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Some Consequences of a Piecewise-Analytic Scattering Amplitude

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In the indefinite-metric quantum field theory formulated by one of the authors and in related work the scattering amplitude does not satisfy the usual Mandelstam analyticity. It becomes piecewise analytic at the threshold for the production of at least one of the negative-metric particles of the theory, i.e., the amplitude above the threshold is not the analytic continuation of the amplitude. Some consequences of such a piecewise-analytic scattering amplitude are investigated. The physical two-body scattering amplitude $G(s,t)$ is taken to be of the form $G(s,t)=F(s,t)+h_1(s,t)\theta(s-s_0)+h_2(s,t)\theta(t-t_0)+h_3(s,t)\theta(u-u_0)$, where $F(s,t)$ and the $h_i(s,t)$ are analytic functions of s and t with the negative-metric thresholds occurring at $s=s_0$, $t=t_0$, and $u=u_0$ in the s, t, and u channels, respectively. The modified forms of the Pomeranchuk theorem, dispersion relations, and finite-energy sum rules due to this general form of piecewise analyticity are derived and the interpretation of experimental results in terms of them are discussed. In particular, the modified forward dispersion relations for π^+p and π^-p scattering differ from the normal forms by a function $\xi(\nu)$ which depends on the piecewise-analytic contributions $h_i(s,t)$ above, where ν is the laboratory momentum of the pion. The forward dispersion relations for the symmetric and antisymmetric combinations of the real part of the π^+p and π^-p scattering amplitudes $D^+(\nu)$ and $D^-(\nu)$, respectively, are tested. The best fits to the latest total cross-section data for $\pi^\pm p$ scattering from 8 to 65 GeV/c which do not satisfy the Pomeranchuk theorem are used. No test for $D^-(\nu)$ which must be twice subtracted is possible since it depends strongly on the πN coupling constant f^2 which is itself determined by dispersion relations. The result for $D^+(\nu)$ allows for a violation, but the evidence is not compelling.

I. INTRODUCTION

The question of introducing negative-metric particles into quantum field theory in order to make it a convergent theory[1] has recently received renewed interest.[2-4] In particular, the question of how unitarity is to be maintained has been discussed extensively. One of the authors[3] has proposed treating the states containing negative-metric particles as "shadow states" which have only principal-value propagators and thus do not contribute to the unitarity sum. Lee[4] has proposed an alternate prescription which he hoped would serve the same purpose. The immediate consequence of the use of such states not contributing to the unitarity sum is that one requires the relaxation of the normal analyticity requirements that have always been assumed[5] for the two-particle elastic scattering amplitude, $G(s, t)$, in terms of the energy squared s and momentum transfer squared $-t$ in the c.m. system. Instead $G(s, t)$ becomes piecewise analytic. That is, it is the boundary value of an analytic function of s and t for real physical values of s and t in a finite domain below the threshold for the production of at least one negative-metric particle and the boundary value of a different analytic function $G_\Delta(s, t)$ above that threshold. The important characteristic is that $G_\Delta(s, t)$ is *not* an analytic continuation of $G(s, t)$.

The circumstances in which shadow states and piecewise analyticity come about are as follows: Local polynomial interactions of fields which are linear operators in a space of positive definite metric are mathematically meaningless because of intrinsic divergences. Any attempt to impose fixed geometric cutoffs in a manifestly covariant manner makes the theory also inconsistent. The natural way out of this dilemma is to make use of spaces with an indefinite metric. To restore the quantum-mechanical probability interpretation in such a theory one must allow this vector space to have a positive-metric subspace which has the status of a physical-state space. The other "states" contribute to the dynamics but are denied the status of physical states; they are the so-called shadow states. The architecture of the theory should be such as to guarantee conservation of probability among the physical states by themselves. The notions of virtual particles and exchanges continue to be valid in this theory, but the "ghost" quanta of fields with negative-metric single-particle states cannot occur in any physical state. Consequently the potential appearance of a virtual ghost on the mass shell is forestalled by the structure of the theory. This automatically leads to the transition amplitudes in different physical regions being boundary values of different analytic functions. Explicit mathematical demonstration of this is given in several recent papers.[3]

We are interested, insofar as this paper is concerned, in searching for possible experimental evidence for or against such a behavior. If our ideas of local polynomial interactions between quantized fields are correct, then it seems to be the inescapable conclusion that such piecewise analyticity must be obtained. Hence the present undertaking.

Here we consider the general elastic scattering process described by four momenta $p_1+p_2 \to p_3+p_4$, where $p_1^2=p_3^2=m_a^2$ and $p_2^2=p_4^2=m_b^2$ with $s=(p_1+p_2)^2=(p_3+p_4)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$, and $s+t+u=2m_a^2+2m_b^2$. We shall investigate some consequences of assuming that the elastic scattering amplitude $G(s, t)$ is piecewise analytic. In particular, for simplicity we consider the case of a single negative-metric threshold (NMT) for each of the s, t, and u channels. That is, there is at most only one break in $G(s, t)$ in each channel. This is contrary to the actual state of affairs in a quantum field theory with negative metric, but would illustrate the departure from analyticity in its essential aspects. We decompose $G(s, t)$ as follows:

$$G(s, t)=F(s, t)+\Delta(s, t), \tag{1.1}$$

where $F(s, t)$ has the usual analyticity attributed to the scattering amplitude, i.e., it is analytic in the cut s plane. $\Delta(s, t)$ is the piecewise-analytic part which is zero below the NMT but nonzero above it in the physical scattering region. Explicitly, we take it to have the form

$$\Delta(s, t)=h_1(s, t)\theta(s-s_0)+h_2(s, t)\theta(t-t_0) + h_3(s, t)\theta(u-u_0), \tag{1.2}$$

where s_0, t_0, and u_0 correspond to NMT's in the s, t, and u channels, respectively. The domains of $G(s, t)$ are conveniently represented in the Kibble-Mandelstam s, t, and u plot of Fig. 1. We see

