PHYSICAL REVIEW A VOLUME 46, NUMBER 1 1 JULY 1992

ARTICLES

Quantum dynamics, metastable states, and contractive semigroups

E. C. G. Sudarshan
Department of Physics, University of Texas, Austin, Texas 78712
and Center for Particle Theory, University of Texas, Austin, Texas 78712
(Received 19 September 1991)

The generalized states of a quantum system obtained by analytic continuation of a dense subset of density operators and their time evolutions are studied. The time evolution of metastable states is computed using the generalized states. Certain model theories are taken for explicit study. For factorizable Liouville operators the asymptotic states are shown to give line shapes. Nonfactorizable Liouville operators and their generators are investigated. The generic cases are shown to be contractions of suitable isometries of enveloping systems.

PACS number(s): 03.65.Ca, 03.65.Bz, 05.70.Ln, 03.65.Db

I. INTRODUCTION: THE TWO ROLES OF TIME

Time enters physics in two different ways: once as "duration" and once as "history." Mechanics and nonequilibrium thermodynamics are the two disciplines where time manifests itself predominantly in these two aspects, respectively.

For thermodynamics in equilibrium Gibbs has introduced the generalized notion of a dynamical state as a phase-space ensemble. Thus, proceeding from mechanics to equilibrium statistical thermodynamics, the states of the "same system" are enlarged. Without such a generalization for a classical statistical system there would not be a state that is invariant under time evolution. For a quantum system with nondegenerate energy levels such a situation does not obtain.

In complex mechanical systems even with a well-defined Hamiltonian, the usual method of integration of the equations of motion fails: as demonstrated by Poincaré [1], this obtains by virtue of multiple resonances and hence vanishing "energy denominators." These nonintegrable "large Poincaré systems" can be integrated [2] if we analytically continue the class of states into a larger class of states that are no longer points in phase space but are fuzzy patches in phase space with dynamical evolution being viewed as a passage from phase-space fuzzy sets to other fuzzy sets. These are expected to yield contraction semigroups [3], except for integrable systems where the time evolution is unitary and objective.

We shall restrict our attention to quantum systems and their time evolution. We want to explore the conditions under which the time symmetry of evolution is broken and metastable states decay. Given a (time-reversal-invariant) Hamiltonian evolution, any metastable state can manifest irreversible decay, yet the irreversibility is "time-reversal invariant." In the course of this investigation we will see how the line shape of an unstable system is obtained, and the class of generalized analytic states [4] that correspond to complex energies is identified.

To get a unique equilibrium (asymptotic) state we need a dynamical evolution that allows the redistribution of energy and corresponds to a nonfactorizable Liouville operator. These dynamical maps [5] may be studied in the context of the generalized states.

II. GROUP OF TIME EVOLUTIONS ON STATE VECTORS AND DENSITY OPERATORS

Consider a generic quantum dynamical system S whose states constitute a Hilbert space $\mathcal{H}$ with vectors ψ. Let the time evolution of S be governed by a Hamiltonian H, which is a self-adjoint operator bounded from below. Without loss of generality we may take

$$H \geq 0 . \tag{1}$$

Consequently, the resolution of the identity associated with H may be written in the form of Stieltje's integrals [6]

$$\mathbb{1}=\int_0^\infty dE_\lambda \ , \quad H=\int_0^\infty \lambda\, dE_\lambda \ , \tag{2}$$

where E_λ are monotonically increasing nested projections. Discrete point eigenvalues of the Hamiltonian may also be included in Stieltje's integrals. We take the generic system to have a continuous spectrum. We may include ideal eigenvectors (without a finite norm), denoted by $|\lambda\rangle$, obeying

$$H|\lambda\rangle=\lambda|\lambda\rangle \ , \quad \langle\lambda|\mu\rangle=\delta(\lambda-\mu) . \tag{3}$$

Then we could write

$$dE_\lambda = d\lambda|\lambda\rangle\langle\lambda| \tag{4}$$

so that

$$\mathbb{1}=\int_0^\infty d\lambda \int d\alpha |\lambda,\alpha\rangle\langle\lambda,\alpha| \ , \qquad H=\int_0^\infty d\lambda\, \lambda \int d\alpha|\lambda,\alpha\rangle\langle\lambda,\alpha| \ , \tag{5}$$

where α is a (possibly continuous) degeneracy index that is summed over.

The time evolution of a generic normalized state given by $|\psi\rangle$ is

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \int_0^\infty d\lambda\, e^{-i\lambda t}\int d\alpha\, \psi(\lambda,\alpha)|\lambda,\alpha\rangle \ , \quad (6)$$

$$\int_0^\infty d\lambda \int d\alpha |\psi(\lambda,\alpha)|^2 = 1 \ ,$$

with

$$\psi(\lambda,\alpha) = \langle\lambda,\alpha|\psi\rangle \ . \quad (7)$$

We note that $|\psi(t)\rangle$ is the boundary value of an analytic function by Titchmarsh's theorem [7] and *cannot vanish* for any t. The time evolutions form a group that is unitarily realized

$$\| |\psi(t)\rangle \|^2 = \|\psi(0)\|^2 = 1 \ . \quad (8)$$

The Hamiltonian evolution is integrable into the unitary time-development operator

$$U(t) = e^{-iHt} \ . \quad (9)$$

Quantum states are rays in the Hilbert space and we may therefore consider the density operator

$$\rho = \sum_\beta |\psi_\beta\rangle\langle\psi_\beta| C_\beta \ , \quad \mathrm{tr}\rho = 1 \ , \quad C_\beta \geq 0: \ \sum_\beta C_\beta = 1 \quad (10)$$

as the representative of the state. For a Hamiltonian system the time evolution is by a factorizable superoperator

$$\rho(t) = e^{i\mathcal{L}t}[\rho(0)] = e^{-iHt}\rho e^{+iHt} \ . \quad (11)$$

For

$$|\psi_\beta\rangle = \int d\lambda \int d\alpha\, \psi_\beta(\lambda,\alpha)|\lambda,\alpha\rangle \quad (12)$$

we have

$$\rho(t) = \sum_\beta C_\beta \int\int\int\int \psi_\beta(\lambda_1,\alpha)\psi_\beta^*(\lambda_2,\alpha') \times e^{-i(\lambda_1-\lambda_2)t} d\lambda_1 d\lambda_2 d\alpha\, d\alpha' \ . \quad (13)$$

Thus the spectrum of the Liouville superoperator $\mathcal{L}$ is the Ritz spectrum $\lambda_1 - \lambda_2$, which stretches from $-\infty$ to ∞ and is infinitely degenerate. The zero eigenvalue corresponds to all the diagonal elements $\lambda_1 = \lambda_2$ and these elements are invariant under time evolution. There are therefore infinitely many fixed points of time evolution: any λ diagonal density operator is invariant for a Hamiltonian evolution. This circumstance is independent of the spectrum of the Hamiltonian including it being bounded from below.

It is convenient to rewrite the time evolution (13) in a slightly different but equivalent form: we choose as labels

$$\nu = \lambda_1 - \lambda_2 \ , \quad \lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2) \ . \quad (14)$$

In terms of these

$$\rho(t) = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu\, e^{-i\nu t} \sum_\beta \mathcal{C}_\beta \int d\alpha_1 \int d\alpha_2 \psi_\beta \left| \lambda + \frac{\nu}{2} \right| \psi_\beta \left| \lambda - \frac{\mu}{2} \right| \equiv \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu\, e^{-i\lambda t}\rho(\lambda,\mu) \ . \quad (15)$$

Thus the Liouville operator spectrum that is degenerate is dependent on the mean energy λ and for each λ it is bounded both above and below:

$$-2\lambda < \nu < 2\lambda \ , \quad 0 < \lambda < \infty \ . \quad (16)$$

For each λ,ν we may have further degeneracies labeled by α_1, α_2 (see Fig. 1).

