large momentum transfers, one invariably obtains case B since P increases with momentum transfer. In the next section we shall see that the

occurrence of P in deciding the configuration prevents us from exploiting the techniques employed for the forward scattering case to deduce dispersion relations for the off-forward scattering of strongly interacting particles.

In the remainder of this section we shall be concerned with the structure of generalized two-point Green's functions, i.e., the matrix elements of the time-ordered products of two field operators between two arbitrary states. Specifically, we wish to relax the restriction that the states are together characterized by only two momenta. In the general case, let p_i $(i=1, 2, \cdots, N)$ be N arbitrary momenta, which together specify the space-time properties of the various states. (We neglect the complications introduced by spin, which are irrelevant to the main argument.) Then if we put

$$P^\mu = \tfrac{1}{2} \sum_{i=1}^{N} p_i{}^\mu, \qquad (23)$$

the mass-spectrum restrictions assume the same form as before. If we now define any two space-like unit vectors Q_1, Q_2 which are mutually orthogonal and orthogonal to the time-like vector P, we can write the inequality

$$4k^2 \leqslant [(2P\cdot k)^2/P^2] - (2Q_1\cdot k)^2 - (2Q_2\cdot k)^2, \qquad (24)$$

which now represents a saddle-shaped three-dimensional hypersurface. By the same methods as used before, for the function

$$f(k) = \int e^{ik\cdot x} d^4x \, \langle \cdots | [j_1(x/2), \, j_2(-x/2)] | \cdots \rangle,$$

one can write the integral representation

$$f(k) = \int d\mu\, d\beta\, d\gamma_1 d\gamma_2 \, H(\mu,\beta,\gamma_1,\gamma_2) \epsilon(P\cdot k + \beta P^2)$$

$$\times \delta(k^2 + 2P\cdot k\beta + 2Q_1\cdot k\gamma_1 + 2Q_2\cdot k\gamma_2 - \mu). \qquad (25)$$

A little reflection shows that this in fact is the general case whenever we have three or more momenta, since k has only four independent components and the quantities k^2, $2P\cdot k$, $2Q_1\cdot k$, $2Q_2\cdot k$ define it completely as soon as the directions of P, Q_1, Q_2 are known. The only exception to this general case is when there are only one or two independent momenta defined, and these cases have already been dealt with.

If we put

$$\gamma_1 = \gamma \cos\alpha, \quad \gamma_2 = \gamma \sin\alpha,$$

we can rewrite $f(k)$ in the form

$$f(k) = \int d\mu\, d\beta\, d\gamma\, d\alpha \, H(\mu,\beta,\gamma,\alpha) \epsilon(P\cdot k + \beta P^2)$$

$$\times \delta(k^2 + 2P\cdot k\beta + 2[Q_1\cdot k \cos\alpha + Q_2\cdot k \sin\alpha]\gamma - \mu), \qquad (26)$$

where $H(\mu,\beta,\gamma,\alpha)$ is suitably defined. The support of $H(\mu,\beta,\gamma,\alpha)$ subject to the mass-spectrum conditions can be found in a manner identical to that we have used in the last section after showing that the α dependence does not significantly enter into the support. The support of H is given by

$$\mu \geqslant \max\{k^2 + 2P\cdot k\beta + 2Q_1\cdot k\gamma \cos\alpha + 2Q_2\cdot k\gamma \sin\alpha\}, \qquad (27)$$

for each value of β, γ, α. The maximum is taken over all k^2, $P\cdot k$, $Q_1\cdot k$, and $Q_2\cdot k$ lying in the region $\Re$:

$$k^2 - n_2{}^2 + P^2 \leqslant 2P\cdot k \leqslant n_1{}^2 - P^2 - k^2,$$
$$k^2 < [(P\cdot k)^2/P^2] - (Q_1\cdot k)^2 - (Q_2\cdot k)^2. \qquad (28)$$

The orientation of the component of k lying in the (Q_1, Q_2) plane is completely arbitrary; thus if the expression in (27) is maximized over this orientation, it will yield

$$\mu \geqslant \max\{k^2 + 2P\cdot k\beta \pm 2\gamma[(Q_1\cdot k)^2 + (Q_2\cdot k)^2]^{\frac{1}{2}}\},$$

which reduces the problem to the simple 3-parameter (off-forward) case, and incidentally exhibits the symmetry in the sign of γ.