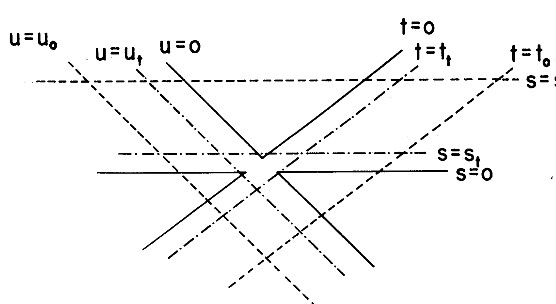

FIG. 1. Kibble-Mandelstam s, t, and u plot showing the NMT triangle defined by NMT's in the s, t, and u channel.

that $G(s, t)=F(s, t)$ inside the NMT triangle defined by s_0, t_0, and u_0. In the domain outside of the triangle, $F(s, t)$ corrected by the $h_i(s, t)$ gives the physical scattering amplitude $G(s, t)$.

In soluble field-theoretic models[6] involving "shadow state" contributions the form of $h_i(s, t)$ can be computed. In general it is an analytic function of s and t with a structure depending on the physical and "shadow state" scattering contributions.

The function $F(s, t)$ is the analytic continuation of $G(s, t)$ above the NMT. We assume in the following that $F(s, t)$ is the analytic function that has the analyticity and crossing properties of the Mandelstam representation, a representation in terms of Regge poles and satisfies the dispersion relations usually attributed to $G(s, t)$.

In the following sections we utilize the analyticity of $F(s, t)$ instead of $G(s, t)$ and thereby obtain corrections to forward dispersion relations in terms of $h(s, t)$ in Sec. II. There we discuss the modification of the Pomeranchuk theorem implicit in our model in terms of $h(s, t=0)$. Then we introduce a modified forward dispersion relation and investigate the experimental deviation from the normal dispersion relations which assume an analytic $G(s, t)$ for the case of $\pi^{\pm}p$ scattering. In Sec. III we apply our model to the finite-energy sum rules (FESR) and interpret their apparent violation in terms of the onset of a NMT. Finally, we summarize our results and present our conclusions in Sec. IV.

II. MODIFICATION OF FORWARD DISPERSION RELATIONS BY PIECEWISE ANALYTICITY

If we give up analyticity of the scattering amplitude then one of the first difficulties we must face is the loss of the standard forward dispersion relation.[7] Furthermore, since we have in mind a quantum field theory with indefinite metric it is not clear that the standard proofs of analyticity, polynomial boundedness, and consequent dispersion relations apply.[8] Hence we investigate the modification of the forward dispersion relation due to the piecewise analyticity introduced in our model by $\Delta(s, t)$ of (1.2) above. The resulting form which depends on the physical amplitude $G(s, t)$ and the correction $h(s, t)$ introduced above is then used to obtain a modified form of the Pomeranchuk theorem. Finally, in this section, we investigate the contribution of the correction $h(s, t)$ from the modified forward dispersion relations using the experimental values of $G(s, t)$ with $t=0$.

A. The Modified Forward Dispersion Relation

We first consider the case of chargeless and spinless pion-nucleon scattering for the sake of simplicity. With the notation introduced above, we take $m_a{}^2=\mu^2$ for the pion mass and $m_b{}^2=M^2$ for the nucleon. The conventional variable for the fixed-t dispersion relation is $\nu=p_1\cdot p_2/M$. The analytic function, $F(\nu,t)$, coincides with the physical amplitude, $G(\nu,t)$, below the NMT, $\nu=\nu_0$, and is the analytic continuation of $G(\nu,t)$ above $\nu=\nu_0$. The NMT is expected to be some distance above the elastic threshold and likely it lies beyond the well-studied resonance region, i.e., beyond $s=4.0\ \mathrm{GeV}^2$.

The unsubtracted dispersion relation for $F(\nu,0)\equiv F(\nu)$ corresponding to this case has the form

$$\mathrm{Re}F(\nu)=\frac{\gamma}{(\nu_B)^2-\nu^2}+\frac{1}{\pi}⨍_{\nu_s}^{\infty}d\nu'\frac{\mathrm{Im}F(\nu')}{\nu'-\nu}+\frac{1}{\pi}⨍_{-\infty}^{\nu_u}d\nu'\frac{\mathrm{Im}F(\nu')}{\nu'-\nu}\ , \tag{2.1a}$$

where $\nu_B=-\mu^2/2M$, $\gamma M/\mu^2$ is the residue of the nucleon pole at $\nu=\pm\nu_B$, ν_s and ν_u refer to the elastic thresholds in the s and u channels, respectively ($\nu_s=-\nu_u=\mu$), and the slash on the integral sign signifies that the principal-value integral is to be taken. The physical amplitude $G(\nu)$ is expressed in terms of $F(\nu)$ using (1.2). We make the simplifying choice that $h_1(s,t)$ and $h_3(s,t)$ are such as to yield the following expression in accordance with crossing symmetry:

$$G(\nu)=F(\nu)+h(\nu)[\theta(\nu-\nu_{s0})+\theta(-\nu+\nu_{u0})]\ , \tag{2.2a}$$

