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Cascade model: A solvable field theory

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A fully solvable quantum theory in which the spectrum of scattering states shifts with the strength of the interaction is presented. Inelastic processes in which two particles go into three are obtained in this cascade model. By using the formalism of analytic continuation of the state spaces, resonances and redundant poles are identified. The completeness of the states is explicitly demonstrated and the eigenchannels when two- and three-particle channels are open are determined. This is one of the few models of scattering of an unstable particle.

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I. INTRODUCTION: SPECTRA, SCATTERING AND SURVIVAL AMPLITUDES

The study of analytic properties of two-particle scattering amplitudes and their resonance structure has been an essential part of quantum theory. Two-particle scattering has been studied using (local) potentials from the early days of quantum mechanics [1]: many relationships between bound states and scattering resonances on the one hand and poles of the scattering amplitude on the other are known, not only in their correspondence but also of the relationships of the scattering phase shifts to the asymptotic bound-state functions [2]. As a natural follow-up, the time evolution of unstable particles can be related to the scattering amplitude structure.

However, the scattering amplitude does not contain all the information contained in the spectrum and (ideal) eigenfunctions. The existence of "redundant poles" in the scattering amplitude was discovered first for local potentials [3] and then in nonlocal (separable) potentials [4]; and the related situation of discrete (normalizable) solutions degenerate in energy with the scattering continuum [5].

These illustrate the desirability of explicit solutions to the Schrödinger equation going beyond the determination of the scattering amplitude. In addition to the limited number of solvable (and solved) analytic potentials [6], including those which are specifically constructed to be phase equivalent, use of models involving separable nonlocal potentials [7] and the Lee model [8] of a mutilated solvable quantum field theory have been most instructive.

One of the lessons learned from the explicit study of models is that the "survival amplitude" of an unstable particle depends on the details of the wave function going beyond the analytic structure of the scattering [9] amplitude. In a similar manner the wave function of a scattering state contains more information than is contained in the S-matrix elements.

On the other hand, the wave functions of genuine three-particle scattering states and even the detailed analytic properties of the three-particle scattering amplitudes are less familiar despite the impressive Faddeev theory of three-particle scattering amplitudes [10]. For the Lee model in the three-particle sector, the scattering amplitude was computed by Amado [11], and the wave functions have been obtained by Bolsterli [12] and by Nelson [13]. Perhaps because of the complicated expressions, there has not been much study of this system. In particular, the possibility of using this model to study the "scattering of an unstable particle" by a fixed target has not been utilized.

Once the wave functions are explicitly determined it becomes possible to consider the analytic continuation of the vector space of states rather than merely of the scattering amplitude. These studies have been instrumental in clarifying several puzzling and often conflicting statements in the literature.

We have constructed a model of three-particle scattering which is simpler than the Lee model but which shares with it a sector in which the essential features of a two-particle system coupled to a three-particle system can be studied. The primary couplings are $A \rightleftarrows B\theta$ and $B \rightleftarrows C\phi$ in such a manner than the $B\theta$ and $C\theta\phi$ states are coupled. We call this the "Cascade model." It would be useful in the study of the cascade decay of a metastable state of a three-level atom. For the present our aim is to illustrate the principles and the structure of the wave functions and the scattering amplitudes, to study the eigenstates and eigenphases of the S matrix, and to study the analytic continuation of the Hilbert space of states.

The plan of the paper is as follows. In Sec. II, we introduce the Cascade model Hamiltonian and show how

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its two-particle and three-particle states can be determined from the (ideal) eigenfunction for the Hamiltonian. The mass renormalization for the B particle and its wave-function renormalization enter in an essential way in the spectrum of the Hamiltonian and the structure of the scattering solutions. This is shown in Sec. III. The generalized Möller matrix and the comparison Hamiltonian are determined in Sec. IV, while Sec. V studies the "in" and "out" states and the S matrix. Section VI deals with the unitarity of the S matrix. Section VII deals with the eigenchannels and eigenphase shifts in the degenerate spectrum. Section VIII deals with the analytic continuation from the Hilbert space H to more general spaces $\mathcal{G}$. The spectra in spaces $\mathcal{G}$ are determined. The modifications introduced by the instability of the B particle are also treated in Sec. VIII. Cases where A is stable and unstable are also discussed. In the concluding section (Sec. IX), the essential results are recapitulated and as yet unsolved problems are enumerated.

II. THE CASCADE MODEL FIELD THEORY

We consider a quantum field theory with five distinct fields A, B, C, θ, ϕ and the corresponding particles (with no antiparticles). The commutation relations are

$$\begin{aligned}
&[A,A^\dagger]=1,\quad [A,B]=[A,B^\dagger]=[A,C]=[A,C^\dagger]=0\ ,\\
&[B,B^\dagger]=1,\quad [B,C]=[B,C^\dagger]=0\ ,\\
&[C,C^\dagger]=1\ ,\\
&[\theta(\omega),\theta^\dagger(\omega')]=\delta(\omega-\omega'),\quad [\theta(\omega),\theta(\omega')]=0\ ,\\
&[\theta(\omega),\varphi(\nu)]=[\theta(\omega),\varphi^\dagger(\nu)]=0\ ,\\
&[\varphi(\nu),\varphi^\dagger(\nu')]=\delta(\nu-\nu'),\quad [\varphi(\nu),\varphi(\nu')]=0\ ,\\
&[A,\theta(\omega)]=[A,\theta^\dagger(\omega)]=[A,\varphi(\nu)]=[A,\varphi^\dagger(\nu)]=0\ ,\\
&[B,\theta(\omega)]=[B,\theta^\dagger(\omega)]=[B,\varphi(\nu)]=[B,\varphi^\dagger(\nu)]=0\ ,\\
&[C,\theta(\omega)]=[C,\theta^\dagger(\omega)]=[C,\varphi(\nu)]=[C,\varphi^\dagger(\nu)]=0\ .
\end{aligned}\tag{2.1}$$

Note that while θ,φ are labeled by continuum parameters $0<\omega$, $\nu<\infty$, the objects A,B,C are treated as single modes ("infinitely heavy"). We choose a total Hamiltonian for this system which allows the transition

$$\begin{aligned}
&A\rightleftarrows B\theta\ ,\\
&B\rightleftarrows C\varphi\ ,
\end{aligned}\tag{2.2}$$

and write

$$H=H_0+V\ ,\tag{2.3}$$

$$H_0=M_0A^\dagger A+\mu_0B^\dagger B+\int_0^\infty d\omega\,\omega\theta^\dagger(\omega)\theta(\omega)+\int_0^\infty d\nu\,\nu\varphi^\dagger(\nu)\varphi(\nu)\ ,\tag{2.4}$$

$$V=\int_0^\infty d\omega\{f^*(\omega)A^\dagger B\theta(\omega)+f(\omega)B^\dagger\theta^\dagger(\omega)A\}+\int_0^\infty d\nu\{g^*(\nu)B^\dagger C\varphi(\nu)+g(\nu)C^\dagger\varphi^\dagger(\nu)B\}\ .\tag{2.5}$$

We adopt the convention that the form factors f and g are for the "one-body to two-body processes:" $A\to B\theta$, $B\to C\phi$, respectively. We may take the labels ω,ν to refer to S waves only. For this Hamiltonian there are, in addition to itself, three further constants of motion:

$$\begin{aligned}
&N_1=A^\dagger A+B^\dagger B+C^\dagger C\ ,\\
&N_2=B^\dagger B+\int_0^\infty\varphi^\dagger(\nu)\varphi(\nu)d\nu\ ,\\
&N_3=A^\dagger A+\int_0^\infty\theta^\dagger(\omega)\theta(\omega)d\omega\ .
\end{aligned}\tag{2.6}$$

Consequently there is no transition between distinct "sectors" labeled by these three quantum numbers. The vacuum state has $N_1=N_2=N_3=0$ and is stable. So are all the states with $N_1=0$ containing only θ and φ "particles." The state $N_1=1$, $N_2=N_3=0$, that of a C particle is also absolutely stable, and has zero energy.

