PHYSICAL REVIEW D VOLUME 47, NUMBER 6 15 MARCH 1993

Analytic continuation of quantum systems and their temporal evolution

E. C. G. Sudarshan and Charles B. Chiu
Center for Particle Physics and Department of Physics, University of Texas, Austin, Texas 78712
(Received 9 March 1992; revised manuscript received 22 October 1992)

The generalized vector space of quantum states is used to study the correspondence between the physical state space $\mathcal{H}$ and its continuation $\mathcal{G}$. Consider the integral representation defined by the scalar product between an arbitrary vector in the dense subset of analytic vectors in $\mathcal{H}$ and its dual vector, where the integration is along the real axis. Keeping the scalar product fixed, the analytic vectors may be continued through the deformation of the integration contour. The deformed contour defines the generalized spectrum of the operator in the continued theory, which typically consists of a deformed contour in the fourth quadrant and the exposed singularities, if any, between the real axis and the deformed contour. Several models are studied with special attention to the unfolding of the generalized spectrum. The two-body models studied are the Lee model in the lowest sector and the Yamaguchi potential model, where the exposed singularities, if present, are simple poles. The three-body model studied is the cascade model, where the exposed singularities may be poles and the branch cuts associated with the quasi-two-body states. We demonstrate that the generalized spectrum obtained leads to the correct extended unitarity relation for the scattering amplitudes. The possibility of having mismatches between poles in the S matrix and the discrete states in the Hamiltonian, which exists in the $\mathcal{H}$ space, obtains also in the generalized $\mathcal{G}$ space. Finally, two distinct views on what constitutes an unstable particle are contrasted. One view is to identify it as a physical state of the system which ceases to exist as a discrete eigenstate in $\mathcal{H}$. Here the survival amplitude of the unstable particle cannot be ever strictly exponential in time. The other view is to identify the unstable particle as a discrete state in the generalized space. It has a pure exponential time dependence. So the corresponding time evolution is realized by a semigroup. While the latter approach appears to be elegant, it is obtained at the expense of giving up the very starting premise of the lower boundedness of the energy spectrum and therefore we consider it to be the less desirable choice.

PACS number(s): 03.65.Db, 11.80.Cr

I. INTRODUCTION

Orthodox quantum mechanics is formulated in a vector space over complex numbers with a sesquilinear inner product [1]. In most applications the vector space is a separable complete space and often taken to be a Hilbert space [2]. The vector space, except in cases of "spin" systems with a finite basis, is made up of L^2 functions of one or more variables or a vector of such functions. The dynamical variables are taken to be linear operators of finite norm. Among them the self-adjoint operators form a preferred class and the observables are usually identified with them.

But it is convenient to deal with unbounded operators such as the canonical coordinate or momentum or the Hamiltonian. Such operators do not have an action on the whole vector space since they could make the length of the image vector unbounded and thus not in the space; so we have to restrict the "domain" of the unbounded operator.

Even a further departure is often needed: when we deal with an operator with a continuous spectrum it is useful to introduce ideal vectors [1] with distribution-valued scalar products.

In the cases where the vector space is realized by functions of a certain class it may be possible to consider analytic continuation of such function spaces with an associated bilinear form but with two analytic vector spaces being defined: the basic vector space and the space of linear functionals on this space. Of course, this generalization could have been considered without analytic continuation. If the base space topology becomes stronger the dual space topology becomes weaker and vice versa. In a Hilbert space the two topologies are the same (completeness of all Cauchy sequences) with a reflexive antilinear transformation connecting the base space (ket) vectors and the dual space (bra) vectors [1]. In the context of density operators this has been emphasized by Segal [3]. In the context of vectors in a Hilbert space this formalism due to Gelfand [4] and amplified by Bohm [5] is called the rigged Hilbert space. While such a generalization is by choice for Hilbert spaces, both in the Segal context and in the course of analytic continuation the dichotomy between the base space and the dual enters automatically.

Nakanishi [6] has employed the notion of an analytically continued set of "wave functions" in the context of a treatment of unstable particles in quantum mechanics. It has also been employed in the context of master analytic representations of noncompact groups[7]. The first systematic generalization of the quantum vector space by analytic continuation was formulated by Sudarshan, Chiu, and Gorini [8]. Rigorous treatment of the problem with careful attention to functional analytic details have since been given [9].

The problem of decaying particles, scattering resonances, and generic metastable states in quantum physics continues to be of current interest. The long-time behavior departing from exponential decay exhibited by Khalfin [10], the short-time Zeno behavior [11], and the detailed transition behavior of quantum metastable excitations constitute a complex of rich phenomenology [12]. It has been further enriched by the multitude of features in the neutral kaon decay and that of other such particles [13] and in the cascade decay phenomena. Recently Yamaguchi [14] has raised important questions about the behavior of decay amplitudes and the possibility that short- and long-lived kaons are orthogonal whether or not *CP* is conserved. From a somewhat different point of view Tasaki, Petrosky, and Prigogine [15] have considered this question with special attention about the breaking of time symmetry in decay.

Apart from these questions there has been some lack of precision about analytic continuations, and about scattering amplitude singularities: not enough attention has been paid to redundant zeros and discrete states buried in the continuum.

Complex variables, analytic functions, and topology are only aids to the mathematical discussion of physical phenomena; an essential part of the task is the proper identification and proper interpretation of mathematical results. Not all quantum theories involving analytic continuations are alike nor are their scope the same. We have found several treatments that are lacking in one aspect or the other. For example many authors act as if poles in the analytic continuation are the only relevant singularities [16]. We show, on the contrary, that the treatment of scattering amplitudes involving unstable particles requires complex branch points. We have therefore paid particular attention to spell out the theory that we introduce. The use of solvable models enables us to illustrate many relevant features of the theory.

The most important point that we emphasize is that suitable dense sets in the analytically continued spaces have corresponding dense set of states in the space with which we start the analytic continuation. Individual states in one space may or may not have analytical partners in the generalized spaces. The analytic continuation is therefore basis dependent.

The outline of our presentations below are as follows. In Secs. II and III, the generalized vector space of quantum states is used to study the correspondence between the physical state space $\mathcal{H}$ and its continuation $\mathcal{G}$. We begin with the observation that the scalar product between an arbitrary vector in the dense subset of analytic vectors in $\mathcal{H}$ and its dual vector has an integral representation. While keeping the scalar product fixed, the analytic vectors may be "analytically continued" through the deformation of the integration contour. A typical analytically continued integral representation of present interest integrates along a deformed contour in the fourth quadrant of the complex energy plane. And it encircles those "exposed" singularities on the second sheet, if any, i.e., those between the real-axis and the deformed contour. The deformed contour together with the exposed singularities constitutes the generalized spectrum of the operator in the continued theory.

Several models are studied with special attention to the unfolding of the generalized spectrum. The two-body systems are studied in Secs. IV and V. There we consider the Lee model and the Yamaguchi potential model. The exposed second sheet singularities, if present at all, are simple poles. The cascade model of a three-body system is studied in Secs. VI and VII, where the exposed second sheet singularities may be poles and branch cuts, with the branch cuts being associated with quasi-two-body states. For the cascade model case, we also show that the generalized spectrum obtained leads to the correct extended unitarity relation for the scattering amplitudes.

In Sec. VIII, we observe that the predictions based on $\mathcal{H}$ and that based on $\mathcal{G}$ are expected to be the same. Since a pure exponential time dependence is not possible for states in $\mathcal{H}$, this then should not be possible for states in $\mathcal{G}$. On the other hand, the Breit-Wigner resonance does correspond to a pure exponential decay and it realizes the semigroups of time evolution. However, here one needs to give up the positivity of energy and define states with all possible values of energy.

In Sec. IX we recall the two possible disparities between poles in the S matrix and the discrete states in the Hamiltonian. In particular, there can be a pole in the S matrix without the corresponding state in the complete states of the Hamiltonian. Conversely there may be a discrete state of the Hamiltonian, which does not have the corresponding pole in the S matrix. We show that these disparities continue to be admissible in the generalized vector space.

Our concluding remarks are given in Sec. X. Two distinct views on what constitutes an unstable particle are contrasted. One view is to identify an unstable particle as a physical state of the system which ceases to exist as a discrete eigenstate of the total Hamiltonian. The survival amplitude of the unstable particle cannot be ever strictly exponential in time. There is no autonomy in its time development. It ages. So the unstable particle does not furnish a representation of the time translation group. The other view is to identify the unstable particle as a discrete state in the generalized space $\mathcal{G}$. It has a pure exponential time dependence. The time evolutions form a semigroup. While the latter appears to be elegant, it is deduced at the expense of giving up the very starting premise of the lower boundedness of the energy spectrum. In this section, based on the generalized vector space framework, we also comment on the neutral keon system of K^0 and $\bar{K}^0$ and the interpretation of "$\langle K_L | K_S \rangle$." We end with a discussion on the formal construction of the intertwining operators which maps $\mathcal{H}$ and $\mathcal{G}$ and vice versa.