where ν_{s0} and ν_{u0} correspond to NMT's in the s and u channels, respectively. We will take $\nu_{s0}=-\nu_{u0}=\nu_0$ in the following. (We have assumed that the t-channel NMT is beyond the physical t-channel threshold.) Using this relation in (2.1a) we arrive at the following modified forward dispersion relation for $G(\nu)$:

$$\mathrm{Re}G(\nu)=\mathrm{Re}h(\nu)[\theta(\nu-\nu_0)+\theta(-\nu-\nu_0)]+\frac{\gamma}{\nu_B{}^2-\nu^2}+\frac{1}{\pi}\left(⨍_{\nu_s}^{\infty}+⨍_{-\infty}^{\nu_u}\right)\frac{\mathrm{Im}G(\nu')}{\nu'-\nu}\,d\nu'-\frac{1}{\pi}\left(⨍_{\nu_0}^{\infty}+⨍_{-\infty}^{-\nu_0}\right)\frac{\mathrm{Im}h(\nu')}{\nu'-\nu}d\nu'\ . \tag{2.1b}$$

Equation (2.1b) clearly differs from the normal form of the dispersion relation everywhere. In the general case that $\mathrm{Re}h(\nu)$ is non-negligible then it comes into play in the physical region for $\nu>\nu_0$, i.e., above the NMT. The effect of $h(\nu)$ would modify the results of the Pomeranchuk theorem and possibly be detectable in tests of forward dispersion relations and finite-energy sum rules at sufficiently high energies.

B. Modification of the Pomeranchuk Theorem

The original demonstration of the equality of the total interaction cross sections for particles and antiparticles at high energies by Pomeranchuk[9] was based on the forward dispersion relations – twice subtracted – and the exponential falloff of the strong interactions between hadrons for large distances. The latter forbids the logarithmic increase of the ratio of the real to the imaginary part of the amplitude with energy in the forward direction.

We consider $\pi^\pm$-proton forward scattering for concreteness. We denote the physical amplitudes by $G_+\equiv G(\pi^+p\to\pi^+p)$ and $G_-\equiv G(\pi^-p\to\pi^-p)$ with $F_\pm(\nu)$ referring to the appropriate analytic part and $h_\pm(\nu)$ the appropriate correction. Using the notation of Sec. IIA we obtain the unsubtracted dispersion relation for $F_\pm(\nu)$,

$$\mathrm{Re}F_\pm(\nu)=\frac{f^2\mu^2}{m\nu_B}\,\frac{1}{\nu_B\pm\nu}+\frac{1}{4\pi^2}⨍_{\mu}^{\infty}d\nu'\,q'\left(\frac{\Sigma_+(\nu')}{\nu'\mp\nu}+\frac{\Sigma_-(\nu')}{\nu'\pm\nu}\right), \tag{2.2b}$$

where $\Sigma_\pm(\nu)=\sigma_\pm(\nu)-4\pi\mathrm{Im}h_\pm(\nu)/q$ and $\sigma_\pm$ refers to the total cross section for the scattering of $\pi^\pm$ off of protons. The analytic contribution $F_\pm(\nu)$ corresponds to the analytic continuation of the local behavior – the behavior below the first NMT – of the scattering amplitude. Thus characteristics of the local $G_\pm(\nu)$ extrapolated to asymptotic ν would hold for $F_\pm(\nu)$. On this basis, at asymptotic energies, $\nu\to\infty$, we take $\Sigma_\pm(\nu)=4\pi\mathrm{Im}F_\pm(\nu)/q$ to be constant with $F_\pm(\nu)$ having the property of exponential falloff of the strong interaction at large distances, i.e., $\mathrm{Re}F_\pm(\nu)/\mathrm{Im}F_\pm(\nu)$ cannot have a $\ln\nu$ increase.

Under the assumption of constant asymptotic $\Sigma_\pm$, two more subtractions must be included in (2.2b) to make it convergent. The twice-subtracted form becomes

$$\mathrm{Re}F_\pm(\nu)=\tfrac{1}{2}(1\pm\nu/\mu)D_+(\mu)+\tfrac{1}{2}(1\mp\nu/\mu)D_-(\mu)+\frac{f^2\mu^2}{m\nu_B}\,\frac{q^2}{(\nu_B{}^2-\mu^2)(\nu_B\pm\nu)}+\frac{q^2}{4\pi^2}⨍_{\mu}^{\infty}\frac{d\nu'}{q'}\left(\frac{\Sigma_+(\nu')}{\nu'\mp\nu}+\frac{\Sigma_-(\nu')}{\nu'\pm\nu}\right), \tag{2.3}$$

where $D_\pm\equiv\mathrm{Re}G_\pm$.

Using the assumed asymptotic behavior of $F_\pm(\nu)$ and (2.3), Pomeranchuk's original proof[9] is applicable, but now produces the result that

$$\Sigma_-(\infty)=\Sigma_+(\infty) \tag{2.4a}$$

or

$$\sigma_+(\infty)-\sigma_-(\infty)=\eta_-(\infty)-\eta_+(\infty)\,, \tag{2.4b}$$

where

$$\eta^\pm(\infty)=\lim_{\nu\to\infty}\frac{4\pi}{q}\,\mathrm{Im}h^\pm(\nu).$$

Thus, the difference between the physical-particle and -antiparticle cross sections is determined by the difference between the particle and antiparticle NMT corrections. If $\eta_-(\infty)=\eta_+(\infty)$, then the normal Pomeranchuk theorem holds. However, their inequality provides an alternative mechanism for the possible violation[10] of the Pomeranchuk theorem that may be indicated by the recent Serpukhov experiments[11] without a logarithmically increasing $\mathrm{Re}G(\nu)/\mathrm{Im}G(\nu)$ as $\nu\to\infty$.