III. GENERALIZED STATES AND SURVIVAL PROBABILITY

So far we have considered the generic state $|\psi\rangle$ to be a superposition of the ideal energy eigenvector $|\lambda\rangle$ and an L^2 weight $\psi(\lambda,\alpha)$. No special wave functions $\psi(\lambda,\alpha)$ were considered. If the state is constructed as the solution to some specific eigenvalue problem there could be a definite structure for $\psi(\lambda,\alpha)$. For the Friedrichs-Lee model [8] in the lowest sector

$$H = \begin{bmatrix} m & f(\omega) \\ f^*(\omega') & \omega\delta(\omega-\omega') \end{bmatrix} \ , \quad \phi_\lambda = \begin{bmatrix} \eta_\lambda \\ \phi_\lambda(\omega) \end{bmatrix} \quad (17)$$

the vector

$$\psi_1 = \begin{bmatrix} a \\ 0 \end{bmatrix} \quad (18)$$

has the specific form [6]

$$\psi_1 = \int_0^\infty d\lambda \frac{f(\lambda)}{D(\lambda+i\epsilon)}\phi_\lambda \ , \quad (19)$$

where the denominator function D is defined by the formula

$$D(z) = z - m - \int \frac{|f(\omega)|^2 d\omega}{z-\omega} \ . \quad (20)$$

This gives a specific function ψ and a corresponding density operator ρ_1, whose time evolution one can study. When $D(z)$ has a zero near enough to the real axis (but not on it) we have a spectral concentration in the neigh-

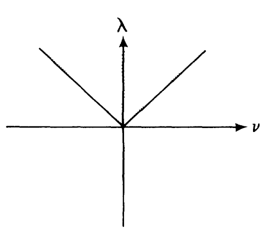

FIG. 1. Spectrum of the Liouville generator for a Hamiltonian system.

borhood governed by the behavior of $D(z)$ and thus a resonance.

The survival probability of the state ρ_1 is defined by

$$P(t)=\mathrm{tr}[\rho_1(t)\rho_1]=\mathrm{tr}(e^{-Ht}\rho_1 e^{iHt}\rho_1) . \tag{21}$$

Clearly

$$P(0)=1 , \quad 1\geq P(t)\geq 0 . \tag{22}$$

The survival probability for small values of t may be expressed in the form

$$P(t)=P(0)+tP'(0)+O(t^2) , \tag{23}$$

provided $P(t)$ is analytic at $t=0$. But if $P'(0)$ exists it is given by

$$P'(0)=-i\,\mathrm{tr}(H\rho_1^2)+i\,\mathrm{tr}(\rho_1 H\rho_1)=0 . \tag{24}$$

So the leading term in t is $O(t^2)$ and the decay rate equal to the negative derivative of $P(t)$ vanishes at t for small t (the Zeno effect [9]). The survival probability $P(t)$ is the square of the absolute value of the survival amplitude

$$A(t)=\psi_1^\dagger e^{-iHt}\psi_1=\int_0^\infty d\lambda\, e^{-i\lambda t}|\psi_1(\lambda)|^2 . \tag{25}$$

For the choice

$$\psi_1(\lambda)=\frac{f(\lambda)}{D(\lambda+i\epsilon)} \tag{26}$$

this can be rewritten

$$\begin{aligned} A(t)&=\int_0^\lambda d\lambda\, e^{i\lambda t}\frac{|f(\lambda)|^2}{D(\lambda+i\epsilon)D(\lambda-i\epsilon)} \\ &=\frac{1}{2\pi i}\int_C dz\frac{e^{izt}}{D(z)} . \end{aligned} \tag{27}$$

By deforming the contour we can reexpress this in terms of a resonance contribution and a background term [10]

$$\begin{aligned} A(t)&=A_{\mathrm{res}}(t)+A_{\mathrm{bgd}}(t) , \\ A_{\mathrm{res}}(t)&=[D'(z_1)]^{-1}e^{-iz_1 t} , \quad D(z_1)=0 , \\ A_{\mathrm{bgd}}(t)&=\frac{1}{2\pi i}\int_{C'}\frac{dz e^{-izt}}{D(z)} , \end{aligned} \tag{28}$$

where C' is a deformation of the contour C (see Fig. 2). Since z_1 has a negative imaginary part, $A_{\mathrm{res}}(t)$ exponentially decreases.

However, this behavior of the survival amplitude is time symmetric: the deformed contour C' is appropriate for $t>0$ since e^{-izt} becomes $e^{-t|z|}$ along it. For $t<0$ we need a contour C'', which is the mirror image of C' with a zero $z_2=z_1^*$ of $D(z)$ in the upper half plane. The resonance contribution would now decrease exponentially with respect to $|t|$.

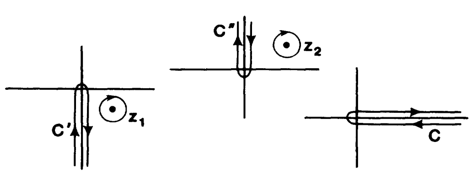

FIG. 2. Contours for computing survival amplitude.

IV. FACTORIZABLE LIOUVILLE DYNAMICS AND THE ANALYTIC CONTINUATION OF VECTORS

The deformation of contours to compute the resonant and background contributions to the survival amplitude and survival probability shows the utility of the construction of generalized states of a quantum system. This would be in the nature of passing from the Hilbert space $\mathcal{H}$ into a set of inner product spaces $\mathcal{J}$ with correspondence between dense sets in $\mathcal{H}$ and $\mathcal{J}$, which respect linearity and inner products. Similarly, we can make correspondence between suitable operators defined in $\mathcal{H}$ with operators defined on the $\mathcal{J}$. The tool for making such a correspondence is analytic continuation.

The notion of analytic continuation of a Hilbert space $\mathcal{H}$ into a family of generalized spaces $\mathcal{J}$ has been described systematically by Sudarshan, Chiu, and Gorini [4] and by Parravicini, Gorini, and Sudarshan [11] (see also M. Nakanishi [12]). It is convenient to restrict ourselves to a system in which the energy states are not degenerate, as for example in the Lee-Friedrichs model in the lowest sector. (For the degenerate case, see Chiu and Sudarshan [13].) Let $|\psi\rangle$ be a state in $\mathcal{H}$:

$$\begin{aligned} |\psi(t)\rangle&=\int_0^\infty d\lambda\,\psi(\lambda)|\lambda\rangle e^{-\lambda t} , \\ \langle\psi|\psi\rangle&=\int_0^\infty d\lambda\,\psi^*(\lambda)\psi(\lambda) . \end{aligned} \tag{29}$$

The set of vectors for which $\psi(\lambda)$ is the boundary value of an analytic function in the lower half plane in the vicinity of the real axis constitutes a dense subset. For such states we can define the generalized ideal state $|z\rangle$ in $\mathcal{J}$ with the property that

$$H|z\rangle=z|z\rangle , \quad \langle z|z'\rangle=\delta(z-z') \tag{30}$$

with $\delta(z-z')$ a delta function along the contour C', generalizing $\delta(\lambda-\lambda')$ along the contour C.

The new spectrum is continuous from 0 to ∞ along the contour C'. If $\psi(z)$ is the analytic continuation of $\psi(\lambda)$ into the lower half plane then the *physical state* $|\psi\rangle$ has the alternate representatives

$$|\psi\rangle=\int_C d\lambda\,\psi(\lambda)|\lambda\rangle e^{-i\lambda t} \tag{31}$$

and

$$|\psi\rangle=\int_{C'} dz\,\psi(z)|z\rangle e^{-izt} \tag{32}$$

in $\mathcal{H}$ and $\mathcal{J}$, respectively (see Fig. 3). There is nothing

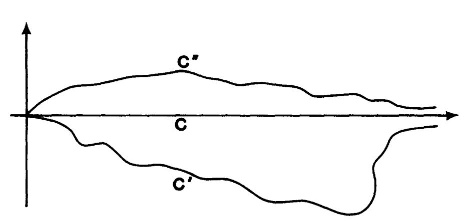

FIG. 3. Spectra in $\mathcal{H}$ and $\mathcal{J}$.

special about the lower half plane, per se: we could have continued the spectrum into the upper half plane provided we do not encounter any obstruction to the analytic continuation. Usually we do, since the spectrum has end points.

Not all operators in $\mathcal{H}$ will take an analytic vector into an analytic vector. However, if they do, such operators also can be defined by their action in $\mathcal{I}$ such that the correspondences are preserved. The Lee-Friedrichs Hamiltonian would have this property if the form factors $f(\omega)$ are analytic in ω. Clearly a generic analytic operator would have some singularity in the complex plane. At this point the analytic continuation should include small contours encircling any isolated singularities and new contours for any branch points that are encountered in the course of analytic continuation. Since these singularities contain dynamical information, this method of analytic continuation would highlight them.

When we deal with a resonant system with a spectral concentration, the time dependence of the survival amplitude is computed best by an analytic continuation into the lower half plane since the time dependence of the background integral along the negative imaginary axis (or any line making a substantial excursion into the lower half plane) is exponentially damped.

A more detailed evaluation of the survival amplitude [10] shows that there are three main regimes: the Zeno regime for small t, the Khalfin regime [14] for very large time where the time dependence is by an inverse power, and a long exponential domain where the resonant contribution dominates. Between the exponential domain and the Khalfin domain there may be further structure, like oscillations [15].