II. VECTOR SPACES AND THEIR ANALYTIC CONTINUATION

Consider an infinite-dimensional vector space $\mathcal{H}$ over the field of complex numbers [1] with vectors $\psi,\phi,\ldots$. Then, if a,b are complex numbers $a\psi+b\phi$ is also a vector, as are finite linear combinations. If $\{|e^{(r)}\rangle\}$ is a countable basis then any vector ψ can be approximated to any desired limit by linear combinations of the form $\sum a_n^{(r)}|e^{(r)}\rangle = |\psi_n\rangle$ where the sequence $\{|\psi_n\rangle\}$ converges to ψ. A linear operator is a linear map from vectors in $\mathcal{H}$ to vectors in $\mathcal{H}$. The linear functional mapping each vector in $\mathcal{H}$ to a complex number constitute the dual vector space $\mathcal{H}'$ to $\mathcal{H}$. A basis $\{\langle f^{(s)}|\}$ in the dual vector space $\mathcal{H}'$ may be obtained by considering the linear functional

$$|e^{(r)}\rangle \xrightarrow{f^{(s)}} \delta_{rs} : |e^{(r)}\rangle \leftrightarrow \langle f^{(r)}| \; . \tag{1}$$

Thus we can put the basis vectors into one-to-one correspondence, but the correspondence is antilinear:

$$a|e^{(r)}\rangle + b|e^{(s)}\rangle \leftrightarrow a^*\langle f^{(r)}| + b^*\langle f^{(s)}| \; . \tag{2}$$

The linear functional can be thought of as the scalar product of vectors in $\mathcal{H},\mathcal{H}'$ bilinear in them:

$$\phi \xrightarrow{\psi} (\psi,\phi) \equiv \langle\psi|\phi\rangle, \quad \psi\in\mathcal{H}', \; \phi\in\mathcal{H} \; , \tag{3}$$

or as a sesquilinear form in $\mathcal{H}$ by making use of the antilinear correspondence (2) between bra and ket vectors.

Given the basis vectors and the notion of scalar products we can introduce the completeness identity. If we have a bra $\langle\psi|$ and a ket $|\phi\rangle$, we can define a linear operator by the vector-valued linear functional,

$$|\chi\rangle \to \langle\psi|\chi\rangle|\phi\rangle \; , \tag{4}$$

and identify it with the linear operator

$$A = |\phi\rangle\langle\psi| \; . \tag{5}$$

In particular we can introduce the linear operator

$$\sum_r |e^{(r)}\rangle\langle e^{(r)}| \; ,$$

which acting on any vector $|\phi\rangle$ reproduces itself:

$$\sum_{r=1}^{\infty} |e^{(r)}\rangle\langle e^{(r)}|\phi\rangle = \sum_{r,s} |e^{(r)}\rangle\langle e^{(r)}|a_s|e^{(s)}\rangle = \sum_{r,s} a_s\delta_{rs}|e^{(r)}\rangle = |\phi\rangle \; .$$

Hence it is the unit operator:

$$\sum_{r=1}^{\infty} |e^{(r)}\rangle\langle e^{(r)}| = \mathbb{1} \; . \tag{6}$$

This is the *completeness identity* and provides a *resolution of the identity.* A linear operator V is *isometric* [17] if, for every vector ϕ,

$$\langle V\phi|V\phi\rangle = \langle\phi|\phi\rangle \; . \tag{7}$$

Given an operator A its *adjoint* operator $A^\dagger$ is defined by

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle \; . \tag{8}$$

An isometric operator V satisfies the relation

$$V^\dagger V = \mathbb{1} \; . \tag{9}$$

The adjoint is an antilinear operator valued function of operators. An operator whose adjoint coincides with itself is called *self-adjoint:*

$$A^\dagger = A \; . \tag{10}$$

An isometric operator is *unitary* if in addition to (9) it satisfies

$$VV^\dagger = \mathbb{1} \; . \tag{11}$$

If a normal linear operator C has the form

$$C = \sum_n c_n |e^{(n)}\rangle\langle e^{(n)}| \tag{12}$$

for some convergent sequence $\{c_n\}$ and some basis, $\{|e^{(n)}\rangle\}$ is said to be *completely continuous.* A completely continuous operator is the discrete (possibly infinite) sum of *projections:*

$$C = \sum_n c_n\Pi_n, \quad \Pi_n = |e^{(n)}\rangle\langle e^{(n)}| \; , \tag{13}$$

with

$$\Pi_n\Pi_m = \Pi_n\delta_{nm}, \quad \sum_n \Pi_n = \mathbb{1} \; . \tag{14}$$

The expressions (12) and (13) also give the *spectral decomposition* of a completely continuous operator [17]:

$$C|e^{(n)}\rangle = c_n|e^{(n)}\rangle \; . \tag{15}$$

For any operator A we can consider the *resolvent* as the analytic operator-valued function

$$R(z;A) = (A - z\mathbb{1})^{-1} \; . \tag{16}$$

$R(z)$ is regular acting on $\mathcal{H}$ everywhere except for the values

$$z = c_n$$

which constitute the spectrum of A. More generally the set of points (discrete or continuous, finite or infinite) where the resolvent operator fails to be regular in $\mathcal{H}$ [i.e., the action of $R(z)$ considered as an analytic function of z is not regular for any vector in $\mathcal{H}$] is called the *spectrum* of A.

For a self-adjoint operator with a continuous spectrum there may be no normalizable eigenvectors in $\mathcal{H}$. In all the explicit examples we have considered that the continuous spectrum has no normalizable eigenvectors. One can either introduce ideal eigenvectors (of infinite length) following Dirac, or consider a continuous family of spectral projections $\Pi(\lambda)$ for eigenvalues "less than" λ by introducing a notion of ordering in the continuous spectrum (when it is possible) and writing a Stieltjes operator-valued integral generalizing the spectral decomposition and completeness identity (12), (13), (14):

$$A = \int \lambda\, d\Pi(\lambda) \; , \tag{13'}$$

$$\int d\Pi(\lambda) = \mathbb{1}, \quad \Pi(\lambda)\Pi(\mu) = \Pi(\lambda), \quad \mu \ge \lambda \; . \tag{14'}$$

So far we have considered the generic form A, the Hilbert space $\mathcal{H}$, and the vectors in $\mathcal{H}$. In the study of quantum systems the space $\mathcal{H}$ is realized in terms of the states of the system and the generic form of the state vectors is in terms of square integrable functions of one or more real variables. A dense subset of such L^2 functions is the class of analytic functions (restricted to real values of the arguments). This dense subset of $\mathcal{H}$ there can be analytically continued. But there are many choices of analytic L^2 functions with varying domains of analyticity and correspondingly many choices of $\mathcal{G}$ and $\mathcal{G}'$. The dense sets of analytic functions form a partially ordered set: continuations using functions analytic in a domain which coincide with the analytic continuation using functions analytic in another domain, and will coincide within their common domain of convergence. Linear relationships are preserved; we can define *analytic linear operators* to be those which acting on an analytic function produces another analytic function. Needless to say the notion of analytic continuation is in terms of the specific L^2 function realization of the space $\mathcal{H}$, and the domain in which $\mathcal{G}$ is defined depends on the dense subset chosen. Since the correspondence between vectors in $\mathcal{H}$ and $\mathcal{H}'$ is antilinear we must analytically continue these spaces separately to produce a family of generalized spaces $\mathcal{G}$ and $\mathcal{G}'$.

The notion of resolvent and spectrum applies to the generalized family of spaces $\mathcal{G},\mathcal{G}'$. The eigenvectors are now *right* eigenvectors in $\mathcal{G}$ and *left* eigenvectors in $\mathcal{G}'$. For every vector in $\mathcal{H}$ we have its dual vector in $\mathcal{H}'$. The product of the analytic continuations of a dense set of vectors in $\mathcal{H}$ (and hence $\mathcal{H}'$) are in $\mathcal{G},\mathcal{G}'$ and it may be called the norm of the vector in $\mathcal{G}$. With respect to this norm we can define Cauchy sequences.

Since the analytic continuation is for both $\mathcal{H}$ and $\mathcal{H}'$ to $\mathcal{G}$ and $\mathcal{G}'$ scalar products and matrix elements of analytic linear operators are preserved. To this extent, the analytic vectors and operators can be thought of as having different representations in the family of spaces $\mathcal{G},\mathcal{G}'$ and these could be put in correspondence with the analytic vectors and linear operators in $\mathcal{H}$. However, the analytic continuation is not of the entire space $\mathcal{H}$ into the completion of $\mathcal{G}$ with the norm as defined as the product of vector in $\mathcal{G},\mathcal{G}'$ associated with the vectors in $\mathcal{H},\mathcal{H}'$. In particular there are vectors in $\mathcal{G}$ which may not have a counterpart in $\mathcal{H}$ and vice versa. We shall find that there are discrete states in $\mathcal{G}$ which have no counterpart in $\mathcal{H}$.