C. High-Energy Tests of Forward Dispersion Relations

The dispersion relations for the symmetric and antisymmetric combinations of the $\pi^\pm$-proton laboratory amplitudes, i.e., $G^\pm\equiv\frac{1}{2}[G_-(\nu)\pm G_+(\nu)]$, which are usually employed assume a modified form based on our results of the previous sections. Using Eq. (2.2b) we find that the corresponding dispersion relation for $F^+(\nu)$ can be written with one subtraction while that for $F^-(\nu)$ does not require any subtractions because of the modified Pomeranchuk theorem. We find that the once-subtracted dispersion relation for $F^+(\nu)$ can be easily separated into the usual once-subtracted form for $G^+(\nu)$ plus a deviation $\xi^+(\nu)$ arising from the piecewise analyticity of $G^+(\nu)$. It has the form

$$D^+(\nu)=D^+(\mu)+\frac{2f^2q^2\nu_B}{(\nu_B^{\,2}-\mu^2)(\nu^2-\nu_B^{\,2})}+\frac{q^2}{4\pi^2}\int_\mu^\infty d\nu'\frac{\nu'}{q'}\,\frac{\sigma_-(\nu')+\sigma_+(\nu')}{\nu'^2-\nu^2}+\xi^+(\nu)\,, \tag{2.5a}$$

where $D^\pm(\nu)\equiv\mathrm{Re}G^\pm(\nu)$ and

$$\xi^+(\nu)=H^+(\nu)[\theta(\nu-\nu_0)+(-\nu-\nu_0)]-\frac{q^2}{4\pi^2}\int_{\nu_0}^\infty d\nu'\frac{\nu'}{q'}\,\frac{\eta_-(\nu')+\eta_+(\nu')}{\nu'^2-\nu^2}\,, \tag{2.5b}$$

with $\eta_\pm(\nu)\equiv(4\pi/q)\mathrm{Im}h_\pm(\nu)$ and $H^\pm(\nu)\equiv\mathrm{Re}h^\pm(\nu)$. In order to express the dispersion relation for $F^-(\nu)$ in terms of the usual form for $G^-(\nu)$ plus a deviation $\xi^-(\nu)$ due to the piecewise analyticity, where the integrals of both contributions are convergent, we must use the twice-subtracted dispersion relation for $F^-(\nu)$. Using Eq. (2.3) we find

$$D^-(\nu)=\frac{\nu}{\mu}D^-(\mu)+\frac{2f^2q^2}{\nu_B^{\,2}-\mu^2}\,\frac{\nu}{\nu^2-\nu_B^{\,2}}+\frac{q^2\nu}{4\pi^2}\int_\mu^\infty\frac{d\nu'}{q'}\,\frac{\sigma_-(\nu')-\sigma_+(\nu')}{\nu'^2-\nu^2}+\xi^-(\nu)\,, \tag{2.6a}$$

where

$$\xi^-(\nu)=H^-(\nu)[\theta(\nu-\nu_0)+\theta(-\nu-\nu_0)]-\frac{q^2\nu}{4\pi^2}\int_{\nu_0}^\infty\frac{d\nu'}{q'}\,\frac{\eta_-(\nu)-\eta_+(\nu)}{\nu'^2-\nu^2}\,. \tag{2.6b}$$

At this stage, we have not confined ourselves to a definite model for $h^\pm(\nu)$ except that it is an analytic function. The deviations $\xi^\pm(\nu)$ càn be investigated without reference to the specific form for $h^\pm(\nu)$ from current experimental measurements of $\sigma_\pm(\nu)$ and $D^\pm(\nu)$ which include the Serpukhov data.[11,12] An analysis of this data was made by Lindenbaum[13] to test the normal forward dispersion relations for $D^+(\nu)$ and $D^-(\nu)$. The former was once subtracted and the latter was unsubtracted taking $\sigma^+(\infty)=\sigma^-(\infty)$. The largest disagreement occurred in the test of the $D^-(\nu)$ relation with better agreement for $D^+(\nu)$. However, the errors in $D^\pm(\nu)$ for large ν as well as the errors in the Serpukhov value of $\sigma_\pm(\nu)$ make it difficult for this analysis to be considered as either a verification of the standard dispersion relations or as giving evidence for any conclusions about the existence of any nonanalytic contribution to the pion-nucleon amplitude at higher energies.

We have repeated the tests of forward dispersion relations using Eqs. (2.5a) and (2.6a) of our model allowing for the violation of the Pomeranchuk theorem. The equation for D^+ is weakly dependent on the subtraction constant, $D^+(\mu)$, and the π-N coupling constant f^2. However, in the equation for $D^-(\nu)$ the low-energy region, specifically, the subtraction constant and the nucleon pole contribution have dominant contributions. The subtraction constants $D^-(\mu)$ and $D^+(\mu)$ have been determined to very good accuracy from low-energy data by Hamilton.[14] However, the only determination of f^2 is by the use of fixed-t or -u dispersion relations.[15] The most recent of these[16] yields

$$f^2=0.0815\pm0.0016\,.$$

Since only constants which are independent of dispersion relations can be used in any test of them, f^2 cannot be specified in our Eqs. (2.5a) and (2.6a). Consequently, the best that one can hope to do in practice is to determine the violation of the normal dispersion relations $\xi^\pm(\gamma)$ as a function of f^2, i.e., $\xi^\pm(\nu,f^2)$.