It is instructive to verify that with an analytic Hamiltonian the ideal energy eigenstates are orthonormal and complete in model theories for which we have explicit solutions. For the Lee-Friedrichs model in the lowest sector the complete set of states are [6]

$$\phi_\lambda = \begin{bmatrix} \eta_\lambda \\ \phi_\lambda(\omega) \end{bmatrix} ,$$
$$\eta_\lambda = \frac{f(\lambda)}{D(\lambda+i\epsilon)} , \tag{33}$$
$$\phi_\lambda(\omega) = \frac{f(\lambda)f(\omega)}{(\lambda-\omega+i\epsilon)D(\lambda+i\epsilon)} + \delta(\lambda-\omega) .$$

The orthonormality is easily verified; the completeness

$$\int_C d\lambda\, \phi_\lambda \phi_l^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & \delta(\omega-\omega') \end{bmatrix} = \int_{C'} dz\, \phi_z \phi_z^* \tag{34}$$

(where ϕ_z^* is the left ideal eigenvector of h with the eigenvalue z in $\mathcal{I}$) can be verified by direct integration. When there is a resonance pole at z_1 the contour C' is to be augmented by a loop integral around z_1. It is the sum of this discrete state together with the continuum that makes up the completeness integral.

To speak of the poles of the S matrix (or, equally, of the poles of the scattering amplitude) is not strictly correct since we could have "redundant poles" of the S matrix [16] that do not correspond to a discrete state entering the completeness relation and that are not in the spectrum of the Hamiltonian in $\mathcal{I}$. Such a situation obtains if the form factor (squared) has poles [17]: they will give rise to poles of the scattering amplitude

$$T(z) = \frac{f^2(z)}{D(z)} \tag{35}$$

but not zeros of $D(z)$. On the other hand, there are exceptional cases in which a complex discrete eigenvalue may obtain with a normalized state entering the completeness relation but for which the scattering amplitude is finite (or possibly vanishes). This will happen when at z_1 for which $D(z)$ vanishes, $f^2(z_1)$ also vanishes. This is a generalization to $\mathcal{I}$ of the possibility of a discrete state buried in the continuum already known in $\mathcal{H}$ [18].

Even though in this simple model we get only one discrete pole (in the lower or upper half plane), depending on the choice between the contours C' or C'' in Fig. 2 there are more elaborate models like the Lee-Friedrichs model in the higher sectors or the cascade model [19], where one can have complex branch cuts or multiple poles or both.

V. GENERALIZED DENSITY OPERATORS

We want to study the time evolution of the density operators $\rho(t)$ and the method of analytic continuation. The obvious method is to generalize

$$\rho = |\psi\rangle\langle\psi^*| \tag{36}$$

and use the analytic continuation for the right and left states ψ, ψ^* in $\mathcal{I}$. This method is adequate for limited exploration like the computation of the survival probability or the spectral line shape. A more natural method is based on the recognition that the time evolution of the density operator $\rho(\lambda,\nu)$ depends only on the Ritz frequency ν and is independent of λ. Therefore we need to analytically continue only ν, keeping λ real. The analytic continuation of the Liouville operator is no longer factorizable [20].

We start from the integral representation for the time evolution of the density operator

$$\rho(t) = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu\, e^{-i\nu t} \int d\alpha\, \rho(\lambda,\nu;\alpha) . \tag{37}$$

For density operators analytic in ν we may define the density operator in $\mathcal{I}$ by

$$\rho(t) = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} dz\, e^{izt} \int d\alpha\, \rho(\lambda,z,\alpha) , \tag{38}$$

where the integration with respect to z is along a complex open contour from -2λ to $+2\lambda$ (see Fig. 4).

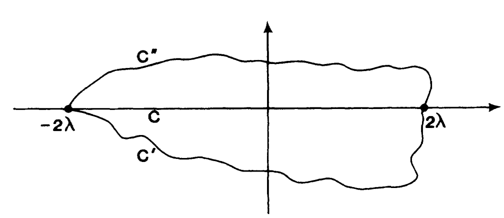

FIG. 4. Contours for analytic density operators.

The spectrum of the Ritz frequencies is infinitely degenerate, once each labeled by the energy λ and possibly other degeneracies indexed by α. The integration is along a sequence of compact arcs from -2λ to $+2\lambda$ with $0<\lambda<\infty$. Thus, though the frequency spectrum ν extends over the entire real axis, the various energy components are restricted in their frequency spectrum.

For a Hamiltonian system, any analytic density operator has a simple temporal evolution as we saw before:

$$\rho(t)=\int_0^{\infty} d\lambda \int_0^{\infty} d\nu\, \rho(\lambda,\nu) e^{-i\nu t} . \tag{39}$$

As $t\rightarrow\infty$, the components with $\nu\neq 0$ all tend to vanish, leading to the asymptotic limit

$$\lim_{t\rightarrow\infty}\rho(t)=\int_0^{\infty} d\lambda\, \rho(\lambda,0) . \tag{40}$$

The state has the *spectral line shape* $\rho(\lambda,0)$. For an initial state that is the projection to a state

$$|m\rangle=\int_0^{\infty} d\lambda \langle\lambda|m\rangle|\lambda\rangle , \tag{41}$$

the line shape is simply

$$\rho(\lambda,0)=\langle\lambda|m\rangle\langle m|\lambda\rangle=|\langle\lambda|m\rangle|^2 . \tag{42}$$

It is the Fourier transform of this line-shape function that gives us the survival *amplitude* [9,10].

The leading corrections to the asymptotic limit are obtained from the kinematic singularities in the ν integration coming from phase-space factors. For motion in three dimensions the phase-space factor is

$$k^2 dk \rightarrow \tfrac{1}{2}\sqrt{\lambda}\, d\lambda \tag{43}$$

so that

$$f(\lambda)=f_1(\lambda)\lambda^{1/4} \tag{44}$$

with $f_1(\lambda)$ a smooth nonvanishing function of λ as $\lambda\rightarrow 0$. Consequently, the density operator $\rho(\lambda,\nu)$ has end-point singularities in the ν integration of the form

$$(\lambda+\tfrac{1}{2}\nu)^{1/4}(\lambda-\tfrac{1}{2}\nu)^{1/4}=(\lambda^2-\tfrac{1}{4}\nu^2)^{1/4} . \tag{45}$$

Hence

$$\left[\int d\nu\, \rho(\lambda,\nu)e^{-i\nu t}-\rho(\lambda,0)\right] =C[\rho(\lambda,2\lambda)+\rho(\lambda,-2\lambda)]t^{-3/2} , \tag{46}$$

where

$$C=\int_0 dx\, e^{it}x^{1/2} \tag{47}$$

is a numerical constant. Note that for the energy-separated components the density operator component decreases at $t^{-3/2}$ in contrast to the survival probability, which decreases as t^{-3}. For $\nu\sim\lambda\ll 1$ we get an integrated term proportional to t^{-3}.

For the next to the leading corrections, which could be exponential or damped oscillatory, we need more dynamical information. Therefore, it would be useful to study some solved models and consider their analytic continuations.

For the Lee-Friedrichs model the decaying state is given by the vector [6]

$$|m\rangle=\int_C d\lambda\, e^{-i\lambda t}\frac{f(x)}{D(\lambda+i\epsilon)} \tag{48}$$

and hence its density operator is

$$\rho(\lambda,\nu)=\int_0^{\infty} d\lambda \int_{-2\lambda}^{2\lambda} d\nu\, e^{-i\nu t} \times\frac{f(\lambda+\frac{1}{2}\nu)f(\lambda-\frac{1}{2}\nu)}{D(\lambda+\frac{1}{2}\nu+i\epsilon)D(\lambda-\frac{1}{2}\nu+i\epsilon)} . \tag{49}$$

The domain of ν integration is given in Fig. 1. The contour of integration can be deformed by moving into the complex plane. The analytically continued integrand has denominator singularities (poles) at

$$z_{1-}=2(\lambda-z_1) ,\quad z_{2+}=-2(\lambda-z_2) , \tag{50}$$

which are symmetrically placed with respect to the imaginary axis. This is for continuation to the lower half plane; if it were into the upper half plane, we would have poles at

$$z_{1+}=2(\lambda-z_2),\quad z_{2-}=-2(\lambda-z_1) . \tag{51}$$

The contours of integration and the pole positions are illustrated in Figs. 5(a) and 5(b). The contour C_1 has not yet encountered any singularities coming from the zeroes of the denominator, but the resonance contribution and the background contribution are not separated. For times that are sufficiently small (Zeno regime) the poles of the denominator are "far away" and the contribution is obtained by

$$\int_0^{\lambda}\frac{f^2(\lambda)}{D^2(\lambda+i\epsilon)}\frac{\sin 2\lambda t}{\lambda t}=\int_0^{\lambda} d\lambda \frac{2f^2(\lambda)}{\lambda D^2(\lambda+i\epsilon)} \times\left[1-\frac{(2\lambda t)^2}{3!}+\cdots\right] . \tag{52}$$