Finally, since the analytic continuation depends on the functional form for the state vectors as a function of its arguments, there is a choice to be made of the relevant dynamical labels. In the study of Hamiltonian systems we often have a "*total energy*" label as well as the values of a *comparison Hamiltonian* energy. On writing the ideal eigenstates of the total Hamiltonian as a function of the comparison Hamiltonian energy we look for analytic vectors; this can be done if the total Hamiltonian-*represented* terms of the functions of comparison Hamiltonian energies is *analytic*. The existence of the comparison ("free") Hamiltonian and its essential role in scattering theory where the "in" and "out" states are defined has been known for some time [18]. Formal theory of scattering does make use of this representation to go "slightly off" the real axis as far as the *scattering amplitude* is concerned. The analytic continuation of scattering amplitude was extended to its various sheets by many authors [19]. However except for the work of Nakanishi [6] and of Sudarshan, Chiu, and Gorini [8] (see also Bohm [20]) there was no intention to consider the analytic continuation of suitable dense sets in the state space $\mathcal{H}$ to the family $\mathcal{G}$.

III. COMPLETE SET OF STATES

If $\{|\lambda\rangle\}$ is the set of ideal eigenvectors for a self-adjoint non-negative (total Hamiltonian) operator so that

$$\Pi(\lambda)=\int_0^\infty |\lambda\rangle\langle\lambda|d\lambda \ , \tag{17}$$

the vector

$$|\phi\rangle=\int_0^\infty \phi(\lambda)|\lambda\rangle d\lambda \tag{18}$$

is a vector in $\mathcal{H}$ if

$$\int_0^\infty |\phi(\lambda)|^2 d\lambda<\infty \ . \tag{19}$$

Provided $\phi(\lambda)$ is analytic in λ in a suitable domain in the complex plane we could deform the contour to write the vector as a vector in $\mathcal{G}$ (see Fig. 1):

$$|\phi\rangle=\int_C \phi(z)|z\rangle dz \ . \tag{20}$$

The analytic continuation includes a simultaneous continuation of the bra vectors

$$\langle\psi|=\int_0^\infty \psi(\lambda)\langle\lambda|d\lambda \tag{21}$$

into a vector in $\mathcal{G}'$:

$$\langle\psi|=\int \tilde{\psi}(z)\langle z|dz \ . \tag{22}$$

The additional closed contours C_1 and C_2 encountered in the continuation (see Fig. 2) are typical of poles and branch cuts. For resonance in scattering we expect to find poles but for multiparticle states involving unstable particles we expect to have branch cuts. While Fig. 2 shows only one pole and one pair of branch points in the finite complex plane we may have more than one; branch points may move to infinity. The completeness identity (6) gets modified to

$$\mathbb{1}=\int_C dz|z\rangle\langle z|+\sum_{\text{poles}}|z_r\rangle\langle z_r|+\int_{C_2}d\xi|\xi\rangle\langle\xi| \ . \tag{23}$$

Further, the scalar product remains unchanged in value:

$$\langle\psi|\phi\rangle=\int_C \tilde{\psi}(z)\phi(z)dz+\sum_{C_1}\tilde{\psi}(z_r)\phi(z_r) + \int_{C_2}\tilde{\psi}(\xi)\phi(\xi)d\xi \ . \tag{24}$$

Here and in (22), $\tilde{\psi}(z)$ is the analytic continuation of the function $\psi^*(z^*)$:

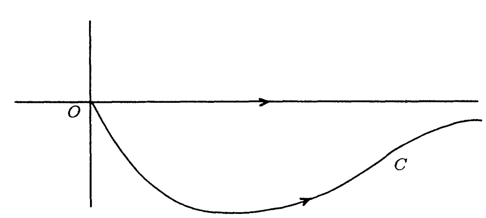

FIG. 1. The z-plane contours defining vectors in $\mathcal{G}$.

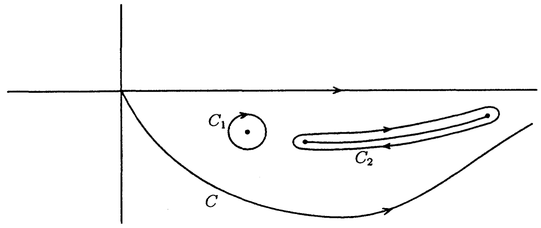

FIG. 2. Possible singularities encountered and the modified contours.

$$\tilde{\psi}(z)=\psi^*(z^*) \; ; \tag{25}$$

and the norm is given by $\langle \psi|\psi\rangle$. If we have a definite state $\psi(\lambda)$ (which may be thought of as the created unstable particle state) then the survival amplitude for the state is given by [11,12]

$$A(t)=\langle \psi|e^{-iHt}|\psi\rangle=\mathrm{tr}(|\psi\rangle\langle\psi|e^{-iHt}) \tag{26}$$

where H is the (total) Hamiltonian and can be expressed in the form of a Fourier integral:

$$A(t)=\int_0^\infty |\psi(\lambda)|^2 e^{-i\lambda t}d\lambda \; . \tag{27}$$

This same survival amplitude can be computed in $\mathcal{G},\mathcal{G}'$ provided $|\psi\rangle$ is an analytic vector:

$$A(t)=\int_{C+C_1+C_2} dz\, \tilde{\psi}(z)\psi(z)e^{-izt} \; . \tag{28}$$

If the analytically continued bilinear quantity is explicitly known the pole contributions and the branch cut contributions can be calculated. This we shall do when we consider solvable models like the Lee model [21] and the Cascade model [22]. Suffice it to say that the survival amplitude can be defined for evolutions both forwards *and* backwards in time; for all times the absolute value of the amplitude is bounded by unity.

For the generic case the poles of the S matrix coincide with the discrete states in the generalized completeness identity (23). However, the existence of a pole in the S matrix is neither sufficient nor necessary to have such additional discrete states in $\mathcal{G}$. This is due to possible existence of redundant poles and of discrete states buried in the continuum. We shall illustrate this in a later section of this paper.

IV. THE LEE MODEL STATES

A simple solvable model [21] is provided by a system with a discrete state and a one-dimensional continuum so that the vectors are of the form

$$(\eta,\phi(\omega))^T=\Phi \tag{29}$$

with

$$\langle\Phi|\Phi\rangle=\eta^*\eta+\int d\omega\,\phi^*(\omega)\phi(\omega) \; . \tag{30}$$

We choose a total Hamiltonian of the form

$$H(\eta,\phi(\omega))^T=\lambda(\eta,\phi(\omega))^T \; ,$$
$$\lambda\eta=m_0\eta+\int_0^\infty g^*(\omega')\phi(\omega')d\omega' \; ,$$
$$\lambda\phi(\omega)=\omega\phi(\omega)+g(\omega)\eta \; .$$

Define the function

$$\alpha(\lambda)=\lambda-m_0-\int_0^\infty \frac{g^*(\omega')g(\omega')}{\lambda-\omega'}d\omega' \; . \tag{31}$$

If $\alpha(\lambda)$ has a real zero it is for a negative value m [unless $g(\omega)$ vanishes some place in the interval $0<\omega<\infty$]. If there is such a zero there is a discrete eigenvalue m for the Hamiltonian H:

$$\phi_0(\omega)=\frac{g(\omega)}{m-\omega}\eta_0, \quad \eta_0=\left[\left.\frac{d\alpha}{d\lambda}\right|_{\lambda=m}\right]^{-1/2} ,$$
$$H(\eta_0,\phi_0(\omega))^T=m(\eta_0,\phi_0(\omega))^T \; . \tag{32}$$

There can at most be one value. No such discrete state exists if

$$\alpha(0)=-m_0+\int\frac{g^*(\omega')g(\omega')d\omega'}{\omega'}<0 \; . \tag{33}$$

On the other hand if for some value $\lambda=M$ we have the twin conditions

$$g(M)=0, \quad \alpha(M)=0 \tag{34}$$

then we can have a discrete state overlapped by the continuum.

There is a continuous spectrum $0<\lambda<\infty$ and a corresponding continuum of scattering states which are ideal states with continuum normalization [18,23]:

$$|\Phi_\lambda\rangle=(\eta_\lambda,\phi_\lambda(\omega))^T\equiv|\lambda\rangle \; , \tag{35}$$

where

$$\eta_\lambda=\frac{g^*(\lambda)}{\alpha(\lambda+i\epsilon)} \; ,$$
$$\phi_\lambda(\omega)=\delta(\lambda-\omega)+\frac{g^*(\lambda)g(\omega)}{(\lambda-\omega+i\epsilon)\alpha(\lambda+i\epsilon)} \; .$$

These states satisfy the orthonormality and completeness relations

$$\langle m|m\rangle=1, \quad \langle m|\lambda\rangle=0 \; ,$$
$$\langle\lambda|\lambda'\rangle=\delta(\lambda-\lambda') \; , \tag{36}$$

and

$$|m\rangle\langle m|+\int d\lambda|\lambda\rangle\langle\lambda|=\mathbb{1} \; . \tag{37}$$

Here

$$|m\rangle=(\eta_0,\phi_0(\lambda))^T \; . \tag{38}$$

These calculations are already available in the literature and involve straightforward contour integration. If there is a discrete state buried in the continuum [24], (34) and (35) show that there are two solutions at this value M: a discrete state of the form (32) with m replaced by M, and an ideal state with $\lambda=M$ which is a pure plane wave:

$$|M\rangle=\left[\left.\frac{d\alpha}{d\lambda}\right|_{\lambda=M}\right]^{-1/2}\left[1,\frac{g(\omega)}{M-\omega}\right]^T \; , \tag{39}$$

$$|M\rangle'=(0,\delta(\lambda-M)+\text{nonsingular terms})^T \; . \tag{40}$$

The state (39) would enter the completeness relation (37) and the orthonormality relations (36).