For this purpose we define the right-hand side of (2.5a) excluding $\xi^+(\nu,f^2)$ as $R^+(\nu,f^2)$, i.e.,

TABLE I. The maximum-likelihood fit of σ_+ and σ_- data of Refs. 11 and 12 to the form $\sigma = A + B(q/q_0)^C$.

	A (mb)	B (mb)	C
σ_+	23.99±0.17	39.6±12.7	−1.33 ±0.18
σ_-	23.63±0.03	27.7±11.7	−0.946±0.006

$$R^+(\nu,f^2) = D^+(\mu) + \frac{2f^2q^2\nu_B}{({\nu_B}^2-\mu^2)(\nu^2-{\nu_B}^2)} + \frac{q^2}{4\pi^2}\int_\mu^\infty d\nu' \frac{\nu'}{q'}\frac{\sigma_-(\nu')+\sigma_+(\nu')}{\nu'^2-\nu^2}. \tag{2.7a}$$

Similarly, we define the right-hand side of (2.6a) except for $\xi^-(\nu)$ by $R^-(\nu,f^2)$,

$$R^-(\nu,f^2) = \frac{\nu}{\mu}D^-(\mu) + \frac{2f^2q^2}{{\nu_B}^2-\mu^2}\frac{\nu}{\nu^2-{\nu_B}^2} + \frac{q^2\nu}{4\pi^2}\int_\mu^\infty \frac{d\nu'}{q'}\frac{\sigma_-(\nu')-\sigma_+(\nu')}{\nu'^2-\nu^2}. \tag{2.7b}$$

In addition to the question of $D^\pm(\mu)$ already discussed, we use experimental data for the total cross sections $\sigma^\pm(\nu)$ to evaluate $R^\pm(\nu,f^2)$. For the region $0 \leq q \leq 5.0$ GeV/c we used the interpolated experimental values for $\sigma^\pm(\nu)$ of Höhler *et al.*[17] The values used in the second region $5.0 \leq q \leq 8.0$ GeV/c were interpolated from the data in the University of Michigan compilation.[18] The integral for the region from $q = 0$ to $q = 8.0$ GeV/c was evaluated by point-to-point integration procedure which fitted six points at a time to a fourth-order polynomial. The integration above $q = 8.0$ GeV/c was evaluated using a maximum-likelihood fit to the available σ_+ and σ_- data of the form

$$c_\pm = A_\pm + B_\pm(q/q_0)^{C_\pm},$$

where $q_0 = 1.0$ GeV/c, C is dimensionless, and A and B are in units of mb. The results of our fits are given in Table I. The present data indicate that $\sigma_-(\infty) - \sigma_+(\infty) \simeq 0.5$ mb and provide the justification for not using a subtracted dispersion relation for $D^-(\nu)$.

The results of our evaluation of Eq. (2.7a) are given in Fig. 2 where we compare the data[12,19] for $\alpha_\pm(\nu,f^2) \equiv \mathrm{Re}G_\pm(\nu,f^2)/\mathrm{Im}G_\pm(\nu)$ to $R_\pm(\nu,f^2)/\mathrm{Im}G_\pm(\nu)$. The latter results are given as solid curves for $f^2 \equiv 0.078$ and 0.084, where each curve is enclosed by a shaded region corresponding to the errors in $D^\pm(\mu)$.[20] The data points for $D^\pm(\nu)$ are determined using the data for $\alpha_\pm$ under the assumption that $\alpha_\pm$ changes negligibly for a laboratory momentum variation of less than 1%. These together with the values for $\sigma^\pm(\nu)$ from Table I enable us to determine the corresponding values of $D^\pm(\nu)$. These

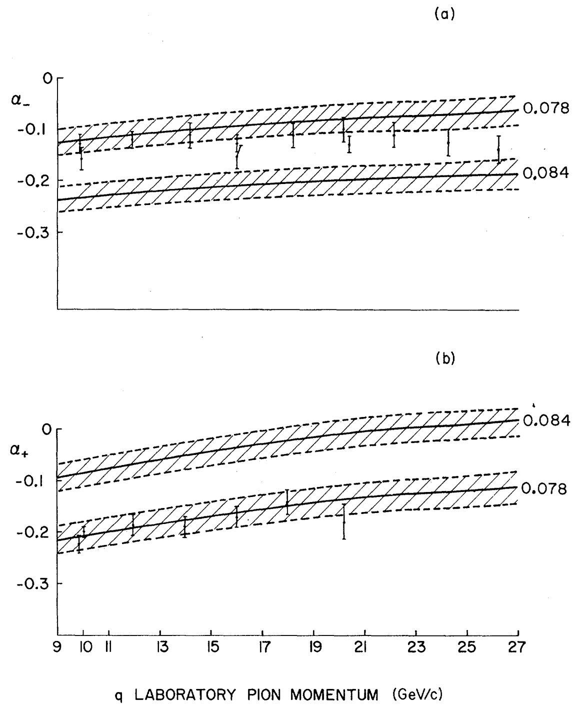

FIG. 2. (a) Plot of $\alpha_- = \mathrm{Re}G_-(q)/\mathrm{Im}G_-(\nu)$ for laboratory pion momentum q in GeV/c. (b) Plot of $\alpha_+ = \mathrm{Re}G_+(q)/\mathrm{Im}G_+(q)$ vs q in GeV/c. The data points of each plot are from the last paper in Ref. 12. The solid curves correspond to $R_-(\nu,f^2)/\mathrm{Im}G_-(\nu)$ and $R_+(\nu,f^2)/\mathrm{Im}G_+(\nu)$, respectively, for $f^2 = 0.078$ and $f^2 = 0.084$ as indicated. The shaded area corresponds to the uncertainty in the solid curves due to errors in $D^-(\mu)$.

points appear in Figs. 3 and 4 (Ref. 21) with the curves for $R^+(\nu,f^2)$ and $R^-(\nu,f^2)$, respectively, for $f^2 = 0.078$ and $f^2 = 0.084$ in the c.m. system in units of $c = \hbar = \mu = 1$. The shaded band in Fig. 4 corresponds to the error introduced by $D^-(\mu)$, as before. Since $D^+(\nu)$ is rather insensitive to f^2 the solid curve in Fig. 3 is essentially unchanged for a wide range of values which includes the range from $f^2 = 0.078$ to 0.084. The dashed curves in Fig. 3 correspond to the upper and lower limits of $R^+(\nu)$ imposed by the errors for A, B, and C given in Table I.

