Inequivalent Quantizations and Fundamentally Perfect Spaces

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(Received 22 June 1987)

We investigate the problem of inequivalent quantizations of a physical system with multiply connected configuration space X. For scalar quantum theory on X we show that state vectors must be single valued if and only if the first homology group $H_1(X)$ is trivial, or equivalently the fundamental group $\pi_1(X)$ is perfect. The θ structure of quantum gauge and gravitational theories is discussed in light of this result.

PACS numbers: 03.65.−w, 02.40.+m, 04.60.+n, 11.15.Kc

In this Letter we consider ordinary scalar, U(1) quantum theory on a general configuration space X. That is, we study quantum systems whose physical state vectors ψ are mappings from a space X into the complex numbers,[1] $\psi: X \rightarrow \mathbb{C}$. It is well known that although the probability densities $|\psi|^2$ must be single valued on X, ψ need not satisfy this requirement. A state vector is, in general, allowed to change by a phase when its argument is transported around a generic loop l in X:

$$\psi(x) \underset{l}{\rightarrow} \alpha(l)\psi(x), \quad \alpha(l) \in \mathrm{U}(1).$$

However, it has been shown by various techniques[2-4] that the mapping α from loops in X into U(1) must be constant on homotopy classes [i.e., $\alpha(l_1)=\alpha(l_2)$ if l_1 can be continuously deformed into l_2]. Therefore we may think of α as a mapping from $\pi_1(X)$, the fundamental (or first homotopy) group of X, into U(1). Moreover, this mapping must be a homomorphism; that is, $\alpha \in \mathrm{Hom}(\pi_1(X),\mathrm{U}(1))$, where $\mathrm{Hom}(A,B)$ is the set of all homomorphisms from the group A to the group B. (This set also forms a group under pointwise definition of the product.)

The linearity of the space of states implies that if one state vector "transforms under loops" according to a given $\alpha \in \mathrm{Hom}(\pi_1(X),\mathrm{U}(1))$, then every other vector from the same Hilbert space must also transform with the *same* α. (This means that there are not transitions between two states which transform differently from each other.) Therefore, for every homomorphism α there will be a distinct Hilbert space of state vectors transforming under loops with that α. In general, each of the above Hilbert spaces will represent a distinct physical system. So, in constructing a scalar, U(1) quantum theory on X (with fixed choice of Hamiltonian operator, boundary conditions, etc.), one has to choose between the different types of allowed multivaluedness of the set of physical state vectors. There are as many such (possibly) inequivalent quantizations as there are elements of $\Omega \equiv \mathrm{Hom}(\pi_1(X),\mathrm{U}(1))$.[5] The existence of this kinematical "quantization ambiguity" has been seen[4] to give rise to various important features of many theories, including the Aharonov-Bohm effect, the different possible statistics for identical particles moving on a given manifold, and the θ structure (or "vacuum angle") of quantum Yang-Mills and gravity theory.

From the above we see that if X is simply connected $[\pi_1(X)=\{e\}]$, than the group Ω is trivial (i.e., contains only one element, namely the trivial homomorphism). This corresponds to the statement that state vectors on simply connected spaces *must* be single valued.[6] The standard intuition is that when X is multiply connected $[\pi_1(X)\neq\{e\}]$, Ω will contain more than one element, so that there will exist a nontrivial set of inequivalent quantizations of the theory. In what follows we will determine the necessary and sufficient conditions for Ω to be trivial and see that this latter statement is incorrect. That is, we will classify those spaces X for which Ω contains only the trivial homomorphism. Finally, the previously mentioned θ structure of quantum gauge and gravitational theories will be reexamined in the light of this result.

The first step is this classification is the realization that since the target group of the homomorphisms in Ω, namely, U(1), is Abelian, each of these maps can be considered as a homomorphism of some Abelian quotient group of $\pi_1(X)$ into U(1). Therefore, we may restrict the domain group of these maps to the *Abelianization*[7] of $\pi_1(X)$, which is isomorphic to $H_1(X)$, the first integral homology group of the space X. That is,[8]

$$\Omega = \mathrm{Hom}(\pi_1(X),\mathrm{U}(1)) \cong \mathrm{Hom}(H_1(X),\mathrm{U}(1)).$$

(This has also been noted by Dowker.[9]) With this characterization of Ω, we can see that if $H_1(X)=\{e\}$, then Ω is trivial. Also, it is straightforward to show that if $H_1(X)$ is nontrivial, then so is Ω. This follows from the fact that $H_1(X)$ is necessarily Abelian and there is *always* a nontrivial homomorphism (as well as the trivial one) from any nontrivial Abelian group into U(1). This is easy to see if $H_1(X)$ is finitely generated (which is always the case if X is a finite-dimensional compact manifold of finite CW complex as well as for the infinite-dimensional or noncompact spaces usually considered in physics), since any finitely generated Abelian group can be written as $Z \oplus \cdots \oplus Z \oplus Z_{d_1} \oplus \cdots \oplus Z_{d_n}$, where Z is the additive group of the integers, and Z_{d_i} is the cyclic group of order d_i.[10] There still exists such a nontrivial homomorphism into U(1) even in the bizarre case when $H_1(X)$ is *not* finitely generated.[11] Thus we have proved the main result of this paper:

Ω is trivial if and only if $H_1(X)=\{e\}$.