The S matrix for the ideal scattering states reduces to a phase

$$S(\lambda)=\alpha(\lambda-i\epsilon)/\alpha(\lambda+i\epsilon), \quad 0<\lambda<\infty \ . \tag{41}$$

If $g(\omega)$ is analytic in ω so is

$$\tilde{g}(\omega)=g^*(\omega^*) \ . \tag{42}$$

Then the continuum ideal states $|\lambda\rangle$ can be replaced by complex eigenvalue ideal states denoted by the same symbol $|\lambda\rangle$ which have branch cuts along a different contour Γ beginning at 0 and ending at infinity. For seeing this we consider the space of analytic functions in the region Δ bounded by Γ and the positive real axis for which the integral

$$\left|\int_\Gamma \phi^*(z^*)\phi(z)dz\right|<\infty \ . \tag{43}$$

The spaces $\mathcal{G},\mathcal{G}'$ consists of vectors $(\eta,\phi(z))^T,(\eta',\phi'(z))$ with such functions $\phi(z)$. We further require that these functions $\phi(z)$ vanish sufficiently fast at infinity so that

$$\int_0^\infty |\phi(\omega)|^2 d\omega=\int_\Gamma \phi^*(z^*)\phi(z)dz \ . \tag{44}$$

Note that the scalar product is between a vector in $\mathcal{G}$ and one in the dual space $\mathcal{G}'$.

Along the contour Γ we can introduce a delta function $\delta(\lambda-z)$ defined by [6,8]

$$\int_\Gamma \phi(z)\delta(\lambda-z)dz=\phi(\lambda) \ . \tag{45}$$

With this definition we can reinvestigate the eigenvalue problem

$$H(\eta,\phi(z))^T=\lambda(\eta,\phi(z))^T \tag{46}$$

with z along the contour Γ. Equation (46) implies

$$\begin{aligned}&(\lambda-m_0)\eta=\int_\Gamma g^*(z'^*)\phi(z')dz' \ ,\\ &(\lambda-z)\phi(z)=g(z)\eta \ .\end{aligned} \tag{47}$$

The continuum ideal vectors have

$$\begin{aligned}&\eta_\lambda=\frac{g^*(\lambda^*)}{\alpha(\lambda+i\epsilon)} \ ,\\ &\phi_\lambda(z)=\delta(\lambda-z)+\frac{g^*(\lambda^*)g(z)}{(\lambda-z+i\epsilon)\alpha(\lambda+i\epsilon)} \ ,\\ &\alpha(z)=z-m_0-\int_\Gamma \frac{g^*(z'^*)g(z')dz'}{z-z'} \ .\end{aligned} \tag{48}$$

These are orthonormal; the computation follows the usual route. They are, together with the possible discrete state

$$\eta_0=[\alpha'(m)]^{-1/2} \ ,\qquad \phi_0(z)=\frac{g(z)\eta_0}{m-z} \ ,$$

also complete, provided $\alpha(m)=0$ for some $m<0$.

In case $m_0\gg 0$, there would be no discrete state $(\eta_0,\phi_0(z))^T$. But if the contour Γ proceeds sufficiently far in the fourth quadrant there would be a complex zero z_1 for $\alpha(z)$ and a discrete state with

$$\begin{aligned}&\eta_1=[\alpha'(z_1)]^{-1/2} \ ,\\ &\phi_1(z)=\frac{g(z)\eta_1}{z_1-z} \ .\end{aligned} \tag{49}$$

This state is orthogonal to the continuum states in $\mathcal{G}$ and enters as a discrete contribution to the completeness relation. Since $\alpha(z)$ is real analytic, if the contour Γ was in the upper half plane there would be a zero z_1^* for $\alpha(z)$ and a corresponding state. In both cases, the discrete state remains fixed and contributes to the complete set of states or not according to whether Γ crosses z_1 (or z_1^*).

The demonstration of the completeness is the resolution of the identity in the form

$$\mathbb{1}=\begin{cases}\int_\Gamma d\lambda|\lambda\rangle\langle\lambda|+|m\rangle\langle m| \ , & m^*=m<0,\ \alpha(m)=0 \ ,\\ \int_{\Gamma'} d\lambda|\lambda\rangle\langle\lambda|+|z_1\rangle\langle z_1|, & \alpha(z_1)=0 \ .\end{cases} \tag{50}$$

In doing the Γ or Γ' integrations we have to compute, for example,

$$\begin{aligned}\int \phi^*_{\lambda^*}(z^*)\phi_\lambda(z')d\lambda=&\ \delta(z-z')+\frac{g^*(z^*)g(z')}{(z'-z-i\epsilon)\alpha(z'-i\epsilon)}\\ &+\frac{g(z')g^*(z^*)}{(z-z'+i\epsilon)\alpha(z+i\epsilon)}+g^*(z^*)g(z')\int_\Gamma \frac{g^*(\lambda^*)g(\lambda)d\lambda}{(\lambda-z-i\epsilon)(\lambda-z'+i\epsilon)\alpha(\lambda+i\epsilon)\alpha(\lambda-i\epsilon)} \ .\end{aligned} \tag{51}$$

The last term can be rewritten as a contour integral encasing the contour Γ (Fig. 3) since

$$g^*(\lambda^*)g(\lambda)=\frac{1}{2\pi i}\{\alpha(\lambda)-\alpha^*(\lambda^*)\} \tag{52}$$

so that the last term becomes

$$g^*(z^*)g(z)\frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{d\lambda}{(\lambda-z'+i\epsilon)(\lambda-z-i\epsilon)\alpha(\lambda)} \ . \tag{53}$$

The poles at $\lambda=z'-i\epsilon$, $z+i\epsilon$ cancel the third and second terms respectively while the remaining contribution would be proportional to the residue at any pole of $1/\alpha(\lambda)$. Note that it is the zeros of $\alpha(z)$ that count, not the blow up of $g^*(z^*)g(z)$.

This conclusion is further demonstrated in the computation of the survival amplitude of the "unstable particle" state $(1,0)^T$. Quite generally,

$$((1,0),e^{-iHt}(1,0)^T)=\int \eta^*_{\lambda^*}\eta_\lambda e^{-i\lambda t}d\lambda \tag{54}$$

$$\begin{aligned}&=\int_\Gamma e^{-i\lambda t}\frac{g^*(\lambda^*)g(\lambda)}{\alpha^*(\lambda^*)\alpha(\lambda)}d\lambda\\ &=\frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{e^{-i\lambda t}d\lambda}{\alpha(\lambda)} \ .\end{aligned} \tag{55}$$

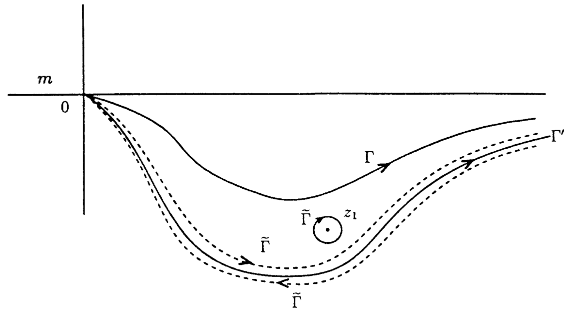

FIG. 3. Contours $\Gamma, \Gamma', \tilde{\Gamma}$ for demonstrating completeness.

Again only the zeros of $\alpha(\lambda)$ contribute, not the singularities of $g^*(\lambda^*)g(\lambda)$. Any such pole of $g^*(\lambda^*)g(\lambda)$ is counterbalanced by a corresponding pole in $\alpha^*(\lambda^*)$.

Here we have acted as if poles are the only singularities encountered in the analytic continuation. But in many contexts there could be branch cuts. We shall discuss such a situation for the cascade model.