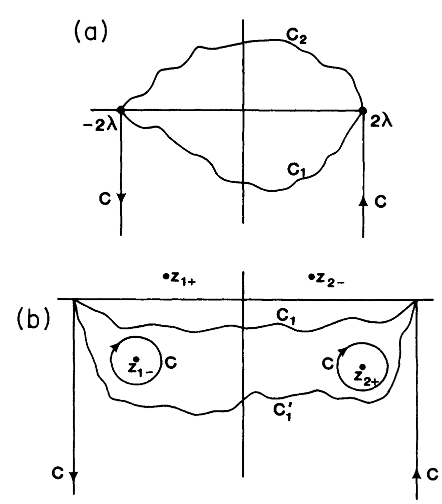

FIG. 5. (a) Integration contours for $\lambda\ll \mathrm{Re}z_1$; (b) integration contours for $\lambda\gg \mathrm{Re}z_2$.

For the contour C the background integral is exponentially damped and contributes the typical $t^{-3/2}$ dependence for the energy-labeled asymptotic dependence. For small values of $\lambda \ll \mathrm{Re} z_1$, the poles $z_{1\pm}$ fall outside the contour C and may be ignored; no resonant contributions are obtained [Fig. 5(a)]. For $\lambda \gg \mathrm{Re} z_1$ the poles $z_{1\pm}$ are encircled by the contour C and they give the exponential contributions

$$\rho(\lambda, z_{1-}) e^{-iz_{1-}t} + \rho(\lambda, z_{1+}) e^{-iz_{1+}t} = \rho(\lambda, 2\lambda - z_2) e^{-(2\lambda - z_2)it} + \rho(\lambda, z_1 - 2\lambda) \times e^{+(2\lambda - z_1)it} . \quad (53)$$

Since z_1 and z_2, respectively, have negative and positive imaginary parts, both terms have exponentially damped oscillations.

VI. CASCADE MODEL GENERALIZED STATES

A richer analytic structure obtains in the cascade model [19]. This model consists of three "particles" A, B, and C with bare energies M_0, μ_0, and 0, respectively, and fields Θ and Φ with quanta θ, and ϕ labeled by energies $0 < \omega, \nu < \infty$. The Hamiltonian is given by

$$H = H_0 + H_{\mathrm{int}} ,$$

$$H_0 = M_0 A^\dagger A + \mu_0 B^\dagger B + \int d\omega\, \omega \theta^\dagger(\omega)\theta(\omega) + \int d\nu\, \nu \phi^\dagger(\nu)\phi(\nu) , \quad (54)$$

$$H_{\mathrm{int}} = \int d\omega\, f(\omega)[A^\dagger B \theta(\omega) + B^\dagger A \theta^\dagger(\omega)] + \int d\nu\, g(\nu)[B^\dagger C \varphi(\nu) + C^\dagger B \varphi^\dagger(\nu)] .$$

In the lowest sector of interest in the present context, we consider the transition

$$A \rightleftarrows B\theta \rightleftarrows C\theta\varphi . \quad (55)$$

If we denote the amplitudes of these true channels by η, $\phi(\omega)$ and $\psi(\omega,\nu)$ the equations of motion can be written

$$(\lambda - M_0)\eta_{\lambda,n} = \int f(\omega')\phi(\omega')d\omega' ,$$
$$(\lambda - \mu_0 - \omega)\phi_\lambda(\omega) = \eta_{\lambda,n} + \int g(\nu')\psi_{\lambda,n}(\omega,\nu')d\nu' , \quad (56)$$
$$(\lambda - \omega - \nu)\phi_{\lambda,n}(\omega,\nu) = \phi_{\lambda,n}(\omega) g(\nu) .$$

For the scattering state in which there is an incoming wave with a φ particle of energy $m > 0$ and a total energy $\lambda > n$ we can solve these equations to obtain

$$\eta_{\lambda n} = \frac{f(\lambda - n)g(n)}{\alpha(\lambda + i\epsilon)\gamma(n + i\epsilon)} \quad (57)$$

$$\phi_{\lambda n}(\omega) = \frac{g(n)\delta(\lambda - \omega - \mu)}{\gamma(n + i\epsilon)} + \frac{f(\omega)}{\gamma(\lambda - \omega + i\epsilon)} \eta_{\lambda n} ,$$
$$\psi_{\lambda n}(\omega,\nu) = \delta(\nu - n)\delta(\lambda - \omega - \nu) + \frac{g(\nu)}{\lambda - \omega - \nu + i\epsilon} \phi_{\lambda n}(\omega) . \quad (58)$$

The functions $\alpha(z)$ and $\gamma(z)$ are defined by

$$\alpha(z) = z - M_0 - \int \frac{f^2(\omega')d\omega'}{\gamma(z - \omega')} ,$$
$$\gamma(z) = z - \mu_0 - \int \frac{g^2(\nu')d\nu'}{\nu' - z} . \quad (59)$$

There may or may not be points M and μ satisfying the equations $\alpha(M) = 0$ and $\gamma(\mu) = 0$. For our present case we will take them not to exist. Then both A and B particles are unstable. The decay of the B particle is just like the Lee-Friedrichs model discussed earlier. There is a complex pole when the amplitude of B is written in terms of the scattering states $C\varphi$. Now we consider A to be unstable as well as B. [In the solution the δ function term in $\phi_{\lambda n}(\omega)$ would be missing.] Then the only wave function we need consider for the unstable particle A is

$$\eta_{\lambda n} = \frac{f(\lambda - n)g(n)}{\alpha(\lambda + i\epsilon)\gamma(n + i\epsilon)} \quad (60)$$

so that

$$|A\rangle = \int_0^\infty d\lambda \int_0^\lambda dn \frac{f(\lambda - n)g(n)}{\alpha(\lambda + i\epsilon)\gamma(n + i\epsilon)} |\lambda n\rangle . \quad (61)$$

The survival amplitude $A(t)$ is given by

$$A(t) = \langle A | e^{-iHt} | A \rangle = \int_0^\infty d\lambda \int_0^\lambda dn \frac{f^2(\lambda - n) g^2(ln) e^{-i\lambda t}}{\alpha^\dagger(\lambda + i\epsilon)\alpha(\lambda + i\epsilon)\gamma^\dagger(n + i\epsilon)\gamma(n + i\epsilon)} . \quad (62)$$

But

$$g^2(n) = \frac{1}{2\pi i}[\gamma(n + i\epsilon) - \gamma^\dagger(n + i\epsilon)] \quad (63)$$

so that

$$A(t) = -\frac{1}{2\pi i} \int_C dz \frac{e^{-izt}}{\alpha(z)} . \quad (64)$$

This is very much the expression that we found for the Friedrichs-Lee model. The entire calculation involved only the space $\mathcal{H}$.

The analytic continuation reveals new structure. The zero of $\alpha(z)$ in the lower half plane will contribute the familiar damped exponential to the survival amplitude. But when the contour is further extended we encounter a new branch point at the complex zero of $\gamma(z)$ corresponding to the existence of a new threshold at this point. There is now a contour integral around this branch cut. So the spectrum of states in $\mathcal{S}$ contains the three-particle branch (which is infinitely degenerate) beginning at 0 and going to ∞, a complex branch from the complex zero μ of $\gamma(z)$ going to infinity, and a discrete complex energy state M corresponding to the complex zero of $\alpha(z)$ [Fig. 6(a)]. Thus the analytic structure of the amplitude can get contributions not only from poles but also from branch cuts.

When the B particle becomes stable, μ creeps up onto the real axis (and below 0). In this case in $\mathcal{H}$ we have two sets of branch cuts: a nondegenerate cut from $\mu<0$ to ∞ and an infinitely degenerate cut from 0 to ∞. The corresponding energy spectra in $\mathcal{H}$ and $\mathcal{J}$ are illustrated in Fig. 6(b). Using the kind of models studied in Ref. [13], we can generate models that have multiple poles and branch cuts in $\mathcal{J}$.