We still see, of course, that if X is simply connected, Ω is trivial, since $\pi_1(X)=\{e\}$ implies $H_1(X)=\{e\}$. However, there *do* exist multiply connected spaces X [$\pi_1(X)\neq\{e\}$] which nevertheless have $H_1(X)$ [the Abelianization of $\pi_1(X)$] trivial. So again on such spaces only single-valued state vectors are allowed. A group whose Abelianization is trivial is called perfect in the mathematical literature.[10,12] So an alternative way of stating our main result is

Ω is trivial if and only if $\pi_1(X)$ is perfect.

We will call a topological space whose fundamental group is perfect a *fundamentally perfect space.* Again we note that the class of fundamentally perfect spaces contains as a proper subclass all simply connected spaces, and that scalar, U(1) quantum mechanics on *any* fundamentally perfect space admits only single-valued state vectors, i.e., it has no quantization ambiguity due to possible multivaluedness of state vectors. We wish to accentuate the fact that our main theorem is extremely general. It applies to the construction of quantum theories on both finite- and infinite-dimensional spaces X: from particle mechanics, to field theory, to string theory, and so on.[13] It therefore has possible applications in every branch of quantum physics and should be of wide interest.

As our first "application" of the above result, we consider pure gauge field theory, with structure (internal symmetry) group G, in $(d+1)$-dimensional flat space-time. We first discuss the *classical* configuration space in the $A_0=0$ gauge (the temporal or Weyl gauge). We impose strong (finite energy) boundary conditions on the gauge potentials and gauge transformations, allowing us to replace the space manifold R^d by its one-point compactification S^d, the d sphere. The classical configuration space can then be written as[14] the set of all time-independent gauge potentials (in Weyl gauge) $\mathcal{A}$, modulo the group $\mathcal{G}$ of time-independent gauge transformations; $\mathcal{G}\equiv\{g: S^d\rightarrow G\}$ is the set of all base-point-preserving differentiable maps from S^d into the structure group G. This space, denoted by $\mathcal{A}/\mathcal{G}$, is called the gauge orbit space. We will assume that the state vectors in the quantum theory have as their domain the classical configuration space $\mathcal{A}/\mathcal{G}$. This assumption is at least valid to all orders of the semiclassical approximation.[4]

The object of interest for us is the fundamental group of the gauge orbit space $\pi_1(\mathcal{A}/\mathcal{G})$. We may write the standard result[14]

$$\pi_1(\mathcal{A}/\mathcal{G})\cong\pi_0(\mathcal{G})\cong\pi_d(G),$$

where the first isomorphism follows since $\pi_n(\mathcal{A})=\{e\}$ for all n and since $\mathcal{G}$ acts freely (i.e., without fixed points) on $\mathcal{A}$. The second isomorphism is a consequence of the definition of $\mathcal{G}$ and the nth homotopy group π_n. So the set Ω labeling the inequivalent quantizations is

$$\Omega=\mathrm{Hom}(\pi_1(\mathcal{A}/\mathcal{G}),\mathrm{U}(1))\cong\mathrm{Hom}(\pi_d(G),\mathrm{U}(1)).$$

If, as an example, we take ordinary (3+1)-dimensional gauge theory with a simple, non-Abelian Lie group G [for which $\pi_3(G)\cong Z$], we find $\Omega=\mathrm{Hom}(Z,\mathrm{U}(1))\cong\mathrm{U}(1)$. Therefore, the set of inequivalent quantizations is labeled by a continuous parameter, the so-called "vacuum angle" θ. In this example $\pi_1(\mathcal{A}/\mathcal{G})\cong Z$, which is Abelian, and thus $H_1(\mathcal{A}/\mathcal{G})\cong\pi_1(\mathcal{A}/\mathcal{G})\cong Z$ [$\pi_1(\mathcal{A}/\mathcal{G})$ is *not* perfect] leading to a nontrivial set Ω. We would like to know whether it is possible to find a gauge theory such that the orbit space $\mathcal{A}/\mathcal{G}$ is multiply connected, yet $H_1(\mathcal{A}/\mathcal{G})=\{e\}$, thereby leading to a trivial Ω (no θ structure) even though $\pi_1(\mathcal{A}/\mathcal{G})$ is nontrivial. Equivalently we ask, can the gauge orbit space be fundamentally perfect but not simply connected [$\pi_1(\mathcal{A}/\mathcal{G})$ perfect *and* nontrivial]? [An obvious example of a *simply connected* $\mathcal{A}/\mathcal{G}$ is QED in 3+1 dimensions where $G=\mathrm{U}(1)$, $\pi_3(G)=\{e\}$, and Ω is trivial.]