V. THE YAMAGUCHI POTENTIAL MODEL STATES

A model related closely to the Friedrichs-Lee model is the separable potential model [25] which in its lowest relevant sector has a one-dimensional continuum. The states in $\mathcal{H}$ are, then, $L^2(0,\infty)$ functions:

$$\left\{ \Phi : \int_0^\infty \phi^*(\omega)\phi(\omega)d\omega < \infty \right\} .$$

We choose a total Hamiltonian of the form

$$(H\phi)(\omega) = \omega\phi(\omega) + \eta h(\omega)\int_0^\infty h^*(\omega')\phi(\omega')d\omega' \tag{56}$$

where $\eta^2 = 1$. Define the function

$$\beta(z) = 1 - \eta\int_0^\infty \frac{h^*(\omega')h(\omega')d\omega'}{z-\omega'} . \tag{57}$$

If $\beta(z)$ has a real zero, it will arise for $\eta < 0$, at $z = z_0 < 0$. In that case there is a discrete solution

$$\phi_0(\omega) = \frac{\eta h(\omega)}{z_0 - \omega}[\beta'(z_0)]^{-1/2} ,$$

$$\eta = -1, \quad z_0 < 0 . \tag{58}$$

There is a continuum of scattering states

$$\Phi_\lambda : \phi_\lambda(\omega) = \delta(\lambda-\omega) + \frac{\eta h^*(\lambda)h(\omega)}{(\lambda-\omega+i\epsilon)\beta(\lambda+i\epsilon)} . \tag{59}$$

These ideal states satisfy orthonormality

$$\langle 0|0\rangle = 1, \quad \langle\lambda|0\rangle = 0 ,$$

$$\langle\lambda|\lambda'\rangle = \delta(\lambda-\lambda') ,$$

and completeness

$$|0\rangle\langle 0| + \int d\lambda|\lambda\rangle| = \mathbb{1} . \tag{60}$$

Of course if $\beta(z)$ has no zero, the discrete state $|0\rangle$ would be missing from this equation.

The S matrix for the ideal scattering states reduce to a phase

$$S(\lambda) = \beta(\lambda - i\epsilon)/\beta(\lambda+i\epsilon), \quad 0 < \lambda < \infty . \tag{61}$$

If $h(\omega)$ is analytic in ω so is $h^*(\omega^*)$, then we can continue the vector space $\mathcal{H}$ into $\mathcal{G}$ and get a spectrum along another contour Γ starting from the origin and going to infinity.

The dimensionless scattering amplitude (in $\mathcal{H}$) is given by

$$T(\omega) = \frac{\pi h(\omega)h^*(\omega)}{\beta(\omega+i\epsilon)} = \exp[i\theta(\omega)]\sin\theta(\omega) , \tag{62}$$

where $\theta(\omega) = \arg\beta(\omega - i\epsilon)$ is the phase shift. If we choose nonrelativistic kinematics so that

$$\omega = k^2/2\mu \tag{63}$$

the more conventional scattering amplitude (with the dimension of a length) is given by

$$\mathcal{T}(k) = \frac{\pi|h(\omega)|^2}{k\beta(\omega+i\epsilon)} = \frac{e^{i\theta(\omega)}\sin\theta(\omega)}{k}$$

$$= [k\cot\theta(\omega) - ik]^{-1} \tag{64}$$

which manifestly satisfies unitarity. The total (S-wave) cross section is given by

$$\sigma(\omega) = \frac{4\pi}{k^2}\sin^2(\omega) . \tag{65}$$

When analytic continuations are carried out the scattering amplitude $T(\omega)$ is continued to yield

$$T(z) = \frac{\pi h(z)h^*(z^*)}{\beta(z+i\epsilon)}, \quad z \text{ on } \Gamma . \tag{66}$$

$T(z)$ so defined may have poles due to complex zeros of $\beta(z)$ or due to poles in $h(z)h^*(z^*)$. The latter do not correspond to extra physical states: they are "redundant poles" (see below in Sec. VIII). If there are no complex zeros of $\beta(z)$ the completeness relation in the analytically continued space $\mathcal{G}$ is

$$\int_\Gamma dz|z\rangle\langle z| = \mathbb{1} . \tag{67}$$

The explicit expression for the ideal states $|z\rangle$ and the proof of the completeness and orthogonality are straightforward. In many contexts there could be branch cuts. We shall discuss such a situation for the cascade model.

VI. THE CASCADE MODEL

We now consider a model [22] with three classes of states for the unperturbed Hamiltonian: a particle A with bare energy M_0; a two-particle continuum with energy $\mu_0 + \omega$, $0 < \omega < \infty$; and a three-particle continuum with energy $\omega + \nu$, $0 \le \omega, \nu < \infty$. We denote the amplitudes for these by η, $\phi(\omega)$, and $\psi(\omega,\nu)$ and the scalar product is given by

$$\eta^*\eta + \int_0^\infty \phi^*(\omega)\phi(\omega)d\omega + \int_0^\infty \int \psi^*(\omega,\nu)\psi(\omega,\nu)d\omega\, d\nu < \infty . \tag{68}$$

The vector space $\mathcal{H}$ of states is the completion of this vector space. The total Hamiltonian and eigenvalue equation are given by

$$\begin{bmatrix} M_0 & f^*(\omega') & 0 \\ f(\omega) & (\mu_0+\omega)\delta(\omega-\omega') & g^*(\nu')\delta(\omega-\omega') \\ 0 & g(\nu)\delta(\omega-\omega') & (\omega+\nu)\delta(\omega-\omega')\delta(\nu-\nu') \end{bmatrix} \begin{bmatrix} \eta_\lambda \\ \phi_\lambda(\omega') \\ \psi_\lambda(\omega'\nu') \end{bmatrix} = \lambda \begin{bmatrix} \eta_\lambda \\ \phi_\lambda(\omega) \\ \psi_\lambda(\omega\nu) \end{bmatrix} . \tag{69}$$

The energy eigenvalues are degenerate and infinitely degenerate once the three-particle channel becomes open. We can enumerate the (ideal) eigenstates of (69) in the form

$$|\lambda n\rangle \equiv \begin{bmatrix} \eta_{\lambda n} \\ \phi_{\lambda n}(\omega) \\ \psi_{\lambda n}(\omega\nu) \end{bmatrix} = \begin{bmatrix} \dfrac{f^*(\lambda-n)}{\alpha(\lambda+i\epsilon)} \dfrac{g^*(n)}{\gamma(n+i\epsilon)} \\ \dfrac{g^*(n)\delta(\lambda-\omega-n)}{\gamma(\lambda-\omega+i\epsilon)} + \dfrac{f(\omega)\eta_{\lambda n}}{\gamma(\lambda-\omega+i\epsilon)} \\ \delta(\nu-n)\delta(\lambda-\omega-n) + \dfrac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\phi_{\lambda n}(\omega) \end{bmatrix} \tag{70}$$

where $0 \le n \le \lambda < \infty$, and

$$\alpha(z) = z - M_0 - \int_0^\infty \frac{f^*(\omega')f(\omega')}{\gamma(z-\omega'+i\epsilon)} , \qquad \gamma(z) = z - \mu_0 - \int_0^\infty \frac{g^*(\nu)g(\nu)}{z-\nu+i\epsilon} d\nu . \tag{71}$$

If there is a real value μ such that

$$\gamma(\mu)=0, \quad \gamma' = \left. \frac{\partial \gamma(z)}{\partial z} \right|_{z=\mu} , \tag{72}$$

there exist a two-particle one-parameter family:

$$|\tau\rangle = \begin{bmatrix} \eta_\tau \\ \phi_\tau(\omega) \\ \psi_\tau(\omega\nu) \end{bmatrix} = \begin{bmatrix} \dfrac{f^*(\tau-\mu)}{\sqrt{\gamma'}\alpha(\tau+i\epsilon)} \\ \dfrac{1}{\sqrt{\gamma'}}\delta(\tau-\mu-\omega) + \dfrac{f(\omega)}{\gamma(\tau-\omega+i\epsilon)}\eta_\tau \\ \dfrac{g(\nu)}{\tau-\omega-\nu+i\epsilon}\phi_\tau(\omega) \end{bmatrix} . \tag{73}$$

Note that λ and τ vary over ranges differing by μ so that

$$0 < \lambda, \quad (\tau-\mu) < \infty .$$

If there is a real value M such that

$$\alpha(M)=0, \quad \alpha' = \left. \frac{\partial \alpha(z)}{\partial z} \right|_{z=M} , \tag{74}$$

then there exists a discrete state

$$|M\rangle = \begin{bmatrix} \eta_0 \\ \phi_0(\omega) \\ \psi_0(\omega,\nu) \end{bmatrix} = \frac{1}{\sqrt{\alpha'}} \begin{bmatrix} 1 \\ \dfrac{f(\omega)}{\gamma(M-\omega)} \\ \dfrac{g(\nu)f(\omega)}{\gamma(M-\omega)(M-\omega-\nu)} \end{bmatrix} . \tag{75}$$