The analytic continuation of factorizable Liouville evolution leads to a well-defined asymptotic state that gives the special line shape of an unstable particle. But if we started from other initial states we will end up again with energy diagonal density matrices

$$\lim_{t\to\infty}\rho(\lambda,\nu,t)=\rho(\lambda,0)\ . \qquad (65)$$

There are therefore infinitely many asymptotic states, each appropriate to a class of initial states. If $\rho(\lambda,\nu;0)$ was a pure state projection operator for any finite t, the same property obtains but the asymptotic state is no longer a projection. Such a behavior is expected for the approach to thermal equilibrium but then for a very general class of initial states we get a definite one-parameter family of density operators of the form

$$\rho(\lambda,\nu;\infty)=Z_{(\beta)}^{-1}e^{-\beta\lambda}\delta(\nu)\ , \qquad (66)$$

where β is the inverse temperature. But such a situation involves the redistribution of energy in the spectrum. We have no mechanism for this energy redistribution as long as the Liouville operator is factorizable. This problem cannot be overcome with analytically continued factorizable Liouville operators. We must go beyond this kind of Liouville operator. The generic Liouville operator would not be factorizable and given a suitable structure we would anticipate that it would lead to an asymptotic thermal distribution. This is the method of dynamical maps [5] and dynamical semigroups [3]. But before I give the general theory, I will outline an attempt to get a semigroup and isolate the effect of resonances. Such a method, if satisfactory otherwise, would also make an unstable particle (or excitation) an autonomous entity [20] with no memory of its past; its decay would then be a quantum Markovian process described by a precise contractive semigroup.

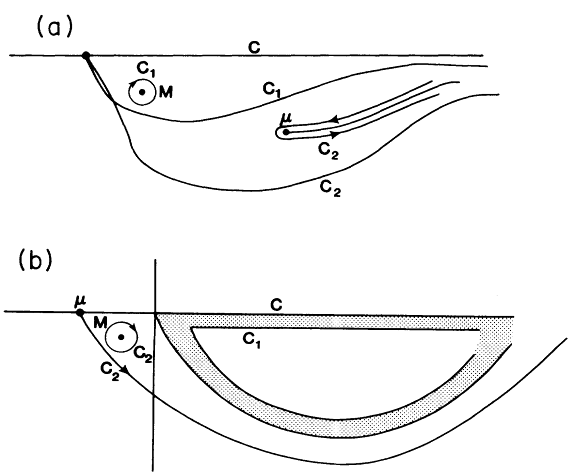

FIG. 6. (a) Spectra in $\mathcal{J}$ for the cascade model with A, B unstable; (b) spectra in $\mathcal{H}$ and $\mathcal{J}$ of the cascade model, B stable.

VII. ANALYTIC DENSITY MATRICES AND QUANTUM SEMIGROUPS

Can we arrive at a quantum semigroup for a factorizable Liouville operator? In the context of nuclear α decay and atomic radiative deexcitation, Gamow [21] and Dirac [22], respectively, developed innovative approximation techniques that led to pure exponential decay. Breit and Wigner [23] formalized these approximation methods into a phenomenological scheme with an unstable state being assigned a complex energy $E_0-i\Gamma/2$. The density operator in this case exponentially shrinks;

$$P(t)=\mathrm{tr}[\rho(t)\rho(0)]=e^{-\Gamma t} \qquad (67)$$

is then strictly exponential without any Zeno regime [9] or Khalfin regime [14]. Further, for $t<0$ this formula is inapplicable, though the Breit-Wigner formalism has a second pole at $E_0+i\Gamma/2$ that leads to

$$P(t)=e^{+\Gamma t}\ . \qquad (68)$$

A contractive semigroup of time evolution can be obtained by taking a class of unphysical states whose spectral amplitude $\psi(\lambda)$, $-\infty<\lambda<\infty$ is the boundary value of an analytic function in the upper half plane and isolated singularities only in the lower half plane. Such an analytic density matrix is obtained by choosing

$$R(\lambda,z)=\frac{1}{2\pi i}\int_{-2\lambda}^{2\lambda}d\nu\frac{1}{\nu-z-i\epsilon}\sigma(\lambda,\nu)\ , \qquad (69)$$

where $\sigma(\lambda,\nu)$ is any density matrix. The limitation for real z

$$-2\lambda<\nu<2\lambda\ , \qquad (70)$$

stemming from the non-negativity of the Hamiltonian, is now relaxed. The Khalfin theorem [14] on survival amplitude no longer obtains. In fact, outside the region the $i\epsilon$ may be omitted and we have

$$R(\lambda,\nu)=\frac{1}{2\pi i}\int d\nu'\frac{1}{\nu'-\nu}\sigma(\lambda,\nu')\ . \qquad (71)$$

For this "density matrix" R the time dependence is

$$R(t)\equiv\int d\lambda\int d\nu\, e^{-i\nu t}R(\lambda,\nu)$$
$$=\begin{cases}0\ , & t<0\\ R_B(t)+\sum_n R_n(t)\ , & t>0\ ,\end{cases} \qquad (72)$$

where the summation is over the discrete poles of $R(\lambda,z)$, and $R_n(t)$ as defined below:

$$R_n(t)=\int d\lambda[R(\lambda,2(\lambda-z_n^*))e^{-2(\lambda-z_n^*)it}$$
$$+R(\lambda,2(z_n-2\lambda))e^{-2(z_n-\lambda)it}] , \tag{73}$$

$$z_n=E_n-\frac{i}{2}\Gamma_n ,$$

characteristic of a decaying state. The expression is further complicated by the presence of a background contribution $R_B(t)$, which is very small, varies as t^{-3}, and is relevant only for $t \gg (1/\Gamma_n)$. In this form all information about the trace of R in $\mathcal{H}$ is lost. There are no diagonal elements in $R_n(t)$ so computed. On the other hand, by virtue of the half-plane analyticity of $R(\lambda,z)$,

$$R(t)=\int d\lambda \int d\nu\, e^{+i\nu|t|}R(\lambda,\nu)=0 , \quad t<0 . \tag{74}$$

Therefore we should consider the state $R(\lambda,\nu)$ as being "created" at time 0. Note that tr$R(t)$ is discontinuous at $t=0$ and tr$R(t)$ decreases with increasing (positive) t.

One could ask whether I could deal with analytic vectors (in place of analytic density operators) and project out the half-plane analytic vectors [24]

$$\Psi_\pm(z)=\frac{1}{2\pi i}\int_0^\infty d\lambda \frac{1}{\lambda-z\pm i\epsilon}\psi(\lambda) , \tag{75}$$

which, by construction, provide functions analytic in half planes. The unitary time evolution on $\psi(\lambda)$ induces two isometric semigroups of evolution on $\Psi_\pm(z)$

$$\Psi_+(z;t)=\frac{1}{2\pi i}\int d\lambda\, e^{-i\lambda t}\frac{1}{\lambda-z+i\epsilon}\sigma(\lambda)$$
$$=T_+(t)\Psi_+(z) , \tag{76}$$

where the time evolutions $T_+(t)$ constitute a contractive semigroup of isometries

$$T_+(t_1)T_+(t_2)\Psi_+(z)=T_+(t_1+t_2)\Psi_+(z) , \quad t_1,t_2>0 ,$$
$$T_+(t)=0 , \quad t<0 , \tag{77}$$
$$T_+(0+)=\mathbb{1} .$$

By the converse of Titchmarsh's theorem [7], the Fourier transform of $\Psi_+(z)$ has support on the positive real line:

$$\tilde{\Psi}_+(\tau)\equiv\int_{-\infty}^{\infty}\Psi_+(\lambda)e^{-i\lambda\tau}d\lambda=0 , \quad \tau<0 . \tag{78}$$

On $\tilde{\Psi}_+(\tau)$ the time evolution semigroup is represented by the contractive semigroup

$$T_+(t)\tilde{\Psi}_+(\tau)=\tilde{\Psi}_T(t+\tau) , \quad t>0 . \tag{79}$$

$T_+(t)$ is not generally defined for $t<0$, though each $\tilde{\Psi}_+(\tau)$ has an "age" τ_0 such that

$$T_+(t)\tilde{\Psi}_+(\tau)=0 , \quad t<-\tau_0 . \tag{80}$$

For $t<0$ we can define the subset of states $\Psi_-(\tau)$ whose Fourier transforms have support over the negative real axis of τ

$$\tilde{\Psi}_-(\tau)\equiv\int_{-\infty}^{\infty}\Psi_-(\lambda)e^{-i\lambda t}d\lambda=0 , \quad \tau>0 , \tag{81}$$

on which the negative time semigroups are

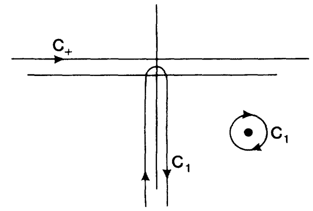

FIG. 7. Contours to compute $A_+(t)$; $t>0$.

$$T_-(t)\tilde{\Psi}_-(\tau)=\Psi_-(\tau+t) , \quad t<0 . \tag{82}$$

From the defining formulas for $\Psi_\pm(\lambda)$ we have the identity

$$\Psi_+(\lambda)+\Psi_-(\lambda)=\psi(\lambda) , \tag{83}$$

so that $\Psi_\pm(\lambda)$ are the projections of $\psi(\lambda)$ into the two classes of functions. While $\psi(\lambda)$ obeys the spectral condition

$$\psi(\lambda)=0 , \quad \lambda<0 , \tag{84}$$

this is true of neither $\psi_+(\lambda)$ nor $\psi_-(\lambda)$. The states $\Psi_\pm(\lambda)$ are therefore not in $\mathcal{H}$ but in a larger space. The analytic continued $\Psi_\pm(x)$ are therefore *not in one-to-one correspondence with the dense subset of analytic vectors* in $\mathcal{H}$.