The lowest nontrivial sector has $N_1=N_2=1$; $N_3=0$ contains the B particle or the $C\varphi$ continuum. This sector has the same structure as the $V-N\theta$ sector of the Lee model [8] and therefore the solutions in this sector can be written down. Define

$$\gamma(\zeta)=\zeta-\mu_0-\int_0^\infty\frac{g^*(\nu')g(\nu')}{\zeta-\nu'}d\nu'\ ,\tag{2.7}$$

which is a real analytic function analytic in the cut plane cut along the real axis from 0 to ∞ and behaves as ζ at infinity. The function $\gamma(\zeta)$ may have at most one zero in the cut plane for $\zeta<0$ provided $g(\nu)$ vanishes nowhere in $0<\nu<\infty$, which we shall assume unless otherwise explicitly stated. If such a zero exists there is a discrete normalized state which may be identified with the physical B; further we have a continuous spectrum $0<\nu<\infty$ which corresponds to the continuum normalizable (ideal) eigenstates of the system. Since these are already available in the literature we do not write them down explicitly. The dimensionless scattering amplitude at energy ν has the explicit form

$$\begin{aligned}
&t(\nu)=|g(\nu)|^2/\gamma(\nu+i\epsilon)=e^{i\chi}\sin\chi\ ,\\
&\exp(2i\chi)=\gamma(\nu-i\epsilon)/\gamma(\nu+i\epsilon)\ .
\end{aligned}\tag{2.8}$$

The physical scattering amplitude with the dimensions of length is obtained by dividing $t(\nu)$ by the momentum k so that the (S-wave) cross section is

$$(4\pi/k^2)\sin^2\chi\ .\tag{2.9}$$

The relation between k and ν depends on the energy function

$$\nu(k)=\begin{cases}\sqrt{k^2+m^2}-m, & \text{relativistic}\ ,\\ \dfrac{k^2}{2m}, & \text{nonrelativistic}\ .\end{cases}\tag{2.10}$$

There is a phase-space factor k^2dk which has been absorbed into $g(\nu)$:

$$|g(\nu)|^2d\nu\sim k^2dk\,|g_1(\nu)|^2\ ,\tag{2.11}$$

where $g_1(\nu)$ is a smooth function of ν without any threshold singularities. We recall that any state in the $N_1=N_3=1$, $N_2=0$ sector may be expanded in terms of the bare B and $C\varphi(\nu)$ states.

$N_1=N_2=N_3=1$ is richer and instructive. There are possibilities of states A, $B\theta(\omega)$, $C\theta(\omega)\varphi(\nu)$ all coupled together. If we denote a generic state by the vector

$$\begin{bmatrix} \eta \\ \phi(\omega) \\ \psi(\omega,\nu) \end{bmatrix} = \Psi , \tag{2.12}$$

the effective Hamiltonian in this sector can be written in matrix form

$$\begin{bmatrix} M_0 & f^*(\omega') & 0 \\ f(\omega) & (\mu_0+\omega)\delta(\omega-\omega') & g^*(\nu')\delta(\omega-\omega') \\ 0 & g(\nu)\delta(\omega-\omega') & (\omega+\nu)\delta(\omega-\omega')\delta(\nu-\nu') \end{bmatrix} = H . \tag{2.13}$$

The eigenvalue equation is

$$H\Psi = \lambda\Psi \tag{2.14}$$

with repeated variables ω', ν' being integrated over. These equations imply

$$\begin{aligned} &(\lambda - M_0)\eta_\lambda = F_\lambda , \\ &(\lambda-\mu_0-\omega)\phi_\lambda(\omega) = G_\lambda(\omega) + f(\omega)\eta_\lambda , \\ &(\lambda-\omega-\nu)\psi_\lambda(\omega,\nu) = g(\nu)\phi_\lambda(\omega) , \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} F_\lambda &= \int_0^\infty f^*(\omega')\phi_\lambda(\omega')d\omega' , \\ G_\lambda(\omega) &= \int_0^\infty g^*(\nu')\psi_\lambda(\omega,\nu')d\nu' . \end{aligned} \tag{2.16}$$

An infinitely degenerate set of continuum (ideal) states are obtained by choosing, $0 < n < \lambda$; expressions obtained from (2.15) and (2.16) through the replacement of subscript "λ" by "λn" are applicable to the ideal states of "λn":

$$\psi_{\lambda,n}(\omega,\nu) = e\,\delta(n-\nu)\delta(\lambda-\omega-\nu) + \frac{g(\nu)\phi_{\lambda,n}(\omega)}{\lambda-\omega-\nu+i\epsilon} . \tag{2.17}$$

Here e is a (real) constant yet to be fixed. Then

$$\begin{aligned} G_{\lambda,n}(\omega) &= \int_0^\infty g^*(\nu')\psi_{\lambda n}(\omega\nu')d\nu' \\ &= eg^*(\lambda-\omega)\delta(\lambda-\omega-n) \\ &\quad + \phi_{\lambda,n}(\omega)\int_0^\infty \frac{|g(\nu')|^2 d\nu'}{\lambda-\omega-\nu'+i\epsilon} . \end{aligned} \tag{2.18}$$

Substituting this value for $G_{\lambda,n}(\omega)$ we may write

$$\begin{aligned} &\gamma(\lambda-\omega+i\epsilon)\phi_{\lambda,n}(\omega) \\ &\qquad = eg^*(n)\delta(\lambda-\omega-n) + f(\omega)\eta_{\lambda,n} , \end{aligned} \tag{2.19}$$

where $\gamma(\zeta)$ is the denominator function (2.7) already defined. A possible choice for $\phi_{\lambda,n}(\omega)$ is

$$\begin{aligned} \phi_{\lambda,n}(\omega) &= e\frac{g^*(n)\delta(\lambda-\omega-n)}{\gamma(n)} + \frac{f(\omega)\eta_{\lambda,n}}{\gamma(\lambda-\omega)} \\ &\quad + b\delta(\lambda-\omega-\mu) \end{aligned} \tag{2.20}$$

and, writing $\alpha(\lambda)$ for $\alpha(\lambda+i\epsilon)$,

$$\eta_{\lambda,n} = \frac{ef^*(\lambda-n)g^*(n)}{\gamma(n+i\epsilon)\alpha(\lambda)} + \frac{bf^*(\lambda-\mu)}{\alpha(\lambda)} , \tag{2.21}$$

where

$$\alpha(\lambda) = \lambda - M_0 - \int_0^\infty \frac{|f(\omega')|^2 d\omega'}{\gamma(\lambda-\omega'+i\epsilon)} . \tag{2.22}$$

In these expressions μ is the real zero of $\gamma(\zeta)$,

$$\gamma(\mu) = 0 ,$$

if it exists; if no zero exists we must choose $b=0$. A set of continuum normalized (ideal) eigenstates of H are obtained as

$$\Psi_{\lambda,n} = \begin{bmatrix} \eta_{\lambda,n} \\ \phi_{\lambda,n}(\omega) \\ \psi_{\lambda,n}(\omega,\nu) \end{bmatrix} = \begin{bmatrix} \dfrac{f^*(\lambda-n)g^*(n)}{\alpha(\lambda)\gamma(n)} \\ \dfrac{g^*(n)\delta(\lambda-\omega-n)}{\gamma(\lambda-\omega)} + \dfrac{f(\omega)}{\gamma(\lambda-\omega)}\eta_{\lambda,n} \\ \delta(n-\nu)\delta(\lambda-\omega-n) + \dfrac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\phi_{\lambda,n}(\omega) \end{bmatrix} \tag{2.23}$$

irrespective of whether $\gamma(\zeta)$ has a zero. If it has, we have an additional continuum of states:

$$\Psi_\lambda = \begin{bmatrix} \eta_\lambda \\ \phi_\lambda(\omega) \\ \psi_\lambda(\omega,\nu) \end{bmatrix} = \begin{bmatrix} \dfrac{f^*(\lambda-\mu)}{\sqrt{\gamma'}\alpha(\lambda)} \\ \dfrac{1}{\sqrt{\gamma'}}\delta(\lambda-\omega-\mu) + \dfrac{f(\omega)\eta_\lambda}{\gamma(\lambda-\omega)} \\ \dfrac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\phi_\lambda(\omega) \end{bmatrix} . \tag{2.24}$$

Here

$$\gamma' = \frac{\partial}{\partial\zeta}\gamma(\zeta)\bigg|_{\zeta=\mu}$$

and the state is chosen to be (ideal) normalized. Note that these states are defined for

$$\mu < \lambda < \infty .$$

Hence the continuous spectrum begins at μ rather than μ_0. Unlike in the lower sector where the continuous spectrum remains unchanged, here the additional continuous spectrum also shifts; and this shift is consistent with the shift of the discrete state in the lower sector. We can then physically interpret this to be the physical $B\theta$ state and the three-particle continuum (λ, n) to be the physical three-particle states $C\theta\varphi$.