These states are (ideally) normalized. By a straightforward calculation they can be shown to be mutually orthogonal. We can also show them to be complete. The best way is to compute $\int\int d\omega' d\nu' \psi^*(\omega'\nu')\psi(\omega'\nu')$ etc. and to convert it into a contour integral. If there are zeros of $\gamma(z)$ they will compensate the one-parameter continuum and so on, and we may obtain

$$\begin{gathered} \langle M|M\rangle = 1, \quad \langle M|\tau\rangle = 0, \quad \langle M|\lambda n\rangle = 0 , \\ \langle \tau'|\tau\rangle = \delta(\tau'-\tau), \quad \langle \tau'|\lambda n\rangle = 0 , \\ \langle \lambda' n'|\lambda n\rangle = \delta(\lambda-\lambda')\delta(n-n') , \end{gathered} \tag{76}$$

and

$$\begin{gathered} \int\int \psi_{\lambda n}(\omega'\nu')\psi^*_{\lambda n}(\omega\nu)d\lambda\, dn + \int \psi_\tau(\omega'\nu')\psi^*_\tau(\omega\nu)d\tau + \psi_0(\omega'\nu')\psi^*_0(\omega\nu) = \delta(\omega-\omega')\delta(\nu-\nu') , \\ \int\int \psi_{\lambda n}(\omega'\nu')\phi^*_{\lambda n}(\omega)d\lambda\, dn + \int \psi_\tau(\omega'\nu')\phi^*_\tau(\omega)d\tau + \phi_0(\omega'\nu')\phi^*_0(\omega) = 0 , \\ \int\int \psi_{\lambda n}(\omega'\nu')\eta^*_{\lambda n}d\lambda\, dn + \int \psi_\tau(\omega'\nu')\eta^*_\tau d\tau + \psi_0(\omega'\nu')\eta^*_0 = 0 , \\ \int\int \phi_{\lambda n}(\omega')\phi^*_{\lambda n}(\omega)d\lambda\, dn + \int \phi_\tau(\omega')\phi^*_\tau(\omega)d\tau + \phi_0(\omega')\phi^*_0(\omega) = \delta(\omega'-\omega) , \\ \int\int \phi_{\lambda n}(\omega')\eta^*_{\lambda n}d\lambda\, dn + \int \phi_\tau(\omega')\eta^*_\tau d\tau + \phi_0(\omega')\eta^*_0 = 0 , \\ \int\int \eta_{\lambda n}\eta^*_{\lambda n}d\lambda\, dn + \int \eta_\tau\eta^*_\tau d\tau + \eta_0\eta^*_0 = 1 . \end{gathered} \tag{77}$$

VII. SCATTERING OF UNSTABLE PARTICLES

To study analytic continuations with complex branch cuts we choose M_0 and μ_0 sufficiently positive so that there is no real zero for $\gamma(z)$ or for $\alpha(z)$. Then the only states in $\mathcal{H}$ which are (ideal) eigenstates are $|\lambda n\rangle$ and these states are complete in the sense of (77). The S-matrix elements are

$$\langle \lambda n,\text{out}|\lambda' n',\text{in}\rangle=\delta(\lambda-\lambda')\{\delta(n-n')+2iT(n,n';\lambda)\}\ , \tag{78}$$

$$T(n,n';\lambda)=-\pi\left\{\alpha(\lambda+i\epsilon)\eta_{\lambda n}\eta_{\lambda n'}+\frac{g^*(n)g(n)}{\gamma(n+i\epsilon)}\delta(n-n')\right\}\ . \tag{79}$$

Both the S-matrix element and the T-matrix element considered as a function of λ can be viewed as analytic functions of (complex) energy z with a branch cut $0<z<\infty$. Since by hypotheses $\gamma(\zeta)$ has no real zero we would find a complex zero at μ_1 in the lower half plane as we deform the branch cut to the contour C of Fig. 1. This pole induces a branch cut in $T(n,n';\lambda)$ from μ_1 to infinity along a contour of our choice. So we can have, as illustrated in Fig. 4, the choice of the contours Γ_1, or $\Gamma_2+\Gamma_2'$, or $\Gamma_3+\Gamma_3'+\Gamma_3''$. For $\Gamma_2+\Gamma_2'$ we have the complex branch cut beginning at μ_1. For $\Gamma_3+\Gamma_3'+\Gamma_3''$ we have the complex branch cut beginning at μ_1 and the pole at M_1.

These analytic properties signal the possibility of analytic continuation of the space $\mathcal{H}$ into $\mathcal{G}$. For the contour Γ_1 we get the complete set of states $|z,\zeta\rangle$:

$$|z,\zeta\rangle=\left[\begin{array}{c}\dfrac{f^*(z^*-\zeta^*)g^*(\zeta^*)}{\alpha(z+i\epsilon)\gamma(\zeta+i\epsilon)}\\ \dfrac{g^*(\zeta^*)\delta(z-\zeta-\xi)}{\gamma(z-\xi+i\epsilon)}+\dfrac{f(\xi)f^*(z^*-\zeta^*)g^*(\zeta^*)}{\alpha(z+i\epsilon)\gamma(\zeta+i\epsilon)\gamma(z-\xi+i\epsilon)}\\ \delta(\zeta-\nu)\delta(z-\xi-\nu)+\dfrac{g(\nu)}{z-\xi-\nu+i\epsilon}\left\{\dfrac{g^*(\zeta^*)\delta(z-\zeta-\xi)}{\gamma(z-\xi+i\epsilon)}+\dfrac{f(\xi)f^*(z^*-\zeta^*)g^*(\zeta^*)}{\alpha(z+i\epsilon)\gamma(\zeta+i\epsilon)\gamma(z-\xi+i\epsilon)}\right\}\end{array}\right]\ , \tag{80}$$

where z lies on the contour Γ_1 and we may choose $\xi+\nu$, ζ, and ν also to lie on this contour. By a lengthy but straightforward calculation using the conversion of open contour integrals into closed contour integrals we can show that (80) constitutes a complete (ideal) orthonormal system. Neither the zeros of α nor of γ are in the complex plane cut along Γ_1, and consequently the closed contour integrals do not enclose any of the related singularities.

If on the other hand we choose the contour Γ_2 we have crossed the branch point at μ_1. This branch point "snags" the closed contour over which we integrate, and completeness is restored only by including the generalized (ideal) states

$$|y\rangle=\left[\begin{array}{c}\dfrac{f^*(y^*-\mu_1)}{\sqrt{\gamma_1'}\alpha(y+i\epsilon)}\\ \dfrac{1}{\sqrt{\gamma_1'}}\delta(y-\mu_1-\xi)+\dfrac{f(\xi)}{\gamma(y-\xi+i\epsilon)}\dfrac{f^*(y^*-\mu_1)}{\sqrt{\gamma_1'}\alpha(y+i\epsilon)}\\ \dfrac{g(\nu)}{y-\xi-\nu+i\epsilon}\left\{\dfrac{1}{\sqrt{\gamma_1'}}\delta(y-\mu_1-\xi)+\dfrac{f(\xi)}{\gamma(y-\xi+i\epsilon)}\dfrac{f^*(y^*-\mu_1)}{\sqrt{\gamma_1'}\alpha(y+i\epsilon)}\right\}\end{array}\right] \tag{81}$$

with

$$\gamma_1'=\left.\frac{\partial\gamma(\zeta)}{\partial\zeta}\right|_{\zeta=\mu_1}\ . \tag{82}$$

Here y and $\xi+\mu_1$ are along Γ_2 and ξ lies on Γ_2'. Γ_2' is obtained from Γ_2 by displacing it by the fixed complex number μ'. The states $|y\rangle$ and $|z,\zeta\rangle$ in Eq. (81) and Eq. (80) now form a complete set. The contour Γ_2' is the spectrum of the "unstable" particle B (which has now become a "stable particle"!) scattering a θ particle with energy ξ. This scattering also obeys in addition to the generalized unitarity relation along Γ_2',

$$T(\zeta,\zeta';z)-T^*(\zeta^*,\zeta'^*;z^*)=\int_{\Gamma_2}d\zeta''T^*(\zeta''^*,\zeta^*;z^*)T(\zeta'',\zeta';z)\ , \tag{83}$$

the unitarity relation

$$T(\xi)-T^*(\xi^*)=T^*(\xi^*)T(\xi) \tag{84}$$

along Γ_2'. There is a technical point here. For the definition of the continued wave functions, the contour Γ_2' is chosen through the "parallel transport" prescription stated above. However, for the continued unitarity relation, it can be shown that it is no longer necessary to be confined to the parallel transported contour Γ_2'.

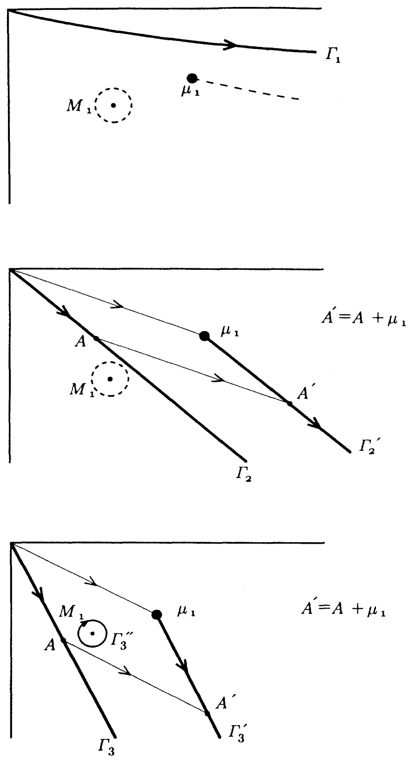

FIG. 4. Spectra and contours for the cascade model with $M_0 \gg \mu_0 \gg 0$.