The survival amplitude of these analytic vectors can be computed by taking their scalar product with their dual at $t=0$. We obtain in this fashion the survival amplitude

$$A_+(t)=\langle\Psi_+^*(0)|\Psi_+(t)\rangle=\frac{1}{2\pi i}\int_{C_+}dz\frac{e^{-izt}}{D(z)} . \tag{85}$$

For $t>0$ (see Fig. 7), we can complete the contour to the lower half plane and do the integration along C_1 to get

$$A_+(t)=A_{\text{res}}(t)+A_{\text{bgd}}(t) . \tag{86}$$

For the case of $t<0$, $A_+(t)$ is not always defined independent of the age τ_0 of the state Ψ_+: if $t<-\tau$, $\Psi_+(t)=0$, but if $0>t>-\tau$ then $\Psi_+(t)$ is defined and the survival amplitude is the continuation of (76). The absolute magnitude of this amplitude is larger than unity since the produced particle at $t=-\tau_0$ had been steadily decaying from then until $t=0$.

VIII. EXPONENTIALLY DECAYING STATES

It is possible to get a pure exponential law for any model? Naturally we have to seek vectors in $\mathcal{J}$. This can be done if I start with a Hamiltonian that is not positive definite in $\mathcal{H}$ and for which the threshold form factors are mutilated. Consider a generalized Lee-Friedrich model for which the denominator function is

$$D(\lambda_\pm)=\lambda-E\pm i\frac{\Gamma}{2}=\frac{1}{2\pi i}\int\frac{\Gamma}{(E-E_0)^2+\frac{1}{4}\Gamma^2}\frac{dE}{E-\lambda\mp i\epsilon} . \tag{87}$$

Such a model can be constructed. The survival ampli-

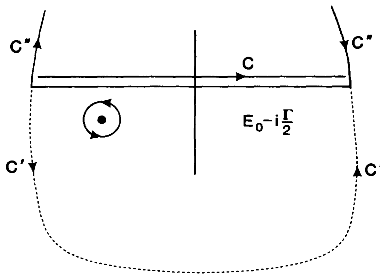

FIG. 8. The upper half-plane function realizing ψ_+ and its survival amplitude.

tude for this model is

$$A_+(t) = \langle \psi^*_{(0)} | \psi(t) \rangle = e^{-iE_0 t - 1/2\Gamma t}, \quad t > 0 . \tag{88}$$

For a ψ_+ with zero age, $A_+(t)$ vanishes for negative t since we can close the contour in the upper half plane (Fig. 8). When the age τ_0 is nonzero, the lower contour C' converges for $t > -\tau_0$ and the upper contour C'' converges for $t < -\tau_0$. So

$$A_+(t) = 0 , \quad t < -\tau_0 . \tag{89}$$

This is valid for all ages.

Since the integrand defining ψ_+ does not have a gap $-\infty < E < \infty$ it follows that the open contour integral

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \frac{\Gamma}{(E-E)^2 + \frac{1}{4}\Gamma^2} \frac{dE}{E-z} \tag{90}$$

defines a piecewise analytic function in the complex plane, analytic in the upper and lower half planes. In the upper half plane it is $z - E_0 + (i\Gamma/2)$ while in the lower half plane it is $z - E_0 - (i\Gamma/2)$ (Fig. 9). So for large negative times ($t < -\tau_0$), while we cannot compute $A_+(t)$ we can define $A_-(t)$ to be

$$A_-(t) = e^{-iEt - (1/2)\Gamma(t)} , \quad t < -\tau_0 . \tag{91}$$

IX. COMPLETELY POSITIVE DYNAMICAL MAPS

Now we return to dynamical maps [5]. A (linear) dynamical map assigns to every density ρ another density operator:

$$\rho \to \mathcal{A}(\rho) ,$$
$$\cos^2\theta \rho_1 + \sin^2\theta \rho_2 \to \cos^2\theta \mathcal{A}(\rho_1) + \sin^2\theta \mathcal{A}(\rho_2) , \tag{92}$$

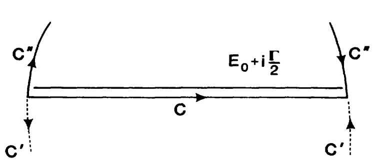

FIG. 9. The lower half-plane function realizing ψ_- and its survival amplitude.

$$\begin{aligned} &\rho \geq 0 \to \mathcal{A}(\rho) \geq 0 , \\ &\mathrm{tr}\rho = 1 \to \mathrm{tr}\mathcal{A}(\rho) = 1 , \\ &\rho = \rho^\dagger \to \mathcal{A}(\rho) = [\mathcal{A}(\rho)]^\dagger . \end{aligned} \tag{93}$$

A generic method of generating such a map is

$$\mathcal{A}(\rho) = \sum_\alpha B(\alpha) \rho B^\dagger(\alpha)$$
$$= \sum_\alpha B^\dagger(\alpha) B(\alpha) = \mathbb{1} . \tag{94}$$

Such maps are called "completely positive" [25]. Not all maps are completely positive. The simplest counterexample is

$$\rho \to \rho^T . \tag{95}$$

Another generic method is the following. Consider the tensor product of $\mathcal{H}$ and another Hilbert space $\mathcal{H}'$. Let us call this tensor product space $\mathcal{K}$. From the density operator ρ in $\mathcal{H}$ and a *fixed* density operator σ in $\mathcal{H}'$ we construct their tensor product $\rho \times \sigma$ in $\mathcal{K}$. Let V be any isometry in $\mathcal{K}$ (if $\mathcal{K}$ is finite dimensional, V could be unitary)

$$V^\dagger V = 1 . \tag{96}$$

Clearly (Tr is trace in $\mathcal{K}$),

$$\mathrm{Tr}[V(\rho \times \sigma)V^\dagger] = \mathrm{Tr}(\rho \times \sigma V^\dagger V)$$
$$= \mathrm{Tr}(\rho \times \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho)\mathrm{tr}_{\mathcal{H}'}(\sigma) = \mathrm{tr}_{\mathcal{H}}(\rho) . \tag{97}$$

So if we define

$$\mathcal{A}(\rho) = \mathrm{tr}_{\mathcal{H}'}(V\rho \times \sigma V^\dagger) , \tag{98}$$

then all the conditions for a dynamical map are met [5,25].

These two generic methods of realizing dynamical maps are equivalent. If the fixed density matrix is diagonalized and the nonzero eigenvalues λ_α are enumerated by $\alpha = 1, 2, \ldots$ and Π_α are the corresponding projectors in $\mathcal{H}'$, then define

$$\mathrm{tr}_{\mathcal{H}'}(V\Pi_\alpha) = B_\alpha , \tag{99}$$

which are operators in $\mathcal{H}$. Then, since

$$V = \sum_\alpha \Pi_\alpha \times \mathrm{tr}_{\mathcal{H}'}(V\Pi_\alpha) ,$$

$$\mathcal{A}(\rho) = \mathrm{tr}_{\mathcal{H}'}(V\rho \times \sigma V^\dagger)$$
$$= \sum_{\alpha,\beta} \mathrm{tr}_{\mathcal{H}}[(V\Pi_\alpha)\rho \times (pi_\alpha \sigma \Pi_\beta)(V^\dagger \Pi\beta)] \tag{100}$$

$$\mathcal{A}(\rho) = \sum_\alpha B_\alpha \rho B_\alpha^\dagger . \tag{101}$$

Conversely, given the map

$$\rho \to \mathcal{A}(\rho) = \sum_\alpha B_\alpha \rho B_\alpha^\dagger , \quad B_\alpha^\dagger B_\alpha = \mathbb{1} , \tag{102}$$

we can construct [26]

$$V=\sum_{\alpha}\Pi_{\alpha}\times B_{\alpha}\ . \tag{103}$$

Then

$$V^{\dagger}V=\mathbb{1} \tag{104}$$

and

$$\mathcal{A}(\rho)=\mathrm{tr}_{\mathcal{H}'}(V_{\rho}\times\sigma V^{\dagger}) \tag{105}$$

with the diagonal matrix

$$(\sigma)_{\alpha\beta}=\mathrm{tr}_{\mathcal{H}}(B^{\dagger}_{\alpha}B_{\alpha})\delta_{\alpha\beta}\ ,\quad \text{no sum over } \alpha\ . \tag{106}$$

So both methods give the same set of completely positive dynamical maps.