In addition to these continuum states there would be a discrete state at $\lambda = M$ provided

$$\alpha(M)=0, \quad \alpha'=\frac{\partial}{\partial z}\alpha(z)\bigg|_{z=M} .$$

This normalized state is

$$\Psi_M=\begin{bmatrix}\eta_M\\ \phi_M(\omega)\\ \psi_M(\omega,\nu)\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{\alpha'}}\\ \dfrac{f(\omega)}{\gamma(M-\omega)}\eta_M\\ \dfrac{g(\nu)}{M-\omega-\nu}\phi_M(\omega)\end{bmatrix} .$$

It is necessary to verify the (ideal) orthonormality and completeness of this set of states so that we can be sure that there are no more eigenstates of H in $\mathcal{H}$. This is done in the next section. Here we conclude by pointing out that having once proved the orthonormality and completeness we can use the Möller matrix [14] of wave functions to diagonalize H and thus determine the *comparison Hamiltonian* H_0' with the correct spectrum. The comparison Hamiltonian would explicitly contain the mass and "wave-function" renormalization. Whether there are stable A and/or B particles, *the Möller matrix would be unitary* [15].

III. STRUCTURE OF THE THREE-PARTICLE SECTOR

We proceed to verify orthonormality and completeness. For the states (2.24) the scalar product $(\Psi_\lambda,\Psi_{\lambda'})$ is given by

$$\Psi_\lambda^\dagger\Psi_{\lambda'}=\eta_\lambda^*\eta_{\lambda'}+\int_0^\infty d\omega'\phi_\lambda^*(\omega')\phi_{\lambda'}(\omega') +\int_0^\infty d\omega'\int_0^\infty d\nu'\psi_\lambda^*(\omega'\nu')\psi_{\lambda'}(\omega'\nu') , \tag{3.1}$$

where the quantities η, $\phi(\omega)$, and $\psi(\omega\nu)$ are defined by (2.24). But

$$\begin{aligned}\int d\nu'\psi_\lambda^*(\omega'\nu')\psi_{\lambda'}(\omega'\nu')&=\phi_\lambda^*(\omega')\phi_{\lambda'}(\omega')\int d\nu'\frac{g^*(\nu')g(\nu')}{(\lambda-\omega'-\nu'-i\epsilon)(\lambda'-\omega'-\nu'+i\epsilon)}\\&=\frac{\phi_\lambda^*(\omega')\phi_{\lambda'}(\omega')}{\lambda'-\lambda+i\epsilon}\int d\nu' g^*(\nu')g(\nu')\left\{\frac{1}{\lambda-\omega'-\nu'-i\epsilon}-\frac{1}{\lambda'-\omega'-\nu'+i\epsilon}\right\}\\&=\frac{\phi_\lambda^*(\omega')\phi_{\lambda'}(\omega')}{\lambda'-\lambda+i\epsilon}\{\gamma(\lambda'-\omega'+i\epsilon)-\gamma(\lambda-\omega'-i\epsilon)+\lambda-\lambda'\} ,\end{aligned}$$

where use has been made of the definition (2.7) The sum of the second and third terms of (3.1) is therefore given by

$$\begin{aligned}&\int_0^\infty d\omega'\frac{\gamma(\lambda'-\omega')-\gamma^*(\lambda-\omega')}{\lambda'-\lambda+i\epsilon}\left\{\frac{\delta(\lambda-\omega'-\mu)}{\sqrt{\gamma'}}+\frac{f^*(\omega')\eta_\lambda^*}{\gamma^*(\lambda-\omega')}\right\}\left\{\frac{\delta(\lambda'-\omega'-\mu)}{\sqrt{\gamma'}}+\frac{f(\omega')\eta_\lambda}{\gamma(\lambda'-\omega')}\right\}\\&\quad=\int d\omega'\frac{\delta(\lambda-\omega'-\mu)\delta(\lambda'-\omega'-\mu)}{\gamma'}\frac{1}{\lambda'-\lambda}[\gamma(\lambda'-\omega')-\gamma^*(\lambda-\omega')]\\&\quad\quad+\frac{f(\lambda-\mu)\eta_{\lambda'}}{(\lambda'-\lambda)\sqrt{\gamma'}}-\frac{f^*(\lambda'-\mu)\eta_\lambda^*}{(\lambda'-\lambda)\sqrt{\gamma'}}+\eta_\lambda^*\eta_{\lambda'}\int_0^\infty\frac{d\omega' f^*(\omega')f(\omega')}{\lambda'-\lambda+i\epsilon}\left[\frac{1}{\gamma^*(\lambda-\omega')}-\frac{1}{\gamma(\lambda'-\omega')}\right]\\&\quad=\delta(\lambda-\lambda')-\eta_\lambda^*\eta_{\lambda'} .\end{aligned}$$

Taking account of the first term also in (3.1) we get

$$\Psi_\lambda^\dagger\Psi_{\lambda'}=\delta(\lambda-\lambda') . \tag{3.2}$$

Now compute $\Psi_\lambda^\dagger\Psi_{\lambda'n'}$ using (2.23) and (2.24):

$$\Psi_\lambda^\dagger\Psi_{\lambda'n'}=\eta_\lambda^*\eta_{\lambda'n'}+\int d\omega'[\phi_\lambda^*(\omega')\phi_{\lambda'n'}(\omega')+\int d\nu'\psi_\lambda^*(\omega'\nu')\psi_{\lambda'n'}(\omega'\nu')] . \tag{3.3}$$

The integral inside the square brackets of (3.3) is

$$\begin{aligned}&\int d\nu'\frac{g^*(\nu')\phi_\lambda^\dagger(\omega')}{\lambda-\omega'-\nu'-i\epsilon}\left\{\delta(\nu'-n')\delta(\lambda'-\omega'-n')+\frac{g(\nu')\phi_{\lambda'n'}(\omega')}{\lambda'-\omega-\nu+i\epsilon}\right\}\\&\quad=\frac{g^*(n')\delta(\lambda'-\omega'-n')\phi_\lambda^*(\lambda'-n')}{\lambda-\lambda'-i\epsilon}+\phi_{\lambda'n'}(\omega')\phi_\lambda^*(\omega')\left\{\frac{\gamma(\lambda'-\omega')-\gamma^*(\lambda-\omega')}{\lambda'-\lambda+i\epsilon}-1\right\} .\end{aligned}$$

The ω' integral now gives a value which cancels the term

$$\frac{g^*(n')\delta(\lambda'-\omega'-n')\phi_\lambda^*(\omega')}{\lambda-\lambda'-i\epsilon}$$

and the term $\eta_\lambda^*\eta_{\lambda'n'}$ so that

$$\Psi_\lambda^\dagger \Psi_{\lambda' n'} = 0 \ . \tag{3.4}$$

The computation of $\Psi_{\lambda n}^\dagger \Psi_{\lambda' n'}$ proceeds along the same lines. Start from

$$\Psi_{\lambda n}^\dagger \Psi_{\lambda' n'} = \eta_{\lambda n}^* \eta_{\lambda' n'} + \int d\omega' [\phi_{\lambda n}^*(\omega')\phi_{\lambda' n'}(\omega') + \int d\nu' \psi_{\lambda n}^*(\omega'\nu')\psi_{\lambda' n'}(\omega'\nu')] \ . \tag{3.5}$$

But

$$\begin{aligned}\phi_{\lambda n}^*(\omega')\phi_{\lambda' n'}(\omega') + \int d\nu' \psi_{\lambda n}^*(\omega'\nu')\psi_{\lambda' n'}(\omega'\nu') \\ = \phi_{\lambda n}^*(\omega')\phi_{\lambda' n'}(\omega') + \int d\nu' \left\{ \delta(\nu'-n)\delta(\lambda-\omega'-n) + \frac{g^*(\nu')\phi_{\lambda n}^*(\omega')}{\lambda-\omega'-\nu'-i\epsilon} \right\} \\ \times \left\{ \delta(\nu'-n')\delta(\lambda'-\omega'-n') + \frac{g(\nu')\phi_{\lambda' n'}(\omega')}{\lambda'-\omega'-\nu'+i\epsilon} \right\} \\ = \delta(n-n')\delta(\lambda-\lambda')\delta(\lambda-n-\omega') + \frac{g^*(n')\phi_{\lambda n}^*(\lambda'-n')}{\lambda-\lambda'-i\epsilon}\delta(\lambda'-n'-\omega') \\ + \frac{g(n)\phi_{\lambda' n'}(\lambda-n)}{\lambda'-\lambda+i\epsilon}\delta(\lambda-n-\omega') + \phi_{\lambda n}^*(\omega')\phi_{\lambda' n'}(\omega')\frac{\gamma^*(\lambda-\omega')-\gamma(\lambda'-\omega')}{\lambda-\lambda'-i\epsilon} \ .\end{aligned}$$

On doing the ω' integration and using the definition of $\alpha(z)$ and $\eta_{\lambda n}$ we get

$$\Psi_{\lambda n}^\dagger \Psi_{\lambda' n'} = \delta(\lambda-\lambda')\delta(n-n') \ . \tag{3.6}$$

When there is a discrete state with energy $\lambda = M < \mu$, the state corresponding to this is given by (2.25). A direct calculation in this case shows that Ψ_M is normalized and orthogonal to Ψ_λ and to $\Psi_{\lambda,n}$.