Two immediate conclusions emerge from our results presented above.

(i) If $\sigma_+(\infty) \neq \sigma_-(\infty)$ as indicated by the above fits to the available data above 8 GeV/c, then the convergent subtracted dispersion relation for $D^-(\nu)$ obtained in (2.6a) cannot be tested nor can $\xi^-(\nu)$ be determined, due to its sensitive dependence on f^2 which in turn depends on the analyticity of the scattering amplitude. (It is interesting to note that a lower value of f^2 than is usually assumed, i.e., $f^2 \simeq 0.078$, is necessary for ξ^- to be consistent with zero.)

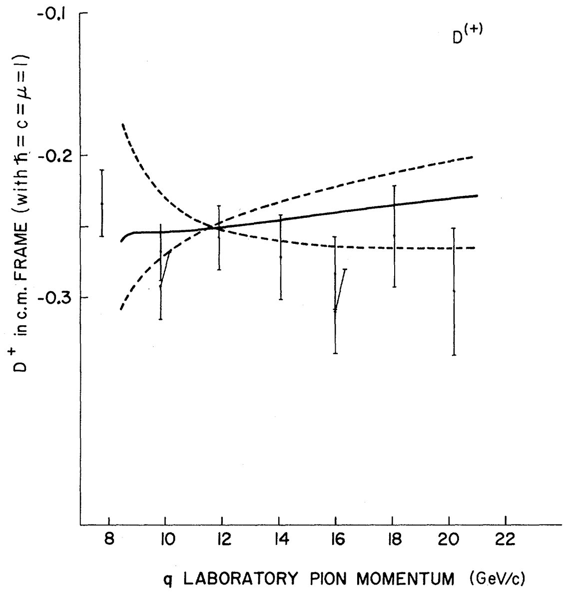

FIG. 3. D^+ in the c.m. system with $\hbar = c = \mu = 1$ vs laboratory pion momentum q in GeV/c. Data points are calculated from $\alpha^\pm$ of the last paper in Ref. 12 and $\sigma^\pm$ from parameters of Table I. We interpolated the data for $\alpha^\pm$ assuming that the values are exchanged for variation in q of less than 1%. The solid curve corresponds to $R^+(\nu)$ and the dashed curves correspond to maximum and minimum values of integrals obtained by taking maximum and minimum error limits of $\sigma_- + \sigma_+$ from Table I.

(ii) The result for $R^+(\nu, f^2)$ which is essentially independent of f^2 compared, in Fig. 3, to $D^+(\nu)$ determined by experiment indicates that it is somewhat more probable that $\xi^+(\nu, f^2)$ is nonzero than zero, but the large errors involved prohibit any definite conclusion.

III. MODIFICATIONS OF FINITE-ENERGY SUM RULES

The FESR's[22] are another set of results which depend crucially on global analyticity of the scattering amplitude. Leading Regge asymptotic behavior of the amplitude together with the normal dispersion relations lead to superconvergence relations which are the basis of the FESR.

Suppose we have an analytic function of ν as defined earlier, $f(\nu, t)$, which is antisymmetric in ν, has a right-hand cut, and satisfies an unsubtracted fixed-t dispersion relation

$$f(\nu, t) = \frac{2\nu}{\pi} \int_0^\infty \frac{\operatorname{Im} f(\nu', t)}{\nu'^2 - \nu^2} d\nu' ,$$

where the integration is to include any Born terms. Then, if $f(\nu, t)$ is dominated by a Regge pole at high energy so as to have the form

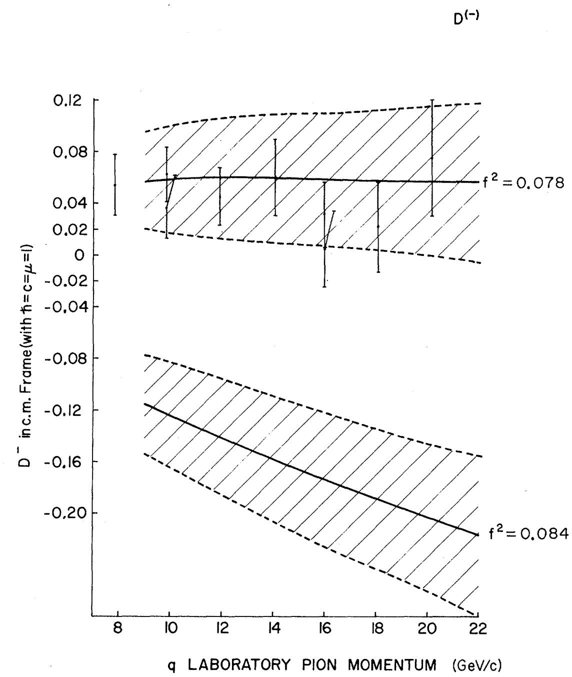

FIG. 4. D^- in the c.m. system with $\hbar = c = \mu = 1$ vs laboratory pion momenta q in GeV/c. The same method used in obtaining data points for D^+ is used here. The solid curves correspond to R^- for $f^2 = 0.078$ and $f^2 = 0.084$ as noted. The shaded region surrounding the solid curves corresponds to their uncertainty because of the error in $D^-(\mu)$.