We note that in these representations, as long as P^μ is taken to be the average momentum of the initial and final states, the choice of the other unit spacelike vectors Q_1, Q_2 are completely at our discretion, and the formal structure is not dependent on whether they label the initial or final state. We have one explicit example in the one-momentum case where the commutators associated with the vertex function and with the forward scattering amplitude have, with appropriate choice of variables, Fourier transforms which have integral representations with the same structure. Similarly in the two-momentum case the commutators associated with the meson propagator in one-nucleon states and with the two-particle wave function have the same structure. The intermediate mass thresholds n_1 and n_2 do depend on the details of the problem as does the explicit expression for the matrix element in terms of the momenta p_1 and p_2. These same properties are true of the more general amplitude.

6. DISPERSION RELATIONS

The Green's functions derived before describe the propagation of the field whose current is denoted by $j(x)$. By suitable restriction of this Green's function, one may obtain the propagation characteristics for physically interesting cases. Thus for example, the physical off-forward scattering amplitude can be obtained by restricting the components of k in the three-parameter Green's function such that the initial and final projectile momenta are on the mass shell. This

implies two restrictions.

$$k^2 = P^2 - \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(q_1^2 + q_2^2),$$

$$2Q \cdot k = \frac{[(q_2^2 - q_1^2) + (P \cdot k/P^2)(p_2^2 - p_1^2)]}{[P^2 - \tfrac{1}{2}(p_1^2 + p_2^2) + (p_1^2 - p_2^2)^2/16P^2]^{\frac{1}{2}}}, \quad (29)$$

where p_1^2, p_2^2 are the squares of the initial and final target masses and q_1^2, q_2^2 are the squares of the initial and final projectile masses; no two of these masses need be equal. With these restrictions the Green's function reduces to a scattering amplitude $T(P \cdot k)$ depending only on the "energy" variable $P \cdot k$.

This amplitude T is singular for such values of $P \cdot k$ for which $k^2 + 2P \cdot k\beta + 2Q \cdot k\gamma - \mu$ vanishes, with μ, β, γ lying in the region S. By virtue of the infinitesimal imaginary terms these singularities, which in general form cut lines in the complex plane, are shifted infinitesimally away from the real axis. The displaced cut lines are in the upper or lower half-planes according to the sign of the imaginary terms in the denominator; and the analytic properties of $T(P \cdot k)$ can then be determined once the region S is known.

There is now, in principle, no reason why one should not study the most general case. In practice, however, the algebraic complications make this study quite involved and most of these are kinematical details. The essential points can be seen in the simplest possible case, which is also the most symmetric, where

$$p_1^2 = p_2^2 = \kappa^2; \quad q_1^2 = q_2^2 = m^2; \quad n_1^2 = n_2^2 = n^2.$$

The on-the-mass-shell values of k^2 and $2Q \cdot k$ are

$$k^2 = P^2 - \kappa^2 + m^2; \quad 2Q \cdot k = 0,$$

and the manifold of singularities are associated with the plane

$$\mu = 2P \cdot k\beta + P^2 - \kappa^2 + m^2.$$

Over this plane the quantity $P^2 + (P \cdot k/\beta)$ changes sign along the lines of intersection of the plane with the parabolic cylinder $\mu = P^2(1 - 2\beta^2) - \kappa^2 + m^2$ the function being negative for the finite strip.

Accordingly, let us split $T(P \cdot k)$ into two parts:

$$T(P \cdot k) = T_1(P \cdot k) + T_2(P \cdot k),$$

which are analytic, respectively, in the upper and the lower half-planes and defined by the integrals

$$T_i(P \cdot k) = \int_i \frac{d\mu d\beta d\gamma \, H(\mu, \beta, \gamma)}{2P \cdot k\beta + P^2 - \kappa^2 + m^2 - \mu + i(P^2 - \kappa^2 + m^2 - 2\beta P^2 - \mu)\epsilon},$$

$$i = 1, 2 \quad (30)$$

where S_1 is that part of S in which the inequality $\mu > 2P \cdot k\beta + P^2 - \kappa^2 + m^2$ is satisfied and S_2 is the complement of S_1 with respect to S. For suitable mass conditions S_2 may be null; in this case the scattering ampli-

tude is analytic in the upper half-plane for the complex variable $P \cdot k$ and dispersion relations can be written down immediately.