The (ideal) states $\Psi_{\lambda,n}$ exist for all values of the parameters M_0, μ_0 and the form factors $f(\omega), g(\nu)$. The states Ψ_λ exist provided $\gamma(\zeta)$ has a real zero; and the state Ψ_M would exist if $\alpha(z)$ has a real zero. If one or more do not exist, the corresponding relations (3.2), (3.3), and (3.6) are empty: in their derivations we had used $\gamma(\mu)=0$ and $\alpha(M)=0$. If Ψ_λ exist, this branch of the spectrum is continuous and nondegenerate with a threshold at $\lambda=\mu<0$ while the $\Psi_{\lambda n}$ spectrum is continuous and infinitely degenerate and has a threshold at $\lambda=0$. If Ψ_M exists, $M<\mu$ below the Ψ_λ threshold. When M_0 increases with μ and the form factors are fixed, so that $\alpha(z)$ no longer has a zero, we expect to find a pole in each of the analytic continuations of the vector spaces. Similarly, when μ_0 increases, keeping $g(\nu)$ unchanged so that $\gamma(\zeta)$ no longer has a zero, the Ψ_λ branch of the *continuous spectrum* moves its threshold to 0 and then disappears; again, analytic continuations would display this as pair complex branches with complex thresholds. We can, of course, have the situation that $\alpha(z)$ has a real zero but $\gamma(\zeta)$ has none. In this case $M<0$ would be a renormalizable state together with a continuum of infinitely degenerate states $0<\lambda<\infty$ with states $\Psi_{\lambda n}$. These results would be consequences of our study of the spectral problem in the continued spaces $\mathcal{G}$.

Given the continuum of orthonormal states $\Psi_{\lambda n}$, we can construct a projection operator

$$\Pi_3 = \int d\lambda \int dn \Psi_{\lambda n}\Psi_{\lambda n}^\dagger \ . \tag{3.7}$$

If neither $\gamma(\zeta)$ nor $\alpha(z)$ have a real zero, we expect these states to be complete and Π_3 to be $\mathbb{1}$. But if there are zeros we need to include

$$\Pi_2 = \int d\lambda \Psi_\lambda \Psi_\lambda^\dagger \tag{3.8}$$

and

$$\Pi_1 = \Psi_M \Psi_M^\dagger \tag{3.9}$$

with

$$\Pi_j \Pi_k = \delta_{jk}\Pi_j \ , \tag{3.10}$$

$$\Pi_3 + \Pi_2 + \Pi_1 = \mathbb{1} \ . \tag{3.11}$$

We should compute the nine matrix elements of Π_j. Let us begin with the relations

$$\begin{aligned}\alpha(\lambda)-\alpha^*(\lambda) &= -\int f^*(\omega')f(\omega') \left\{ \frac{1}{\gamma(\lambda-\omega'+i\epsilon)} - \frac{1}{\gamma(\lambda-\omega'-i\epsilon)} \right\} d\omega' \\ &= -\int f^*(\omega')f(\omega') \left\{ -2\pi i \frac{\delta(\lambda-\omega'-\mu)}{\gamma'(\mu)} - 2\pi i \frac{g^*(\lambda-\omega')g(\lambda-\omega')}{\gamma(\lambda-\omega'+i\epsilon)\gamma(\lambda-\omega'-i\epsilon)} \right\} d\omega' \\ &= +2\pi i \int f^*(\omega')f(\omega') \left\{ \frac{\delta(\lambda-\omega'-\mu)}{\gamma'} + \frac{g^*(\lambda-\omega')g(\lambda-\omega')}{\gamma(\lambda-\omega')\gamma^*(\lambda-\omega')} \right\} d\omega' \\ &= 2\pi i \left\{ \frac{f^*(\lambda-\mu)f(\lambda-\mu)}{\gamma'} + \int dn' \frac{f^*(\lambda-n')f(\lambda-n')g^*(n')g(n')}{\gamma(n')\gamma^*(n')} \right\} \ .\end{aligned} \tag{3.12}$$

In case $\gamma(\xi)$ has no (real) zero, the first term would be missing. Further, if $\alpha(z)$ has a zero at M,

$$\frac{1}{\alpha(\lambda)}-\frac{1}{\alpha^*(\lambda)}=-2\pi i\frac{\delta(\lambda-M)}{\alpha'}+\frac{\alpha^*(\lambda)-\alpha(\lambda)}{\alpha^*(\lambda)\alpha(\lambda)} . \tag{3.13}$$

We can now compute the various matrix elements of Π_j:

$$\int d\lambda\left[\int dn\; \eta^*_{\lambda n}\eta_{\lambda n}+\eta^*_\lambda\eta_\lambda\right]=\int d\lambda\frac{1}{2\pi i}[\alpha(\lambda)-\alpha^*(\lambda)]\frac{1}{\alpha(\lambda)\alpha^*(\lambda)}$$
$$=\frac{1}{2\pi i}\int d\lambda\left\{\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right]-\frac{2\pi i\delta(\lambda-M)}{\alpha'}\right\} \tag{3.14}$$

so that the (1,1) matrix element of $\Pi_2+\Pi_3$ is

$$(\Pi_2+\Pi_3)_{0,0}=\frac{-1}{2\pi i}\int_{C_2+C_3}\frac{dz}{\alpha(z)} , \tag{3.15}$$

where the subscripts 0,0 are the kinematic label for the (1,1) or (A,A) element, and

$$(\Pi_1)_{0,0}=\frac{1}{\alpha'}=-\frac{1}{2\pi i}\int_{C_1}\frac{dz}{\alpha(z)} .$$

We now explain the contour labels used. Let the contour slightly above the real axis, from 0 to ∞, be C_+. We label the contour deformed from C_+ in the fourth quadrant of the second sheet by C. When there is no singularity between C_+ and C, such as the situation in Fig. 1, the relation $C_+-C=0$ holds. For the situation in Fig. 2, two additional contours C_1 and C_2 need to be taken into account. This leads to $C_+-C=C_1+C_2$.

Let the usual counterclockwise contour on the physical sheet at infinity be C_∞. The absence of singularities on the physical sheet leads to the relation $C_\infty-C_-+C_+=0$, with C_- being the contour, slightly below the real axis, from 0 to ∞. For the situation of Fig. 2,

$$-C_\infty=-C_-+C_+=-C_-+(C_1+C_2+C)=C_1+C_2+(-C'_-+C)=C_1+C_2+C_3 ,$$

where C'_- is the continuation of C_- and C_3 is the counterclockwise contour encircling the three-body branch cut along C.

The (1,1) element of Π is

$$\Pi_{0,0}\equiv(\Pi_1+\Pi_2+\Pi_3)_{0,0}$$
$$\equiv-\frac{1}{2\pi i}\int_{C_1+C_2+C_3}\frac{dz}{\alpha(z)}=+\frac{1}{2\pi i}\int_{C_\infty}\frac{dz}{z}=1 , \tag{3.16}$$

where the relation $C+C_\infty=0$ (see Fig. 1) is used.