When the dimension of $\mathcal{H}$ is finite, $\mathcal{K}$ can be chosen to be finite dimensional and consequently V is unitary. But for $\mathcal{H}$ infinite dimensional, $\mathcal{K}$ is also infinite dimensional, there are isometric operators that are not unitary. The elementary isometric operator with deficiency index 1 in the space $\mathcal{K}$ (indexed by a denumerable index $k=1,2,\ldots$) is [27]

$$V_{k,\ell}=\delta_{k,\ell+1}\ . \tag{107}$$

It is useful to note that if H is the Hamiltonian in $\mathcal{H}$, the operators B_{α} may not all commute with $\mathcal{H}$. Therefore, there is a possibility of energy redistribution in the dynamical map, and the generic map is a contraction map with possibly one fixed point. This is in direct contrast with factorizable Liouville operators, which have a one-parameter family of fixed states. All such dynamical maps are contractions of an isometry of a larger system [26]. The dimension of $\mathcal{H}'$ need not be any larger than the dimension of $\mathcal{H}$ squared. For the generating element of the convex set of maps one can show that the dimension of $\mathcal{H}'$ is less than or equal to the dimension of $\mathcal{H}$ [25].

The quantum dynamical maps are finite time developments [5]. But the existence of the temporal group in $\mathcal{H}$ and the semigroup in $\mathcal{J}$ suggests that we look at their generators. In $\mathcal{H}$ this generator is nothing but the Hamiltonian in the factorizable case. In $\mathcal{J}$, however, we have semigroups. If we take the Kronecker product $\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{H}'$ we write the isometric operator $V(t)$ in the form

$$V(t)=\exp(-itJ)=1-itJ+\frac{(it)^2}{2!}J^2+\cdots\ . \tag{108}$$

As long as we keep only the linear terms in t for small t we have an effective Hamiltonian group of evolution of ρ in $\mathcal{H}$ given by

$$\rho\to\rho(t)=e^{-iHt}\rho e^{iHt}\ , \tag{109}$$

where

$$H\ \mathrm{tr}_{\mathcal{H}'}(J)\ . \tag{110}$$

To get any result going beyond this, we need to retain the quadratic term

$$\rho\to\rho(t)=\rho-it[(\mathrm{tr}_{\mathcal{H}'}J),\rho] -\frac{t^2}{2!}[(\mathrm{tr}_{\mathcal{H}'}J^2)\rho+\rho(\mathrm{tr}_{\mathcal{H}'}J^2)-2H\rho H]+\cdots\ . \tag{111}$$

The third term can be written in the form

$$-\frac{t^2}{2!}\{[J^{\dagger}_{\alpha}J_{\alpha},\rho]-2H\rho H\}=\frac{t^2}{2}\{[J_{\alpha},\rho J^{\dagger}_{\alpha}]+[J_{\alpha}\rho,J^{\dagger}_{\alpha}]\}\ . \tag{112}$$

The term $+t^2H\rho H$ is the term quadratic in t of $e^{-iHt}\rho e^{iHt}$. Here J_{α}, and $J^{\dagger}_{\alpha}$ are the $(1,\alpha)$ and $(\alpha,1)$ blocks of J and H is the (1, 1) element partitioned with respect to a basis in $\mathcal{H}'$. Kossakowski had shown that for a finite-dimensional system the most general dynamical semigroup generator has the form [3,28,29]

$$\dot{\rho}=i[\rho,H]+\sum_{\alpha}\{J^{\dagger}_{\alpha}J_{\alpha},\rho\}\ . \tag{113}$$

The generalization of this to an infinite-dimensional system has been made by Lindblad [30].

We can go in the opposite direction and demonstrate that all dynamical semigroups have generators that can be obtained from semigroup generators for an extended system. Construct

$$\Lambda_{11}=H\ ,\quad \Lambda_{1\alpha}=J_{\alpha}\ ,\quad \Lambda_{\alpha 1}=J^{\dagger}_{\alpha}\ ,\quad \Lambda_{\alpha\beta}=(\Lambda_{\beta\alpha})^{\dagger} \tag{114}$$

arbitrary. Then Λ so constructed is Hermitian symmetric. To order t^2 the change in ρ is

$$it[\rho,H]+\frac{(it)^2}{2!}(J^{\dagger}_{\alpha}J_{\alpha}\rho+\rho J^{\dagger}_{\alpha}J_{\alpha})+(it)^2H\rho H\ . \tag{115}$$

For $t=\tau$ sufficiently small we absorb $\sqrt{\tau}$ into the definition of J_{α} and write the approximate evolution equation

$$\dot{\rho}=i[\rho,H]+\{J^{\dagger}_{\alpha}J_{\alpha}\ ,\rho\}\ , \tag{116}$$

which is equivalent to

$$\dot{\rho}=i[\rho,\mathrm{tr}_{\mathcal{H}'}J]+\{\mathrm{tr}_{\mathcal{H}'}(J^{\dagger}J),\rho\}\ . \tag{117}$$

For the generic case, this evolution does not leave the energy distribution unchanged since $\sum_{\alpha}J^{\dagger}_{\alpha}J_{\alpha}$ may not commute with H and a unique equilibrium state may result.

I seek the generator of the semigroups of time evolution in J obtained by the open contour construction of generalized state vectors

$$R(\lambda,\nu,t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\nu'\frac{\rho(\lambda,\nu')e^{i\nu t}}{\nu-\nu'+i\epsilon} \tag{118}$$

and its analytic continuation into operators in $\mathcal{J}$. For $t>0$ we can close the contour in the lower half plane. By distortion of the contour we may express it in terms of an integral over complex frequencies with one or more dominating poles.

Analytic density operators can be expressed in terms of contour integrals in the complex plane; the time evolution of the density operator is then expressible as a direct integral of damped exponentials. Since two distinct contours may have only their end points in common, the two direct integrals may not have much in common, yet realize the time dependence of one state [31]. For contours in the lower half plane one may or may not have discrete "pole" contributions. For the generalized half-plane analytic density operators, the end points recede to infinity and for one sign of time the direct integral of damped harmonic dependences vanish.

X. CONCLUDING DISCUSSION

The generalized ("macroscopic") states and the physical ("microscopic") states of a physical system are both representatives of the same system as long as we deal with the dense subset of analytic density operators. A physical system that is isolated ought to have a Hamiltonian that is bounded from below and the energy should be invariant under time evolution. But when the system is in interaction with other systems or with unobserved degrees of freedom inherent in the same system, the (reduced) Hamiltonian may not be positive definite, or even specific. It is in this context that we consider dynamical maps and dynamical semigroups.

Time manifests itself in two distinct aspects: reversible time for Hamiltonian systems and historical time for nonfactorizable Liouvillian systems [32]. The emergence of historical time does not involve the breaking of time-reversal symmetry considered by Wigner [33]. Rather, most irreversible systems are time-reversal symmetric (that is, invariant when momenta and angular momenta are reversed and an antilinear transformation is implemented).

The introduction of generalized states by the analytic continuation of a dense set of analytic vectors in the physical space serves to display the approximate exponential decay as due to the dominance of a complex discrete generalized state; but as long as one starts with a physical state and continues it into the generalized spaces, the discrete complex energy state should always be accompanied by a background integral that necessitates the Khalfin and Zeno regimes. Exact semigroups are obtained only when one starts with unphysical states with an energy spectrum unbounded from below.

Such a behavior may be appropriate for dealing with partial subsystems of a larger (unobserved) system. The effective dynamics is, in such cases, realized by a linear trace, hermiticity, and positivity properties of the subsystem density matrix and is, as a rule, nonfactorizable. This structure is necessary to bring about the redistribution of energies of the subsystem: the dynamics cannot be Hamiltonian. This generic dynamical law gives the notion of dynamical maps. It is shown that the contraction of the system dynamics into the subsystem dynamics leads to nonfactorizable dynamical maps. The generic form of dynamical maps is presented and it is shown that one can always realize it as the contraction of a unitary map.