In the context of the continuation of wave functions, further deformation of the contour does alter the states $|\tau\rangle$. When z and $\xi+\zeta$ are along the contour Γ_3, τ is along $\Gamma_3'=\Gamma_3+\mu_1$, see Fig. 4. It could also uncover the discrete state $|M_1\rangle$ with

$$|M_1\rangle=\frac{1}{\sqrt{\alpha_1'}}\begin{bmatrix} 1 \\ \dfrac{f(\xi)}{\gamma(M_1-\xi)} \\ \dfrac{g(\nu)f(\xi)}{\gamma(M_1-\xi)(M_1-\xi-\nu)} \end{bmatrix}, \tag{85}$$

which then needs to be included in the completeness relation.

Unitarity relations for the T matrix are energy-local relations [26] and as such do *not mix the unstable particle scattering and stable particle scattering.*

VIII. EXTENDED SPACES AND SEMIGROUP OF TIME EVOLUTION

We have so far formulated the passage from $\mathcal{H}$ to $\mathcal{G}$ as a correspondence between dense sets in $\mathcal{H}$ and $\mathcal{G}$. With this understanding the basis in $\mathcal{G}$ is "the same" as in $\mathcal{H}$. Therefore when we know that a pure exponential-decay time dependence is not possible for states in $\mathcal{H}$ (with a nonnegative spectrum for the total Hamiltonian) the same should also be obtained for corresponding states in $\mathcal{G}$. Furthermore since the time evolution (and regression) are implemented by unitary family of linear operators realizing the time translation *group,* the same would also be true of the states in $\mathcal{G}$. *A pure exponential decay* or a Stieltjes integral over damped exponentials would then *not* be *possible with states obtained by analytic continuation of physical states.*

One can, however, ask what property has to be relaxed to realize an extended space $\tilde{\mathcal{H}}$ and its corresponding continuation $\tilde{\mathcal{G}}$ so that a semigroup of time evolutions can be realized. These semigroups would, generally, be realized by an isometry which is not, however, unitary. After all, an unrestricted Breit-Wigner resonance [27] with its Lorentz line shape does correspond to pure exponential decay (for positive time). We need to relax the positivity of energy and define states with all possible values of energy. In this case we can realize semigroups of time evolution [28].

Let $\psi(\lambda)$ be a vector in a Hilbert space $\mathcal{H}$:

$$\int_0^\infty |\psi(\lambda)|^2 d\lambda=1, \quad \psi(\lambda)=0, \ \lambda<0 \ . \tag{86}$$

We enlarge it into $\mathbb{H}_\pm$, where $\Psi(\lambda)$ is defined for negative values of λ also, in such a fashion that it is analytic in a half-plane:

$$\Psi_\pm(z)=\frac{\mp}{2\pi i}\int_0^\infty d\lambda \frac{1}{\lambda-z\pm i\epsilon}\psi(\lambda) \ . \tag{87}$$

These functions are analytic in the two half-planes and their sum is equal to $\psi(\lambda)$:

$$\psi(\lambda)=\Psi_+(\lambda)+\Psi_-(\lambda) \ . \tag{88}$$

On $\Psi_+(\lambda)$ the time evolution for positive times is realized by a contractive semigroup:

$$\begin{aligned}\Psi_+(z;t)&=T_+(t)\Psi_+(z)\\ &=-\frac{1}{2\pi i}\int_0^\infty d\lambda\, e^{-i\lambda t}\frac{1}{\lambda-z+i\epsilon}\psi(\lambda) \ ,\end{aligned} \tag{89}$$

$$\begin{aligned}&T_+(t_1)T_+(t_2)=T_+(t_1+t_2), \quad t_1,t_2>0 \ ,\\ &T_+(t)=0, \quad t<0, \quad T_+(0+)=1 \ .\end{aligned} \tag{90}$$

By the converse of a theorem of Titchmarsh,

$$\tilde{\Psi}_\pm(\tau)\equiv\int_{-\infty}^\infty \Psi_\pm(\lambda)e^{-i\lambda\tau}d\lambda=0, \quad \pm\tau<0 \ . \tag{91}$$

Then

$$T_+(t)\tilde{\Psi}_+(\tau)=\tilde{\Psi}_+(\tau+t), \quad t>-\tau \ , \tag{92}$$

$$T_+(t)\tilde{\Psi}_+(\tau)=0, \quad t<-\tau \ . \tag{93}$$

Thus a semigroup evolution obtains on the half-plane analytic function $\Psi_+(\lambda)$. A similar conclusion obtains the backward tracing of $\Psi_-(\lambda)$.

Given $\Psi_+(\lambda)$ we can continue it to a vector $\Psi_+(z)$ in $\tilde{\mathcal{J}}$ and the semigroup acts in $\tilde{\mathcal{J}}$ in the same fashion.

The functions $\Psi_+(z)$ are analytic in the half-plane by construction. They constitute the Hardy class of functions [29] which are square-integrable along Rez for any negative imaginary part. *None* of this class *is a physical state* (expressible as linear combinations of states of nonnegative total energy). But many familiar unphysical states such as the Breit-Wigner function

$$\Psi_+(\lambda)=\frac{\Gamma}{\sqrt{\pi}}\frac{1}{\lambda-\lambda_0+(i/2)\Gamma} \tag{94}$$

are included in this Hardy class. In addition to such a single pole we could also have multiple poles and/or branch points. To obtain them we can use a perfectly physical state obtained as a linear combination of states like (35), (80), or (81) and carry out the linear maps (87) into the two Hardy class functions.

IX. REDUNDANT AND DISCRETE STATES IN THE CONTINUUM

For the model discussed in the preceding section, when the contour Γ passes through $z=M_1$ the continuum wave function (45) exhibits singularity at $z=M_1$, a complex eigenvalue. There is, when the contour justifies it, a discrete eigenstate with eigenvalue M_1. The scattering amplitudes also have singularities (poles) at the same point. People often take the poles of the scattering amplitude to correspond to unstable particles. It has, however, been known [30] that poles appear in the S matrix (or the scattering amplitude) which do not correspond to discrete eigenstates of the Hamiltonian in $\mathcal{H}$. This is true of the (repulsive) exponential potential; a number of phase-equivalent potentials [31] have been known for some of which the S-matrix poles are bound (discrete) states while for others it is not. The circumstances obtain in the context of the Lee model and other such models. In the Lee model this corresponds to the distinction between the zeros of the denominator function $\alpha(z)$ and the poles of the form factor $f^*(z^*)f(z)$. Nor are these redundant singularities restricted to being isolated poles; for example the S-wave Yukawa potentials give a branch cut [32], but with no continuum of (ideal) states entering the description. In all such cases the redundant singularities of the S matrix do not correspond to states entering the complete set of states.

A similar situation is obtained in the case of analytic continuation of the vector space $\mathcal{H}$ to $\mathcal{G}$. Consider the Lee model wave functions (48). They would develop singularities not connected with the spectrum of the Hamiltonian in $\mathcal{G}$ if the form factor $g(z)$ develop singularities. But these singularities do not give any contributions to the completeness identity since in these calculations we obtain the contour integrals involving $1/\alpha(z)$. The poles in $g^*(z^*)g(z)$ are matched by corresponding terms in $\alpha(z)$ and they disappear from the contour integral. As the contour Γ smoothly deforms itself, it is not snagged by singularities of $g^*(z^*)g(z)$. The same situation is obtained for the Cascade model; only the zeros of $\alpha(z)$ contribute to discrete state and only the branch cuts in $\gamma(\zeta)$ contribute to the scattering states involving an unstable particle.

A related phenomenon is that of states which contribute to the complete set of states which are located in the continuum but which do not contribute any singularity for the S matrix [24]. This occurs when a zero of $\alpha(z)$ coincides with a zero of the form factor $g(z)$ as far as the Lee model is concerned. The spectrum is degenerate at this point M, $\alpha(M)=0$ with a discrete state in $\mathcal{H}$ and an ideal state belonging to the continuum. In analytic continuation we can have complex zeros of $\alpha(z)$ where the scattering amplitude vanishes; nevertheless the complete set of states include these states. They also enter the computation of survival amplitudes (53).