The (2,2) or the $B\theta(\omega),B\theta(\omega')$ matrix element of $\Pi_2+\Pi_3$ is

$$(\Pi_2+\Pi_3)_{\omega,\omega'}=\int d\lambda\left[\phi_\lambda(\omega)\phi^*_\lambda(\omega')+\int dn\,\phi_{\lambda n}(\omega)\phi^*_{\lambda n}(\omega')\right]$$
$$=\frac{1}{\gamma'}\int d\lambda\left[\delta(\lambda-\mu-\omega)+\sqrt{\gamma'}\frac{f(\omega)}{\gamma(\lambda-\omega)}\eta_\lambda\right]\left[\delta(\lambda-\mu-\omega')+\sqrt{\gamma'}\frac{f^*(\omega')\eta^*_\lambda}{\gamma^*(\lambda-\omega')}\right]$$
$$+\int d\lambda\,dn\left[\frac{g^*(n)\delta(\lambda-\omega-n)}{\gamma(\lambda-\omega+i\epsilon)}\frac{f(\omega)\eta_{\lambda n}}{\gamma(\lambda-\omega+i\epsilon)}\right]\left[\frac{g(n)\delta(\lambda-n-\omega')}{\gamma^*(\lambda-\omega')}+\frac{f^*(\omega')\eta^*_{\lambda n}}{\gamma^*(\lambda-\omega')}\right]$$
$$=\int d\lambda\{\delta(\omega-\omega')\}\left[\frac{\delta(\lambda-\mu-\omega)}{\gamma'}+\int dn\frac{g^*(n)g(n)}{\gamma(n)\gamma^*(n)}\delta(\lambda-n-\omega')\right]$$
$$+\frac{f(\omega)f^*(\omega')}{2\pi i\gamma^*(\lambda-\omega')\alpha^*(\lambda)}\left[\frac{1}{\gamma^*(\lambda-\omega')}-\frac{1}{\gamma(\lambda-\omega)}\right]+\frac{f(\omega)f^*(\omega')}{2\pi i\gamma(\lambda-\omega)\alpha(\lambda)}\left[\frac{1}{\gamma^*(\lambda-\omega')}-\frac{1}{\gamma(\lambda-\omega)}\right]$$
$$+\frac{f(\omega)f^*(\omega')}{2\pi i\gamma(\lambda-\omega)\gamma^*(\lambda-\omega')}\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right] .$$

Carrying out the dn integration and rewriting the $d\omega$ integral as a contour integral along C_∞, the first term gives $\delta(\omega-\omega')$ while the remaining terms together yield

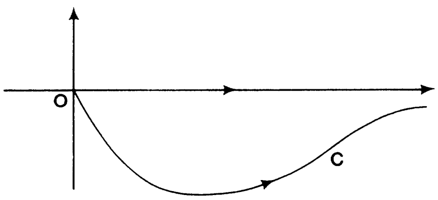

FIG. 1. Contour for integration over λ in demonstrating completeness identity.

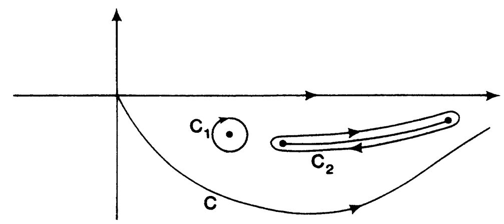

FIG 2. $\gamma^*(n)/\gamma(n)$ as a function of n.

$$\frac{+f(\omega)f^*(\omega')}{2\pi i}\int_{C_\infty}\frac{d\lambda}{\gamma(\lambda-\omega)\gamma(\lambda-\omega')\alpha(\lambda)}=0 .$$

Thus,

$$\phi_M(\omega)\phi_M^*(\omega')+\int d\lambda\phi_\lambda(\omega)\phi_\lambda^*(\omega')+\int dn\,\psi_{\lambda n}(\omega)\psi_{\lambda n}^*(\omega')=\delta(\omega-\omega') . \tag{3.17}$$

For the (3,3) or $[C\theta(\omega)\varphi(\nu),C\theta(\omega')\varphi(\nu')]$ element of Π_3 we compute for the case when neither $\alpha(z)$ nor $\gamma(\zeta)$ have a zero; then we have the expression

$$\begin{aligned}\int d\lambda\int dn\;\psi_{\lambda n}(\omega,\nu)\psi_{\lambda n}^*(\omega',\nu')\\ =\delta(\omega-\omega')\delta(\nu-\nu')+g(\nu)g^*(\nu')\delta(\omega-\omega')\int d\lambda\Bigg\{&\delta(\lambda-\omega-\nu)\frac{1}{(\lambda-\omega'-\nu'-i\epsilon)\gamma^*(\lambda-\omega')}\\ &+\delta(\lambda-\omega'-\nu')\frac{1}{(\lambda-\omega-\nu+i\epsilon)\gamma(\lambda-\omega)}\\ &+\frac{1}{(\lambda-\omega-\nu+i\epsilon)(\lambda-\omega'-\nu'-i\epsilon)}\left[\frac{1}{\gamma^*(\lambda-\omega')}-\frac{1}{\gamma(\lambda-\omega)}\right]\Bigg\} .\end{aligned} \tag{3.18}$$

But the $d\lambda$ integral terms cancel each other so that

$$\Pi_3(\omega,\nu;\omega',\nu')=\delta(\omega-\omega')\delta(\nu-\nu') . \tag{3.19}$$

If $\gamma(\mu)=0$ so that Π_2 is nontrivial, the additional terms

$$\int d\lambda\psi_\lambda(\omega,\nu)\psi_\lambda^*(\omega',\nu')$$

would have to be added to obtain (3.18) and (3.19) would be amended by

$$(\Pi_2+\Pi_3)(\omega,\nu;\omega',\nu')=\delta(\omega-\omega')\delta(\nu-\nu') . \tag{3.20}$$

The same comment applies if $\alpha(M)=0$; then

$$(\Pi_1+\Pi_2+\Pi_3)(\omega,\nu;\omega',\nu')=\delta(\omega-\omega')\delta(\nu-\nu') . \tag{3.21}$$

Next we compute the off-diagonal elements. The (2,1) element of $\Pi_2+\Pi_3$ is

$$\begin{aligned}(\Pi_2+\Pi_3)_{\omega,0}&=\int d\lambda\left[\phi_\lambda(\omega)\eta_\lambda^*+\int dn\,\phi_{\lambda n}(\omega)\eta_{\lambda n}^*\right]\\ &=\int\frac{d\lambda}{\gamma'}\left[\delta(\lambda-\mu-\omega)+\frac{f(\omega)}{\gamma(\lambda-\omega)}\frac{f^*(\lambda-\mu)}{\alpha(\lambda)}\right]\frac{f(\lambda-\mu)}{\alpha^*(\lambda)}\\ &\quad+\int d\lambda\,dn\left[g^*(n)\frac{\delta(\lambda-n-\omega)}{\gamma(\lambda-\omega)}+\frac{f(\omega)}{\gamma(\lambda-\omega)}\frac{f^*(\lambda-n)}{\alpha(\lambda)}\frac{g^*(n)}{\gamma(n)}\right]\frac{f(\lambda-n)g_n}{\alpha^*(\lambda)\gamma^*(n)}\\ &=\int d\lambda\left[\frac{\delta(\lambda-\omega-\mu)}{\gamma'}\frac{f(\lambda-\mu)}{\alpha^*(\lambda)}+\int dn\,g(n)g^*(n)\frac{\delta(\lambda-\omega-n)}{\gamma(n)\gamma^*(n)}\frac{f(\lambda-n)}{\alpha^*(\lambda)}\right]\\ &\quad+\int d\lambda\left[\frac{1}{\gamma'}\frac{f(\omega)f^*(\lambda-\mu)f(\lambda-\mu)}{\gamma(\lambda-\omega)\alpha(\lambda)\alpha^*(\lambda)}+\frac{f(\omega)}{\gamma(\lambda-\omega)}\frac{1}{\alpha(\lambda)\alpha^*(\lambda)}\int dn\frac{|f(\lambda-n)|^2|g(n)|^2}{\gamma(n)\gamma^*(n)}\right]\\ &=\int d\lambda\left\{\frac{f(\omega)}{2\pi i\alpha(\lambda)^+}\left[\frac{1}{\gamma^*(\lambda-\omega)}-\frac{1}{\gamma(\lambda-\omega)}\right]+\frac{f(\omega)}{\gamma(\lambda-\omega)}\frac{1}{2\pi i}\left[\frac{1}{\alpha(\lambda)^*}-\frac{1}{\alpha(\lambda)}\right]\right\} ,\end{aligned} \tag{3.22}$$

where, for the second time, the discontinuity identity of $\alpha(\lambda)$ of (3.12) was used. So

$$(\Pi_2+\Pi_3)_{\omega,0}=\frac{f(\omega)}{2\pi i}\int d\lambda\left[\frac{1}{\alpha^*(\lambda)\gamma^*(\lambda-\omega)}-\frac{1}{\alpha(\lambda)\gamma(\lambda-\omega)}\right]. \tag{3.23}$$