Similar considerations are standard for dynamical semigroups and the generic Kosakowski form of the dynamical semigroup generators is rederived. The converse is also shown, that the Kossakowski form is realizable as the (limit of a) contraction of a Hamiltonian evolution of a larger system.

Arriving at a semigroup and introducing historical time by themselves do not lead a complete understanding of statistical thermodynamics. There has to be an approach to a specific equilibrium; this requires a change in the energy distribution. For a decaying particle we have a line shape computed dynamically, but approach to thermal equilibrium involves a new kind of dynamics.

With regard to the question of whether there are autonomous pure exponentially decaying particles: Are we forever stuck with approximate exponential decay law with "aging" of the unstable particles, the Zeno regime in infancy, and the Khalfin regime in old age? The answer lies in the choice of the states [34]. If we use physical states that are composed of non-negative energy states we have aging particles, but we can choose (if we choose) generalized states that have pure exponential decay. Which is the better choice: exponential decay and generalized states, or aging particles and conventional states? Yamaguchi [34] and Tasaki, Petrosky, and Prigogine [35] have recently called attention to the implications of the notion of generalized (complex pole) states for the kaon decay complex and to the possibility of experimental tests to delimit the choice.

One could use the quantum envelope of a classical system [36] to deal with the time evolution and generalized states of a classical system. This would be treated in a separate paper.

ACKNOWLEDGMENTS

Discussions with Ioannis Autoniou and my colleagues Charles Chiu, Arno Bohm, Gopalakrishnan Bhamathi, and Tomio Petrosky have helped clarify my ideas. To Chiu and Bhamathi I am further indebted for permission to quote from yet unpublished work. Over the years I have been inspired by the work of Ilya Prigogine and I am pleased to acknowledge extended discussions with him; this paper has benefited from his insights and sentiments. The work was supported by the U.S. Department of Energy Grant No. DOE-FG05-85ER-40-200.

[1] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste* (Dover, New York, 1957); E. Whitaker, *A Treatise on the Analytical Dynamic of Particles and Rigid Bodies* (Cambridge University Press, London, 1937); V. I. Arnold, *Dynamical Systems* (Springer-Verlag, Berlin, 1988).

[2] I. Antoniou and I. Prigogine (unpublished); T. Petrosky and I. Prigogine, Physica A **147**, 439 (1988).

[3] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976); V. Gorini, F. Verri, and A. Kossakowski, Rep. Math. Phys. **13**, 149 (1978).

[4] E. C. G. Sudarshan, C. B. Chiu, and V. Gorini, Phys. Rev. D **18**, 2914 (1978); A. Bohm, J. Math. Phys. **21**, 1040 (1980); **22**, 2813 (1981).
[5] P. M. Matthews, J. Rau, and E. C. G. Sudarshan, Phys. Rev. **121**, 920 (1961); K. Kraus, Ann. Phys. (N.Y.) **64**, 311 (1971).
[6] E. C. G. Sudarshan, in *Relativistic Particle Interactions,* Proceedings of the 1961 Brandeis Summer Institute (Benjamin, New York, 1962).
[7] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon, Oxford, 1937).
[8] K. O. Friedrichs, Commun. Pure Appl. Math. **1**, 361 (1948); T. D. Lee, Phys. Rev. **95**, 1329 (1954); P. A. M. Dirac, Proc. R. Soc. London **114**, 243 (1927).
[9] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).
[10] C. B. Chiu, E. C. G. Sudarshan, and B. Misra, Phys. Rev. D **16**, 520 (1977).
[11] G. Parravicini, V. Gorini, and E. C. G. Sudarshan, J. Math. Phys. **21**, 2208 (1980); A. Bohm, *ibid.* **21**, 1040 (1980); **22**, 2813 (1981).
[12] M. Nakanishi, Prog. Theor. Phys. **19**, 607 (1958).
[13] C. B. Chiu and E. C. G. Sudarshan, Phys. Rev. D **42**, 3712 (1990).
[14] L. A. Khalfin, Zh. Eksp. Teor. Fiz. **33**, 1053 (1958) [Sov. Phys.—JETP **6**, 1053 (1958)].
[15] David Onley and Anita Kumar (unpublished).
[16] S. T. Ma, Phys. Rev. **69**, 668 (1946); **71**, 195 (1947); V. Bargman, Rev. Mod. Phys. **21**, 488 (1949); R. G. Newton, J. Math. Phys. **1**, 319 (1960).
[17] S. N. Biswas, T. Pradhan, and E. C. G. Sudarshan, Nucl. Phys. B **50**, 269 (1972).
[18] E. P. Wigner and J. von Newmann, Z. Phys. **30**, 465 (1929); Barry Simon, Commun. Pure Appl. Math. **22**, 531 (1967); E. C. G. Sudarshan, *Field Theory, Quantization, and Statistical Physics,* edited by E. Tirapegui (Reidel, Dordrecht, 1981).
[19] C. B. Chiu, E. C. G. Sudarshan, and G. Bhamati, Phys. Rev. D (to be published).
[20] I. Prigogine (private communication).
[21] G. Gamow, Z. Phys. **51**, 204 (1928).
[22] P. A. M. Dirac, Proc. R. Soc. London **114**, 243 (1927).
[23] G. Breit and E. P. Wigner, Z. Phys. **63**, 54 (1930).
[24] A. Bohm, J. Math. Phys. **21**, 1040 (1980); **22**, 2313 (1981); Physica **124A**, 103 (1984).
[25] M. D. Choi, Can. J. Math. **24**, 520 (1972); Illinois J. Math. **18**, 565 (1974).
[26] E. C. G. Sudarshan, in *From SU(3) to Gravity,* edited by E. Gotsman and G. Tauber (Cambridge University Press, London, 1985), p. 33.
[27] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Ungar, New York, 1955), Chap. VIII.
[28] A. Kossakowski, Rep. Math. Phys. **3**, 247 (1972).
[29] E. B. Davies, Commun. Math. Phys. **15**, 277 (1969); **19**, 83 (1970); **22**, 51 (1971).
[30] G. Lindblad, Commun. Math. Phys. **40**, 147 (1975).
[31] I. Prigogine, *From Being to Becoming,* (Freeman, New York, 1980); G. Nicolis and I. Prigogine, *Exploring Complexity* (Freeman, New York, 1989).
[32] E. P. Wigner, Gott. Nachr. **31**, 546 (1932).
[33] A. Bohm and M. Gadella, in *Dirac Kets, Gamow Vectors, and Gelfand Triplets,* edited by A. Bohm and J. D. Dollard, Lecture Notes in Physics Vol. 348 (Springer-Verlag, Berlin, 1989), Chap. IV.
[34] Y. Yamaguchi, J. Phys. Soc. Jpn. **57**, 1525 (1958); **57**, 3339 (1958); **58**, 4375 (1989); **60**, 1545 (1991).
[35] S. Tasaki, T. Petrosky, and I. Prigogine (unpublished).
[36] E. C. G. Sudarshan, in *From Symmetries to Strings,* edited by Ashok Das (World Scientific, Singapore, 1990).

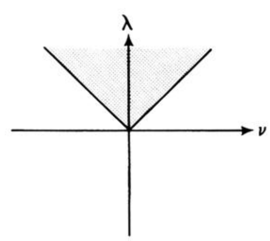

FIG. 1. Spectrum of the Liouville generator for a Hamiltonian system.

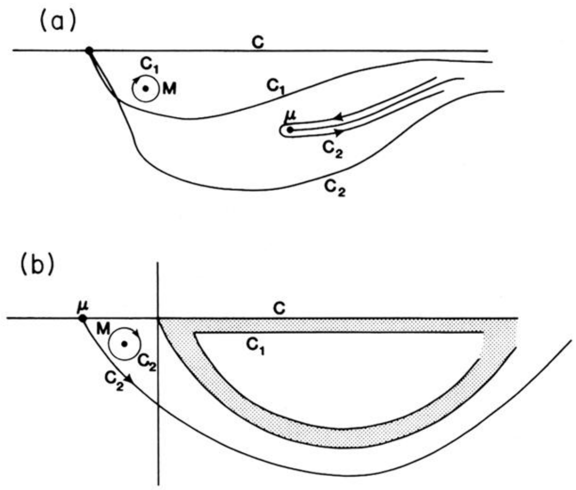

FIG. 6. (a) Spectra in $\mathcal{J}$ for the cascade model with A, B unstable; (b) spectra in $\mathcal{H}$ and $\mathcal{J}$ of the cascade model, B stable.