For the Lee model we choose a form factor $g^*(z^*)g(z)$ and an $\alpha(z)$ such that

$$\alpha(M_1)=0,\quad g^*(z^*)g(z)\sim(z-M_1)^2G(z) \tag{95}$$

for some complex M_1. Then the scattering amplitude vanishes at this point:

$$T(z)\sim(z-M_1)t(z)\ . \tag{96}$$

The (ideal) state of this point is a "plane wave"

$$\tilde{\eta}_1=0,\quad \phi_1(z)=\delta(z-M_1)+\text{nonsingular terms}\ , \tag{97}$$

(with no asymptoting diverging wave) which is degenerate in energy with the proper state in $\mathcal{G}$ with

$$\eta_1=[\alpha'(M_1)]^{-1/2},\quad \phi_1(z)=\frac{g(z)\eta_1}{M_1-z}\ . \tag{98}$$

In a similar manner for the Cascade model, if the form factors have zeros along the cut beginning at the branch point μ_1 then the scattering amplitude vanishes at these points on the branch cut, but the (ideal) states $|z\rangle$ in (81) beginning at μ_1 exist and contribute to the completeness (and to the survival amplitude for the unstable A particle).

Thus the S-matrix singularities and the spectrum of states are not necessarily in correspondence.

Along with redundant poles we could also have redundant branch cuts from the "geometry of the potential." There will be no contribution from these to the completeness identity. Such branch cuts are familiar as the left hand (and the short- and circle-) cuts in partial-wave dispersion relations.

X. DISCUSSION: TWO CHOICES FOR UNSTABLE PARTICLE STATES

In our study of generalized quantum state spaces we have given exposition to analytic continuation of state spaces, correspondence between dense sets of states in $\mathcal{H}$ and in $\mathcal{G}$. For analytic Hamiltonians the spectrum can be "analytically continued" in $\mathcal{G}$. The resolution of unity embodied in the completeness identity has alternate expressions. Incidentally this is an example of reducible representations of the (time) translation group having different decompositions in which no component of one decomposition is equivalent to any component of the other one. The notions of discrete states, of continuous spectra, of "in" and "out" states and exact expressions for the (ideal) states are all obtained for these generalized spaces.

There are two views that one could take about what is an unstable particle. One is that it is a physical state of the system which is normalizable and which ceases to exist as a discrete eigenstate of the total Hamiltonian. If $|M\rangle$ denotes this normalized state, the survival amplitude is

$$A(t)=\langle M|e^{-iHt}|M\rangle$$
$$=\int d\lambda\, e^{-i\lambda t}\langle M|\lambda\rangle\langle\lambda|M\rangle \ . \qquad (99)$$

This amplitude cannot be ever strictly exponential in t and is bounded in absolute value by unity for all t, positive or negative. It exhibits a Khalfin regime where it has an inverse power dependence and a Zeno regime where the departure of its absolute value from unity is quadratic in t. But for much of the intermediate region it is approximately exponential in $|t|$. One of the drawbacks of this picture of an unstable particle is that its survival amplitude does not furnish a representation of the time translation group or semigroup. The unstable particle so defined is not "autonomous." It *ages*.

The other picture of the unstable particle is as a discrete state in the generalized space $\mathcal{G}$ and as such having a pure exponential dependence. The time evolutions form a semigroup (for $t>0$) the absolute value steadily decreasing exponentially. Such a state cannot have a counterpart physical state in $\mathcal{H}$. For negative values of t the state tends to blow up. If we start from any state in $\mathcal{H}$ which can be continued into $\mathcal{G}$, the result so obtained would never be a pure discrete decaying state, but that plus remnants of a continuum. We could extend $\mathcal{H}$ to $\tilde{\mathcal{H}}$ by relaxing the spectral condition $H\geq 0$ and obtain a state in $\mathbb{H}_{\pm}$ as in (87); then we could obtain a semigroup evolution law (92). While this choice appears to be elegant, it is deduced at the expense of giving up the lower boundedness of the energy spectrum. We consider it to be the less desirable choice.

Anyhow, the distinction between these two choices for the "unstable particle" becomes even more pronounced when we consider two distinct lifetimes for two unstable particles coupled to common continuum (ideal) states. Such a situation is obtained for positronium coupled to two- and three-photon continuum states: ortho- and parapositronium have lifetimes which differ by three orders of magnitude [33]. If charge conjugation invariance is invoked one of them (*orthopositronium*) decays only into two photons and the other into three photons. The positronium states may be classified by their charge conjugation properties. And the two sets of states would be orthogonal.

A more interesting case is provided by the K^0-$\bar{K}^0$ system [34] (and also by B^0-$\bar{B}^0$ and D^0-$\bar{D}^0$ systems) where there are two different lifetimes which differ by two orders of magnitude and which share common decay channels. These two "particles" K_1 and K_2 would have been eigenstates of combined inversion (CP) if that were a symmetry operation. But since CP is not conserved in the phenomenological Lee-Oehme-Yang [35] (see also Wu-Yang [36]) generalization of the Weisskopf-Wigner [37] phenomenological non-Hermitian Hamiltonian model the physical states K_S and K_L which are the "decaying particles" are not orthogonal. Khalfin [38] had shown that the Lee-Oehme-Yang version predictions are not consistent with the non-negativity of Hamiltonian if CP is not conserved; we have verified this by detailed calculations [13]. Experimental results on kaon decay with appropriate phenomenological definitions of K_S, K_L have determined that $|\langle K_S|K_L\rangle|\sim 10^{-3}$ rather then zero.

On the other hand if we identify the decaying particles K_S and K_L with two distinct discrete states in $\mathcal{G}$ they will be at different energy values; since K_S and K_L differ in lifetime they have different imaginary parts, while $K^0,\bar{K}^0$ oscillations [39] show that they have different real parts. Hence in $\mathcal{G}$, K_S, and K_L are orthogonal and have strictly exponential time evolutions. It is not exactly news that a discrete complex energy eigenvalue state cannot be obtained from a Hamiltonian in $\mathcal{H}$ bounded from below but can correspond, at most, to a state in the extended space $\tilde{\mathcal{H}}$. The additional spurious contributions from $-\infty$ to 0 for K_S and K_L states in $\tilde{\mathcal{H}}$ would cancel the overlap of K_S and K_L states in $\mathcal{H}$. Recently the question of whether $\langle K_S|K_L\rangle$ should vanish or not have been reexamined by Y. Yamaguchi [14] and by Tasaki, Petrosky, and Prigogine [15]. In our view there is no discrepancy between the expectation of the vanishing scalar product found by these authors and the nonzero value found in measurements. This matter will be analyzed in detail elsewhere [40].

Finally, we observe that the spaces $\mathcal{H}$ and $\mathcal{G}$ that we have used are distinct spaces though there is one-to-one correspondence between dense sets of analytic vectors in $\mathcal{H}$ and $\mathcal{G}$. This correspondence can be implemented by an intertwining operator $V:\mathcal{H}\rightarrow\mathcal{G}$ with its inverse $V^{-1}:\mathcal{G}\rightarrow\mathcal{H}$ given by the formal Stieltjes integral

$$V(z,x)=\int d\alpha\, \psi_\alpha(z)\psi_\alpha^*(x) \ ,$$
$$V^{-1}(z,x)=\int d\alpha\, \psi_\alpha(x)\psi_\alpha^*(z^*)=\int d\alpha\, \psi_\alpha(x)\tilde{\psi}_\alpha(z) \ ,$$

where $\{\psi_\alpha(x)\}$ is an analytic basis in $\mathcal{H}$ and $\{\psi_\alpha(z)\}$ its counterpart in $\mathcal{G}$. Any analytic operator, including the Hamiltonian in $\mathcal{H}$ has the counterpart in $\mathcal{G}$ defined by

$$A\rightarrow VAV^{-1} \ .$$

These operators V, V^{-1} are intertwinning between the spaces $\mathcal{H}$ and $\mathcal{G}$.

ACKNOWLEDGMENTS

This work was supported by U.S. Department of Energy Grant No. DOE-FG05-85ER-40-200. We thank G. Bhamathi for her participation in the present work and for valuable discussions. Also we wish to thank John Dollard and Charles Radin for comments regarding the work of Wigner and von Neumann and of Simon. Lastly we thank Tom Kibble for his critical comments on the present work.

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[40] Denote

$$|K_L\rangle = p_L|K^0\rangle + q_L|\bar{K}^0\rangle + \text{continuum} ,$$

$$|K_S\rangle = p_S|K^0\rangle + q_S|\bar{K}^0\rangle + \text{continuum} .$$

In the approximation of setting $p_L = p_S \equiv p$, $q_L = q_S \equiv q$, and ignoring the continuum contributions, the experimental quantity

$$\text{“}\langle K_S|K_L\rangle\text{”} \approx \langle K^0|K_S\rangle^* \langle K^0|K_L\rangle + \langle \bar{K}^0|K_S\rangle^* \langle \bar{K}^0|K_L\rangle \propto |p|^2 - |q|^2 ,$$

while in the same approximation the scalar product in the generalized vector space

$$\langle \tilde{K}_S|K_L\rangle \approx pq - qp = 0 .$$

For the exact treatment where $p_L \neq p_S$, $q_L \neq q_S$, and the continuum contributions are taken into account, see Charles B. Chiu and E. C. G. Sudarshan, "Theory of the neutral kaon system", University of Texas Report No. DOE-40200-296, 1992 (unpublished).