In general the region S_2 is not null and we have analyticity in neither half-plane. To obtain the precise structure of the cut lines observe that the planes $\mu = 2P \cdot k\beta + P^2 - \kappa^2 + m^2$ form an axial pencil with the line Λ: $\mu = P^2 - \kappa^2 + m^2$, $\beta = 0$ as axis. To every plane of this pencil we can associate a value of the "energy" $P \cdot k$. Two essentially distinct cases arise according as Λ has not (or has) any common points with S; these correspond to the conditions

$$P^2 - \kappa^2 + m^2 < (>)(n - \kappa)^2: \quad \text{case } A,$$
$$P^2 - \kappa^2 + m^2 < (>)n^2 - \kappa^2: \quad \text{case } B. \quad (31)$$

In either case A or B, if Λ does not have any points in S, then the function $T(P \cdot k)$ has cut lines in the upper half-plane only for a finite range of values of $P \cdot k$ and these are in the unphysical region since $P \cdot k$ is proportional to the average energy of the projectile in the initial and final states in the Breit coordinate system. This makes it possible for us to define a continuation of the physical scattering amplitude by changing the sign of $i\epsilon$ in $T_2(P \cdot k)$. By virtue of the reality of the weight function $H(\mu, \beta, \gamma)$ consequent on the charge-conjugation invariance of the theory, this is equivalent to the redefinition

$$\tilde{T}(P \cdot k) = T_1(P \cdot k) + T_2^*(P \cdot k),$$

which leaves the physical scattering amplitude unaltered:

$$\tilde{T}(P \cdot k) = T(P \cdot k) \quad \text{for} \quad (P \cdot k)^2 > P^2 m^2.$$

The function so defined is analytic in the upper half-plane and constitutes the analytic continuation of the physical scattering amplitude to the unphysical region; the dispersion relations for these (redefined) amplitudes follow in the standard fashion.

If on the other hand, Λ has a finite segment lying inside S, there will be cut lines in both half-planes for all energies in the amplitude $T(P \cdot k)$; and these are no simple analytic properties, at least for a general $H(\mu, \beta, \gamma)$. In this case we can obtain no dispersion relations for the physical scattering amplitude.

For the amplitude to have the required analyticity properties it is thus advantageous to have the intermediate mass threshold as high as possible; and needless to say this is true in the general nonsymmetric case. If the state of least mass occurring in any order of the commutator is a single-particle state separated by a finite gap from the beginning of the continuum function, then $f(k)$ vanishes for physical values of the invariants lying between an isolated plane and the continuum. This additional information cannot be expressed as support conditions on $H(\mu, \beta, \gamma)$ but it takes the form of additional integrability conditions enabling us to write a modified Green's function with a smaller

support. This bound state may be removed and re-expressed as in II, and the analytic information of interest may be deduced from the beginning of the continuum of mass values.

When the external particle (the projectile) is strongly interacting, the restriction to the mass shell makes Λ have a finite segment lying inside $\mathcal{S}$ for *all* nonvanishing momentum transfers, since $n \leqslant \kappa + m$. Thus it is not possible to derive dispersion relations for nonforward scattering of physical particles from this representation. When the momentum transfer tends to zero, Λ moves over to the boundary of $\mathcal{S}$ and in the limit of the forward-scattering amplitude we obtain the dispersion relations derived in II.

Here one needs to point out that it is not essential to take both the projectile states to be on the mass shell. One may consider those specializations of the Green's function which correspond to off-the-mass-shell functions in a completely analogous manner and study the analyticity properties of these amplitudes for virtual processes. But since the consideration of these various restrictions of the Green's function adds nothing essentially new in principle we shall not devote any further attention to them. Similar considerations apply to the three-momentum and multi-momentum amplitudes considered in Sec. 5.

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