The answer to this question is negative. This is easy to see for $d\geq 2$ since $\pi_1(\mathcal{A}/\mathcal{G})\cong\pi_d(G)$, and π_d is always Abelian for $d\geq 2$,[15] while if $\pi_1(\mathcal{A}/\mathcal{G})$ were perfect and nontrivial it would have been *non-Abelian.* The remaining case $d=1$ is taken care of by the use of another theorem from algebraic topology[15] which states that the fundamental group of a *topological group* is always Abelian. So we have seen that for pure gauge theories with space manifold S^d, $\pi_1(\mathcal{A}/\mathcal{G})$ is always Abelian and thus $\pi_1(\mathcal{A}/\mathcal{G})\cong H_1(\mathcal{A}/\mathcal{G})$ regardless of the structure group G and spatial dimensionality d. Therefore the usual intuition that multiple connectedness of $\mathcal{A}/\mathcal{G}$ implies nontrivial Ω has proven correct here despite our result that in general this is not true. Allowing the space manifold to have a more nontrivial compact topology does not seem to change this negative result.[16]

Interestingly, the situation in gravitational theories is different. Let us assume the space-time manifold M to have the structure $M=\Sigma\times\mathbb{R}$, where Σ is a compact three-manifold with fixed topology. The configuration space relevant to semiclassical canonical quantum gravity theory is[14] the set of all Riemannian metrics on Σ, RiemΣ, modulo the group $\mathrm{Diff}_F(\Sigma)$ of diffeomorphisms of Σ which leave a point in Σ and a frame at that point fixed. We denote this orbit space by $\mathrm{Riem}(\Sigma)/\mathrm{Diff}_F(\Sigma)\equiv R/D$ and wish to find its fundamental group. As with gauge theories we may write[14]

$$\pi_1(R/D)\cong\pi_0(\mathrm{Diff}_F(\Sigma)).$$

Therefore $\Omega\cong\mathrm{Hom}[\pi_0(\mathrm{Diff}_F(\Sigma)),\mathrm{U}(1)]$ which when nontrivial leads to the gravitational analog of the θ structure in gauge theories.

Now $\mathrm{Diff}_F(\Sigma)$ is an infinite-dimensional Lie group, and the calculation of $\pi_0(\mathrm{Diff}_F(\Sigma))$ is in general very difficult. However, in some recent work,[17] this zeroth homotopy group has been computed for three-manifolds of the form $\Sigma = S^2/H$, where H is a finite group acting freely on S^3. Interestingly, for three-manifolds of this type there is exactly one which has $\pi_0(\mathrm{Diff}_F(\Sigma))$ perfect and nontrivial. This is the space $\Sigma = S^3/I^*$, for which $\pi_0(\mathrm{Diff}_F(\Sigma)) \cong I^*$ (I^* is the binary icosahedral group[12,18]). Therefore, scalar, U(1) quantum gravity on R/D for this Σ will only allow single-valued state vectors even though R/D is multiply connected [$\pi_1(R/D) \simeq I^*$]. Other space manifold topologies for which this is true may also exist. The above example shows the danger of trusting the standard folklore on the existence of θ structure.

In closing, it is important to note that the above results may change if the state vectors are mappings from the space X into some Hermitean vector space other than $\mathbb{C}$. For instance, if we have n-component complex state vectors, $\psi: X \to \mathbb{C}^n$, then there is an extension of the above U(1) quantization ambiguity, the analog of Ω being $\mathrm{Hom}(\pi_1(X), \mathrm{U}(n))$.[19] Now, although for $n=1$ this set is trivial if and only if $\pi_1(X)$ is perfect, this may not be so for $n>1$ since such groups will generally possess more-dimensional nontrivial unitary representations.

We are very thankful to Chandni Shah Imbo, Cecile DeWitt-Morette, Steve Carlip, Gary Hamrick, and Guiseppe Marmo for much help with mathematical and physical questions. It is also a pleasure to thank Duane A. Dicus, Joseph Polchinski, A. P. Balachandran, and Greg Nagao for useful discussions and interesting comments. One of us (T.I.) would like to thank the Graduate School of the University of Texas at Austin for support through a fellowship. This work was supported in part by the U.S. Department of Energy under Contract No. DE-FG05-85ER40200.