From (2.25),

$$\begin{aligned}(\Pi_1)_{\omega,0}&=\phi_M(\omega)\eta_M^*=\frac{f^*(\omega)}{\gamma(M-\omega)}\eta_M\eta_M^*\\&=\frac{f^*(\omega)}{\gamma(M-\omega)}\frac{1}{\alpha'}=-\frac{f^*(\omega)}{2\pi i}\int_{C_1}d\lambda\frac{1}{\alpha(\lambda)\gamma(\lambda-\omega)}.\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24) and using $C_\infty=-(C_1+C_2+C_3)$ leads to

$$\begin{aligned}(\Pi)_{\omega,0}=(\Pi_1+\Pi_2+\Pi_3)_{\omega,0}&=-\frac{f^*(\omega)}{2\pi i}\int_{C_\infty}d\lambda\frac{1}{\alpha(\lambda)\gamma(\lambda-\omega)}\\&=-\frac{f^*(\omega)}{2\pi i}\int_{C_\infty}\frac{d\lambda}{\lambda^2}=0.\end{aligned} \tag{3.25}$$

The matrix element (3.1) for the three-particle continuum is given by

$$\begin{aligned}(\Pi_3)_{\omega\nu,0}&=\int d\lambda dn\,\psi_{\lambda n}(\omega\nu)\eta_{\lambda n}^*\\
&=\int d\lambda\,dn\left[\delta(n-\nu)\delta(\lambda-n-\omega)\eta_{\lambda n}^*+\frac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\left[g^*(n)\frac{\delta(\lambda-n-\omega)}{\gamma(\lambda-\omega)}+\frac{f(\omega)}{\gamma(\lambda-\omega)}\eta_{\lambda n}\right]\eta_{\lambda n}^*\right]\\
&=\int d\lambda\left[\delta(\lambda-\omega-\nu)\frac{f(\omega)g(\nu)}{\alpha^*(\lambda)\gamma^*(\lambda-\omega)}+\frac{g(\nu)f(\omega)}{\lambda-\omega-\nu+i\epsilon}\frac{g(\lambda-\omega)g^*(\lambda-\omega)}{\gamma(\lambda-\omega)\gamma^*(\lambda-\omega)}\frac{1}{\alpha^*(\lambda)}\right.\\
&\qquad\left.+\frac{g(\nu)f(\omega)}{\lambda-\omega-\nu+i\epsilon}\frac{1}{2\pi i}\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right]\frac{1}{\gamma(\lambda-\omega)}\right]\\
&=\frac{g(\nu)f(\omega)}{2\pi i}\int d\lambda\left[\left[\frac{1}{\lambda-\omega-\nu-i\epsilon}-\frac{1}{\lambda-\omega-\nu+i\epsilon}\right]\frac{1}{\gamma^*(\lambda-\omega)\alpha^*(\lambda)}\right.\\
&\qquad+\frac{1}{\lambda-\omega-\nu+i\epsilon}\left[\frac{1}{\gamma^*(\lambda-\omega)}-\frac{1}{\gamma(\lambda-\omega)}\right]\frac{1}{\alpha^*(\lambda)}\\
&\qquad\left.+\frac{1}{\lambda-\omega-\nu+i\epsilon}\frac{1}{\gamma(\lambda-\omega)}\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right]\right]\\
&=\frac{g(\nu)f(\omega)}{2\pi i}\int d\lambda\left[\frac{1}{\lambda-\omega-\nu-i\epsilon}\frac{1}{\gamma^*(\lambda-\omega)\alpha^*(\lambda)}-\frac{1}{(\lambda-\omega-\nu+i\epsilon)\gamma(\lambda-\omega)\alpha(\lambda)}\right]\\
&=-\frac{g(\nu)f(\omega)}{2\pi i}\int_{C_3}d\lambda\frac{1}{(\lambda-\omega-\nu+i\epsilon)\gamma(\lambda-\omega)\alpha(\lambda)}.\end{aligned} \tag{3.26}$$

We leave it as an exercise for the reader to show that, with the replacement of C_3 by C_2 and C_1, the same expression is also applicable for the two-particle continuum states and the discrete state M, respectively. Since $C_\infty=-(C_3+C_2+C_1)$, we arrive at

$$\begin{aligned}(\Pi)_{\omega\nu,0}=(\Pi_1+\Pi_2+\Pi_3)_{\omega\nu,0}&=-\frac{g(\nu)f(\omega)}{2\pi i}\int_{C_\infty}d\lambda\frac{1}{(\lambda-\omega-\nu)\gamma(\lambda-\omega)\alpha(\lambda)}\\&=-\frac{g(\nu)f(\omega)}{2\pi i}\int_{C_\infty}d\lambda\frac{1}{\lambda^3}=0.\end{aligned} \tag{3.27}$$

The element (3,2) for the 3-particle continuum is given by

$$(\Pi_3)_{\omega\nu,\omega'}=\int d\lambda\, dn\, \psi_{\lambda n}(\omega\nu)\phi^*_{\lambda n}(\omega')$$
$$=\int d\lambda\, dn\left[\delta(n-\nu)\delta(\lambda-n-\omega)\phi^*_{\lambda n}(\omega')+\frac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\phi_{\lambda n}(\omega)\phi^*_{\lambda n}(\omega')\right]$$
$$\equiv\int d\lambda\, dn\left\{\delta(n-\nu)\delta(\lambda-n-\omega)\left[\frac{g(\nu)\delta(\lambda-\omega'-n)}{\gamma^*(\lambda-\omega')}+\frac{f^*(\omega')}{\gamma^*(\lambda-\omega)}\frac{f(\lambda-n)g(n)}{\alpha^*(\lambda)\gamma^*(n)}\right]\right.$$
$$+\frac{g(\nu)}{\lambda-\omega-\nu+i\epsilon}\frac{1}{2\pi i}\delta(\omega-\omega')\left[\frac{1}{\gamma^*(\lambda-\omega)}-\frac{1}{\gamma(\lambda-\omega)}\right]$$
$$\left.+f(\omega)f^*(\omega')\left[\frac{1}{\gamma^*(\lambda-\omega)\gamma^*(\lambda-\omega')\alpha^*(\lambda)}-\frac{1}{\gamma(\lambda-\omega)\gamma(\lambda-\omega')\alpha(\lambda)}\right]\right\}$$
$$=\frac{\delta(\omega-\omega')}{2\pi i}\int d\lambda\left[\left[\frac{1}{\lambda-\nu-\omega-i\epsilon}-\frac{1}{\lambda-\nu-\omega+i\epsilon}\right]\frac{g^*(\nu)}{\gamma^*(\lambda-\omega')}\right.$$
$$\left.+\frac{g^*(\nu)}{\lambda-\omega-\nu+i\epsilon}\left[\frac{1}{\gamma^*(\lambda-\omega)}-\frac{1}{\gamma(\lambda-\omega)}\right]\right]$$
$$=\frac{\delta(\omega-\omega')g(\nu)}{2\pi i}\int d\lambda\left[\frac{1}{\lambda-\nu-\omega+i\epsilon}\frac{1}{\gamma^*(\lambda-\omega')}-\frac{1}{\lambda-\nu-\omega-i\epsilon}\frac{1}{\gamma(\lambda-\omega')}\right]$$
$$=-\frac{\delta(\omega-\omega')g(\nu)}{2\pi i}\int_{C_3}d\lambda\frac{1}{(\lambda-\nu-\omega)\gamma(\lambda-\omega')}\ . \tag{3.28}$$

Again we leave it as an exercise to the reader to show that the same expression is also applicable for the two-body continuum and the discrete state M with the replacement of the contour C_3 by the corresponding contours C_2 and C_M. Using the relation, $-C_\infty=C_3+C_2+C_M$, it leads to

$$\Pi_{\omega\nu,\omega'}=(\Pi_1+\Pi_2+\Pi_3)_{\omega\nu,\omega'}$$
$$=+\frac{\delta(\omega-\omega')g(\nu)}{2\pi i}\int_{C_\infty}\frac{d\lambda}{(\lambda-\nu-\omega)\gamma(\lambda-\omega')}$$
$$=\frac{\delta(\omega-\omega')g(\nu)}{2\pi i}\int_{C_\infty}\frac{d\lambda}{\lambda^2}=0\ . \tag{3.29}$$