$$f_R(\nu, t) = \beta(t) \frac{(\pm 1 - e^{i\pi\alpha(t)})\nu^{\alpha(t)}}{\sin\pi\alpha(t)\Gamma(\alpha + 1)} ,$$

with $\alpha < -1$, then it satisfies the superconvergence relation

$$\int_0^\infty \operatorname{Im} f(\nu', t) d\nu' = 0 .$$

Finally, if we can represent $\operatorname{Im} f(\nu', t)$ by a sum of Regge poles above an energy $\nu = N$, then the superconvergence relation can be written as

$$\frac{1}{N} \int_0^N \operatorname{Im} f(\nu', t) d\nu' = \sum_i \frac{\beta_i N^{\alpha_i(t)}}{\Gamma(\alpha_i + 2)} .$$

In general, for the same properties for $f(\nu, t)$ one has the set of FESR's of the form

$$\frac{1}{N^{n+1}} \int_0^N \nu^n \operatorname{Im} f(\nu, t) d\nu = \sum_i \frac{\beta_i(t) N^{\alpha_i(t)}}{(\alpha_i + n + 1)\Gamma(\alpha_i + 1)} . \tag{3.1}$$

The form of the FESR for an $f(\nu, t)$ which satisfies a subtracted dispersion relation can be determined in a similar manner.[22]

In the event that a NMT exists, the general form of the FESR, (3.1), is no longer valid since global analyticity is not true. In terms of our model, however, the physical amplitude, $G(s,t)=F(s,t)+\Delta(s,t)$, as discussed in the previous sections, contains an analytic part $F(s,t)$ which is the analytic extension of $G(s,t)$ above the NMT. $F(s,t)$ is the object that satisfies the conditions necessary for the superconvergence relations and the FESR. In addition to satisfying a dispersion relation it can be described in terms of Regge poles for large s. If we use ν instead of s and consider the $F(\nu,t)$ contribution to $\pi^- p$ scattering of Sec. II, it will satisfy Eq. (3.1). We extend our earlier relation between $G(\nu,t)$ and $F(\nu,t)$ to physical t, i.e.,

$$G(\nu,t)=F(\nu,t)+h(\nu,t)[\theta(\nu-\nu_0)+\theta(-\nu-\nu_0)]\,,$$

with the NMT's at $\nu=\pm\nu_0$. Assuming Regge behavior corresponding to $F(\nu,t)$ *only*, the resulting modified FESR is

$$\frac{1}{N^{n+1}}\int_0^N \nu^n\,\mathrm{Im}G(\nu,t)d\nu=\frac{1}{N^{n+1}}\int_{\nu_0}^N \nu^n\,\mathrm{Im}h(\nu,t)d\nu+\sum_i\frac{\beta_i(t)N^{\alpha_i(t)}}{(\alpha_i+n+1)\Gamma(\alpha_i+1)}\,. \tag{3.2}$$

The term by which (3.2) differs from the usual FESR depends only on $\mathrm{Im}h(\nu,t)$ and is zero for $\nu_0 \geq N$. This is in contrast to the modified forward dispersion relation discussed in Sec. II where we found that the correction depended upon both $\mathrm{Re}h(\nu)$ and $\mathrm{Im}h(\nu)$ in the general case.

The presence of the correction term in (3.2) might produce violations of the usual FESR for $\nu>\nu_0$ despite the fact that it is satisfied below ν_0. According to the form of the correction term such a violation might be interpreted as a fixed pole at an α whose exact value depends on the ν dependence of $\mathrm{Im}h(\nu,t)$. The location of the NMT is also important in determining the appropriate Regge parameters from measurements of various angular distributions. Measurements above and below the NMT at $\nu=\nu_0$ will yield different sets of parameters; in the former case we are computing the parameters of $G(\nu,t)$ while the latter case corresponds to $F(\nu,t)$ whose Regge parameters are those in Eq. (3.2)

Clearly, NMT effects would not upset the FESR if $\mathrm{Im}h(\nu,t)=0$. That is, provided that the Regge parameters used were those of $F(\nu,t)$. However, soluble model calculations suggest that $\mathrm{Im}h(\nu,t)\neq 0$ is most probable. In this case one would discover that the normal FESR would not be satisfied above $\nu=\nu_0$.

Numerous tests of FESR using either the fixed-t or fixed-u sum rules have been made. The early tests by Dolen, Horn, and Schmid[22] were performed on $\pi^- p$ charge-exchange scattering with a cutoff at $N=1.5$ GeV for the $B^{(-)}$ amplitude and $N=2.5$ GeV for the $A'^{(-)}$ amplitude to test the consistency of the ρ trajectory. They discovered that an additional ρ trajectory was needed to satisfy the FESR, 0.4 lower than the ρ. It was suggested that this could be a manifestation of a cut. They also seemed to require a fixed pole in the $n=0$ FESR for the $B^{(-)}$ amplitude at $j=0$ which they argue would not contribute to the physical amplitude because it is at a wrong-signature–nonsense point. This analysis was subsequently redone by Aviv and Horn[23] using better low-energy data. They again needed a fixed pole to satisfy the $n=0$ FESR in addition to the ρ in the $B^{(-)}$ amplitude but this time at $\alpha=-1$. The fact that fixed poles are needed to satisfy the fixed-t FESR's might be interpreted as NMT effects in our model.