[1]The results in this paper are strictly valid only for topological spaces X which possess universal covering spaces. The restriction to such X is not severe and basically serves to eliminate physically unreasonable spaces. See M. G. G. Laidlaw and C. M. DeWitt, Phys. Rev. D **3**, 1375 (1971); M. G. G. Laidlaw, Ph.D. thesis, University of North Carolina, 1971 (unpublished).

[2]Laidlaw and DeWitt, in Ref. 1.

[3]J. S. Dowker, J. Phys. A **5**, 936 (1972).

[4]See C. J. Isham, in *Relativity, Groups and Topology II*, edited by B. S. DeWitt and R. Stora (Elsevier, New York, 1984), and references therein.

[5]$\mathrm{Hom}(G,\mathrm{U}(1))$, for any group G, is also just the set of one-dimensional unitary representations of G. It is often called the character group of G.

[6]This much has been known for a long time. See, for instance, E. Merzbacher, Am. J. Phys. **30**, 237 (1962).

[7]The Abelianization of a group G is the group $G/[G,G]$ where $[G,G]$ is the commutator subgroup of G. It is the maximal Abelian quotient group of G and has as quotient groups of itself all other Abelian quotient groups of G. Clearly, an Abelian group is its own Abelianization.

[8]$\mathrm{Hom}(H_1(X),\mathrm{U}(1))$ is also called the first cohomology group of X with values in U(1) and denoted by $H^1(X,\mathrm{U}(1))$. Therefore $\Omega \cong H^1(X,\mathrm{U}(1))$.

[9]J. S. Dowker, J. Phys. A **18**, 3521 (1985) and **19**, L415 (1986).

[10]See, for example, D. J. S. Robinson, *A Course in the Theory of Groups* (Springer-Verlag, New York, 1982).

[11]The proof of this proceeds as follows. The group U(1) is injective (divisible) which means that given any two Abelian groups A and B, a *one-to-one* homomorphism α from A to B, and a homomorphism β from A into U(1), then there exists a homomorphism γ from B into U(1) such that $\beta = \gamma \circ \alpha$ (see Ref. 10). Now, let B be our nonfinitely generated $H_1(X)$, A a nontrivial finitely generated subgroup of B (which always exists), α the natural injection of A into B, and β any nontrivial homomorphism from A into U(1) (which always exists since A is finitely generated Abelian; see text). By the above there exists a homomorphism γ from B into U(1) such that $\beta = \gamma \circ \alpha$. Now, α and β are both nontrivial, and therefore γ must be nontrivial. This completes the proof. We thank Gary Hamrick for showing us this.

[12]Here are some miscellaneous facts about perfect groups. All non-Abelian simple groups are perfect. The smallest nontrivial perfect group is A_5, the alternating group on five elements, whose order is 60. The smallest nonsimple perfect group is I^*, the binary icosahedral group, whose order is 120. It is a central extension of Z_2 by A_5.

[13]It is important to note, however, that when one is quantizing a system which does not possess a global Langrangean description (such as systems with Wess-Zumino–type terms in the action), special care must be taken in constructing the space X. See F. Zaccaria, E. C. G. Sudarshan, J. S. Nilsson, N. Mukunda, G. Marmo, and A. P. Balachandran, Phys. Rev. D **27**, 2327 (1983); A. P. Balachandran, H. Gomm, and R. D. Sorkin, Nucl. Phys. **B281**, 573 (1987); A. P. Balachandran, Nucl. Phys. **B271**, 227 (1986).

[14]See, for example, C. J. Isham, in *Quantum Structure of Space and Time,* edited by M. J. Duff and C. J. Isham (Cambridge Univ. Press, New York, 1982).

[15]M. J. Greenberg and J. R. Harper, *Algebraic Topology* (Benjamin-Cummings, Reading, MA, 1981).

[16]For some of the mathematics relevant to seeing this, consult, C. J. Isham, in *Old and New Questions in Physics, Cosmology, Philosophy and Theoretical Biology,* edited by A. Van der Merwe (Plenum, New York, 1983).

[17]D. M. Witt, J. Math. Phys. **27**, 573 (1986).

[18]It is interesting that the space manifold S^3/I^*, called the Poincaré three-sphere, is *itself* fundamentally perfect. That is $\pi_1(S^3/I^*) \cong I^*$ which is perfect. We also note that in Ref. 17 the result $\pi_0(\mathrm{Diff}_F(S^3/I^*)) \cong I^*$ is proved under the assumption of a weak form of the so-called Hatcher conjecture.

[19]For some work relevant to these considerations, see R. Sorkin, in *Topological Properties and Global Structure of Space-Time,* edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1986); F. J. Bloore, I. Bratley, and J. M. Selig, J. Phys. A **16**, 729 (1983); Balachandran, in Ref. 13, and Syracuse University Report No. SU-4428-361, 1987 (unpublished); Dowker, Ref. 3.