The off-diagonal elements $\Pi_{ij}=\Pi^*_{ji}$. Using (3.25), (3.27), and (3.29), the remaining elements are

$$\Pi_{0,\omega}=\Pi^*_{0,\omega}=0\ , \tag{3.30}$$

$$\Pi_{\omega\nu,0}=\Pi^*_{0,\omega\nu}=0\ , \tag{3.31}$$

and

$$\Pi_{\omega',\omega\nu}=\Pi^*_{\omega\nu,\omega'}=0\ . \tag{3.32}$$

IV. THE MÖLLER MATRIX AND THE COMPARISON HAMILTONIAN

The matrix (with continuous indices) of the (ideal) eigenfunctions including the discrete solutions, if any, by virtue of the results demonstrated on orthonormality and completeness, furnishes us the generalized Möller matrix:

$$\Omega=(\Psi_M,\Psi_\lambda,\Psi_{\lambda n}) \tag{4.1}$$

so that

$$\Omega(M_0;M)=\eta_M,\quad \Omega(M_0;\lambda)=\eta_\lambda,\quad \Omega(M_0;\lambda,n)=\eta_{\lambda n}\ ,$$
$$\Omega(\omega;M)=\phi_M(\omega),\quad \Omega(\omega;\lambda)=\phi_\lambda(\omega),\quad \Omega(\omega;\lambda,n)=\phi_{\lambda n}(\omega)\ , \tag{4.2}$$
$$\Omega(\omega,\nu;M)=\psi_M(\omega,\nu),\quad \Omega(\omega,\nu;\lambda)=\psi_\lambda(\omega,\nu),\quad \Omega(\omega,\nu;\lambda,n)=\psi_{\lambda n}(\omega,\nu)\ .$$

The generalized Möller matrix is *unitary*

$$\Omega\Omega^\dagger=\mathbb{1},\quad \Omega^\dagger\Omega=\mathbb{1}\ . \tag{4.3}$$

Further, it diagonalizes the Hamiltonian,

$$H\Omega=\Omega H_C,\quad \Omega^\dagger H\Omega=H_C\ , \tag{4.4}$$

where H_C is the *comparison Hamiltonian:*

$$H_C=\begin{bmatrix} M & 0 & 0 \\ 0 & (\mu+\zeta)\delta(\zeta-\zeta') & 0 \\ 0 & 0 & (\zeta+n)\delta(\zeta-\zeta')\delta(n-n') \end{bmatrix}\ . \tag{4.5}$$

where $\lambda=\mu+\zeta$ and $\zeta+n$. This is to be contrasted with "free Hamiltonian" H_0 obtained from (2.13) by setting $f=g=0$:

$$H_0=\begin{bmatrix} M_0 & 0 & 0 \\ 0 & (\mu_0+\omega)\delta(\omega-\omega') & 0 \\ 0 & 0 & (\omega+\nu)\delta(\omega-\omega')\delta(\nu-\nu') \end{bmatrix}\ . \tag{4.6}$$

When we compare H_C and H_0 we see that, while they have quantitatively the same structure [provided $\alpha(M)=0$, $\gamma(\mu)=0$], their *spectra are different* [15]. There is the mass renormalization of the discrete eigenvalue from M_0 downwards to M; and the continuous spectrum $\mu<\lambda<\infty$ is also shifted downward from $\mu_0<\omega<\infty$. Only the double continuum $0<n<\lambda<\infty$ is coextensive and of the same multiplicity as the double continuum $0<\nu<(\omega+\nu)<\infty$ of H_0. The Möller matrix intertwines H_C and H, but not H_0 and H.

H_C could be identified as the free Hamiltonian if we include *mass and wave-function renomalization* terms in the interaction. The mass renormalization requires that we add to H_0 the quantity Δ given by

$$\Delta=\begin{bmatrix} M-M_0 & 0 & 0 \\ 0 & (\mu-\mu_0)\delta(\omega-\omega') & 0 \\ 0 & 0 & 0 \end{bmatrix}\ , \tag{4.7}$$

which is negative definite. There is the need for a wave-function renormalization and a consequent coupling constant renormalization in view of the structure of (2.24) and (2.23). The A,B fields have wave-function renormalizations

$$\begin{aligned} &B\to Z_B^{-1/2}B=(\gamma')^{1/2}B\ ,\\ &A\to Z_A^{-1/2}A=(\alpha')^{1/2}A\ ,\\ &\theta\to\theta,\quad \Phi\to\Phi\ . \end{aligned} \tag{4.8}$$

Since there are no proper vertex corrections the coupling constant renormalizations reflect the wave-function renormalizations:

$$\begin{aligned} &f(\omega)\to(\gamma'\alpha')^{-1/2}f(\omega)\ ,\\ &g(\nu)\to(\gamma')^{-1/2}g(\nu)\ . \end{aligned} \tag{4.9}$$

Since there are no divergences in the problem, the wave function and coupling constant renormalizations are inessential and the mass renormalization making $H_0+\Delta$ identifiable with H_C is only necessary in this sector.

These renormalizations are sufficient for the higher sectors, e.g., $N_1=1$, $N_2=1$, $N_3=2$, where the mass renormalizations alter the continuum thresholds from M_0 to M and μ_0 to μ but the four-particle threshold at 0 is left unaffected.

The relation (4.4) can be obtained through the definition of the eigenvalue equations,

$$H\Omega=(M\Psi_M,\lambda\Psi_\lambda,\lambda\Psi_{\lambda n})\ ,$$

and the use of the orthogonality relation:

$$\begin{aligned}\Omega^\dagger H\Omega&=\begin{bmatrix} M\langle M|M\rangle & \lambda\langle M|\lambda\rangle & \lambda\langle M|\lambda n\rangle \\ M\langle\lambda'|M\rangle & \lambda\langle\lambda'|\lambda\rangle & \lambda\langle\lambda'|\lambda n\rangle \\ M\langle\lambda'n'|M\rangle & \lambda\langle\lambda'n'|\lambda\rangle & \lambda\langle\lambda'n'|\lambda n\rangle \end{bmatrix}\\ &=\begin{bmatrix} M & 0 & 0 \\ 0 & \lambda\delta(\lambda'-\lambda) & 0 \\ 0 & 0 & \lambda\delta(\lambda'-\lambda)\delta(n'-n) \end{bmatrix}\\ &=H_C\ .\end{aligned}$$

Notice for the (2,2) element, with $\lambda=\mu+\zeta$, $\lambda\delta(\lambda'-\lambda)=(\mu+\zeta)\delta(\zeta'-\zeta)$.

Thus, the unitary transformation, taken in reverse, can convert the comparison Hamiltonian to the total (interacting) Hamiltonian:

$$\Omega H_C\Omega^\dagger=H\ . \tag{4.10}$$

The *notion of interaction is basis dependent.* However, the *distinction* between H_C and H is obvious.

V. THE *S* MATRIX

The solutions we have obtained for the continuum eigenstates of H are not the only ones we could obtain. If we changed reciprocals of the singular operators from

$$(\lambda-\omega-\nu+i\epsilon)^{-1}\to(\lambda-\omega-\nu-i\epsilon)^{-1} \tag{5.1}$$

in (2.17), the solutions (2.23) and (2.24) would acquire the form

$$\Psi'_{\lambda n}=\begin{bmatrix}\eta'_{\lambda n}\\ \phi'_{\lambda n}(\omega)\\ \psi'_{\lambda n}(\omega,\nu)\end{bmatrix}=\begin{bmatrix}\dfrac{f^*(\lambda-n)g^*(n)}{\alpha^*(\lambda)\gamma^*(n)}\\ \dfrac{g^*(n)\delta(\lambda-\omega-n)}{\gamma^*(\lambda-\omega)}+\dfrac{f(\omega)}{\gamma^*(\lambda-\omega)}\eta'_{\lambda,n}\\ \delta(n-\nu)\delta(\lambda-\omega-n)+\dfrac{g(\nu)}{\lambda-\omega-\nu-i\epsilon}\phi'_{\lambda,n}(\omega)\end{bmatrix},$$

$$\Psi'_{\lambda}=\begin{bmatrix}\eta'_{\lambda}\\ \phi'_{\lambda}(\omega)\\ \psi'_{\lambda}(\omega,\nu)\end{bmatrix}=\begin{bmatrix}\dfrac{f^*(\lambda-\mu)}{\sqrt{\gamma'}\alpha^*(\lambda)}\\ \dfrac{1}{\sqrt{\gamma'}}\delta(\lambda-\omega-\mu)+\dfrac{f(\omega)}{\gamma^*(\lambda-\omega)}\eta'_{\lambda}\\ \dfrac{g(\nu)}{\lambda-\omega-\nu-i\epsilon}\phi'_{\lambda}(\omega)\end{bmatrix} \tag{5.2}$$

with

$$\Psi'_M\equiv\Psi_M . \tag{5.3}$$

The Möller matrix

$$\Omega'=(\Psi'_M,\Psi'_\lambda,\Psi'_{\lambda n}) \tag{5.4}$$

is also unitary and intertwines H and H_C:

$$\Omega'\Omega'^{\dagger}=\Omega'^{\dagger}\Omega'=\mathbb{1} ,$$
$$H\Omega'=\Omega' H_C,\quad \Omega'^{\dagger}H\Omega'=H_C . \tag{5.5}$$