A particular set of fixed-u FESR's with $u=M^{*2}=(1238\text{ MeV})^2$ were investigated by Kayser[24] for πN scattering using the resonance dominance approximation for the s-channel processes and obtaining the Regge parameters directly from the πN spectrum. Here the effect of varying the cutoff was investigated. It was discovered that for cutoffs for $s\leq(1.808\text{ GeV})^2$ the sum rules were reasonably satisfied. However, for cutoffs above this and specifically at $s=(2.313\text{ GeV})^2$ the FESR's failed badly. If one accepts this analysis, at least two explanations of this effect are possible. One is the failure of the resonance approximation because of significant background contributions for $s>(1.808\text{ GeV})^2$. The other is the violation of the FESR for $s>(1.808\text{ GeV})^2$ because of a breakdown of analyticity such as we have described above. For the latter explanation we would expect a NMT somewhere above $s=(1.8\text{ GeV})^2$ for the unphysical point corresponding to $u=M^{*2}=(1238\text{ MeV})^2$.

The FESR provide a promising avenue for the detection of NMT effects provided that such thresholds fall within the range of detailed low-energy phase-shift analysis. The recently completed phase-shift analyses which extend to 3 GeV for πN scattering should be used in fixed-t FESR's with the view of seriously investigating the presence of the low NMT that may be indicated by Kayser's fixed-u FESR test.

IV. CONCLUSIONS

We have investigated some consequences of a scattering amplitude which is piecewise analytic due to the presence of at least a negative-metric threshold in the s channel. The loss of global analyticity which is implied was seen to lead to a mechanism for violating the Pomeranchuk theorem without a logarithmic increase with energy in the ratio of the real to the imaginary part of the scattering amplitude.

Since any violation of the global dispersion relations for the scattering amplitude is a necessary condition for the presence of nonanalytic contribution, we reanalyzed the dispersion relations for $\pi^{\pm}p$ scattering in terms of our model allowing for the violation of the Pomeranchuk theorem. We discovered that the piecewise-analytic contribution to $D^-(\nu)$ determined by analyzing the appropriately subtracted dispersion relations is crucially dependent upon f^2, the πN coupling constant which in turn depends on dispersion relations for its evaluation. Thus, we are unable to unambiguously determine the possible piecewise-analytic contribution to $D^-(\nu)$. On the other hand, our analysis of the $D^+(\nu)$ amplitude allows for the existence of a nonanalytic contribution. However the errors involved in the data points make it difficult to make any definite conclusion.

But, by the same token, we must recognize that the experimental basis for concluding that we have a globally analytic transition amplitude is at best somewhat flimsy. We must therefore consider the question of global analyticity to be an open question.

We have also discussed the possible implications of a piecewise-analytic scattering amplitude for FESR. Results of analysis of FESR by other authors can be interpreted as due to the presence of a simple piecewise-analytic contribution as described by our model. Such an interpretation is not unique since errors involved in the analysis might also be causing the apparent violation of FESR's.

The tests of dispersion relations and FESR's are sensitive to the global analytic behavior. In addition to these there are also possible "local" tests. For example when a NMT in s is encountered the partial-wave (or fixed-t) amplitude would exhibit a discontinuity in analyticity. In the various model theories studied the onset of this change in analytic behavior comes weighted with the relevant phase space. We expect, therefore, that the s-wave amplitude would be continuous but its first derivative discontinuous at the NMT. In other words, we expect a cusp in the two-body s-wave amplitude at a (two-particle) NMT. For higher partial waves the discontinuities would be of higher order.

The direct test of such a cusp behavior is usually difficult since with the kind of statistics and energy intervals used even in the best experiments this could be easily missed. But if other analyses point to a particular NMT at a specific value of s then the search for cusps may corroborate such a result.

The analysis we presented above can be readily extended to the case of more than one NMT by representing the physical amplitude $G(s, t)$ in terms of a sum of NMT contributions. For example, for the forward amplitude $G(\nu)$, Eq. (2.2a) can be written as

$$G(\nu) = F(\nu) + \sum_i h_i(\nu)[\theta(\nu - \nu_{si}) + \theta(-\nu + \nu_{ui})],$$

where $F(\nu)$ and $h_i(\nu)$ are analytic functions of ν and the subscript i labels different NMT. Then, e.g., the piecewise-analytic contribution $\xi^+(\nu)$ to the $D^+(\nu)$ dispersion relations can be expressed as

$$\xi^+(\nu) = \sum_i \left(\mathrm{Re}\,h_i^+(\nu)[\theta(\nu - \nu_{i0}) + \theta(-\nu - \nu_{i0})] - \frac{q^2}{\pi}\int_{\nu_{i0}}^{\infty} d\nu' \frac{\nu'}{q'^2}\frac{\mathrm{Im}h_{i-} + \mathrm{Im}h_{i+}}{\nu'^2 - \nu^2}\right).$$

The complication of successive NMT thus in turn complicates the explicit form of the piecewise-analytic contributions in question.

In conclusion, we see that a piecewise-analytic structure of the scattering amplitude which is necessary for a convergent field theory cannot yet be distinguished experimentally from a structure satisfying Mandelstam analyticity. More accurate experiments at intermediate as well as higher energies for improved tests of dispersion relations, FESR's as well as the detection of cusps will resolve this question.

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[21]There is a systematic scale error of ±0.008 in the determination of $\alpha_\pm$ described in Ref. 12 which is opposite in sign for α_+ and α_-. The effect of this error almost cancels for D^+ but adds for D^-. We have not included its effect in Figs. 2 and 4.

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