Hence,

$$\Omega'^{\dagger}\Omega H_C(\Omega'^{\dagger}\Omega)^{\dagger}=H_C . \tag{5.6}$$

So, if the spectrum of H_C is nondegenerate $\Omega'^{\dagger}\Omega$ is almost a phase: We write

$$S=\Omega^{\dagger}\Omega' , \tag{5.7}$$

$$SH_C=H_CS . \tag{5.8}$$

So for the discrete state S is unity, and for $\lambda<0$ it is, at most, a phase. But, for $\lambda>0$, we have a matrix for S diagonal in energy.

The states (2.23) and (2.24) are so chosen that

$$\lim_{t\to-\infty} e^{iH_Ct}e^{-iHt}\Psi_{\lambda,n}=\begin{bmatrix}0\\0\\ \delta(n-\nu)\delta(\lambda-\omega-n)\end{bmatrix} \tag{5.9}$$

and

$$\lim_{t\to-\infty} e^{iH_Ct}e^{-iHt}\Psi_{\lambda}=\begin{bmatrix}0\\ (\gamma')^{-1/2}\delta(\lambda-\omega-\mu)\\ 0\end{bmatrix}, \tag{5.10}$$

which, apart from the need for the wave-function renormalized (4.8), are the "plane-wave" ideal eigenstates of the comparison Hamiltonian (4.5) with the renormalized threshold for Ψ_λ. These states are, therefore, called the "in" states of scattering.

The vanishing of the first components in (5.9) and (5.10) are attributable to the rapid oscillatory phase factor in the limit where $t\to-\infty$. The equalities for the second and the third components in (5.9) and (5.10) are to be understood for scalar multiplication of both sides by bra vectors whose components are smooth functions of ω. The integration over ω in evaluating the scalar product is along an open contour; for $t\to-\infty$ the contour could be closed in the upper half-plane with the results given in (5.9) and (5.10).

We could equally well evaluate the lim when $t\to+\infty$ but in this case the contour has to be closed in the lower half-plane and thus enclosing the zero at $\lambda=\omega+\nu$.

For $t\to+\infty$ we get

$$\lim_{t\to+\infty} e^{iH_Ct}e^{-iHt}\Psi_{\lambda,n}=\begin{bmatrix}0\\ -2\pi i\delta(\lambda-\omega\mu)\dfrac{f(\lambda\mu)}{\gamma'}\eta_{\lambda n}\\ \delta(\lambda-\omega-\nu)\left[\delta(n-\nu)\left[1-2\pi i\dfrac{g^*(n)g(n)}{\gamma(n)}\right]-\dfrac{2\pi i g(\nu)f(\omega)\eta_{\lambda n}}{\gamma(\lambda-\omega)}\right]\end{bmatrix} \tag{5.11}$$

and

$$\lim_{t\to+\infty} e^{iH_Ct}e^{-iHt}\Psi_{\lambda}\equiv\begin{bmatrix}0\\ \dfrac{1}{\sqrt{\gamma'}}\delta(\lambda-\omega-\mu)\left[1-\dfrac{2\pi i f^*(\lambda-\mu)f(\lambda-\mu)}{\gamma'\alpha(\lambda)}\right]\\ -2\pi i\delta(\lambda-\omega-\nu)\dfrac{g(\nu)f(\lambda-\nu)f^*(\lambda-\mu)}{\sqrt{\gamma'}\gamma(\nu)\alpha(\lambda)}\end{bmatrix}, \tag{5.12}$$

while

$$\lim_{t\to+\infty} e^{iH_C t}e^{-iHt}\Psi_M=\Psi_M \ . \tag{5.13}$$

In contrast with these are the "out" solutions which behave simply as $t\to+\infty$ but have a more elaborate structure as $t\to-\infty$. These also form a complete orthonormal set.

Given the "in" and "out" states, we can compute the scattering amplitude from

$$\Psi_{\text{scattered}}=\lim_{t\to\infty}[\Psi(t)-\Psi(-t)] \tag{5.14}$$

or by considering the scalar product of the "in" and "out" states to get the S matrix:

$$(\Psi',\Psi)=S \ . \tag{5.15}$$

Of course, both these should give the *same* result for the scattering amplitude.

If the form factors are real,

$$f^*(\omega)=f(\omega),\quad g^*(\nu)=g(\nu)\ , \tag{5.16}$$

then the "out" states are simply the complex conjugates of the "in" states. On the other hand, the time-reversed "out" state is the complex conjugate of the "in" state whether or not the form factors are real. Consequently, the S matrix would be *symmetric* for real form factors, that is *for time-reversal-invariant* Hamiltonians.

For convenience, in the following calculations in this section we will assume the form factors to be real so that

$$\Psi'=\Psi^* \ . \tag{5.17}$$

We now proceed to compute the S-matrix element. By virtue of (5.4),

$$S(M;M)=1,\quad S(M;\lambda)=0,\quad S(M;\lambda,n)=0 \ . \tag{5.18}$$

Using (5.17) we can write

$$\begin{aligned} S(\lambda;\lambda')&=(\Psi'_\lambda,\Psi_{\lambda'})\\ &=(\Psi^*_\lambda,\Psi_{\lambda'})\\ &=\eta_\lambda\eta_{\lambda'}+\int d\omega\,\phi_\lambda(\omega)\phi_{\lambda'}(\omega)\\ &\quad+\int d\omega\int d\nu\,\psi_\lambda(\omega,\nu)\psi_{\lambda'}(\omega,\nu)\ . \end{aligned} \tag{5.19}$$

But

$$\int \psi_\lambda(\omega,\nu)\psi_{\lambda'}(\omega,\nu)d\nu =\left[-1+\frac{\gamma(\lambda-\omega)-\gamma(\lambda'-\omega)}{\lambda-\lambda'}\right]\phi_\lambda(\omega)\phi_{\lambda'}(\omega)\ ,$$

so that the $d\omega$ integrals can be written

$$\int d\omega\frac{\gamma(\lambda'-\omega)-\gamma(\lambda-\omega)}{\lambda'-\lambda}\left\{\frac{\delta(\lambda-\omega-\nu)}{\sqrt{\gamma'}}+\frac{f(\omega)\eta_\lambda}{\gamma(\lambda-\omega)}\right\}\left\{\frac{\delta(\lambda'-\omega-\mu)}{\sqrt{\gamma'}}+\frac{f(\omega)\eta_{\lambda'}}{\gamma(\lambda'-\omega)}\right\} =-\eta_\lambda\eta_{\lambda'}-2i\pi\delta(\lambda-\lambda')\alpha(\lambda)\eta_\lambda^2+\delta(\lambda-\lambda')\ .$$

Hence

$$S(\lambda;\lambda')=\delta(\lambda-\lambda')[1-2\pi i\alpha(\lambda)\eta_\lambda^2]\ . \tag{5.20}$$

Below the three-particle threshold

$$S(\lambda;\lambda')=\delta(\lambda-\lambda')\frac{\alpha^*(\lambda)}{\alpha(\lambda)},\quad \lambda<0\ . \tag{5.21}$$

In a similar fashion the production amplitudes for the three-particle channel from a two-particle channel

$$S(\lambda;\lambda',n')=\eta_\lambda\eta_{\lambda'n'}+\int d\omega\left[\phi_\lambda(\omega)\phi_{\lambda'n'}(\omega)+\int d\nu\,\psi_\lambda(\omega,\nu)\psi_{\lambda',n'}(\omega,\nu)\right]\ . \tag{5.22}$$

But

$$\int d\nu\,\psi_\lambda(\omega,\nu)\psi_{\lambda'n'}(\omega,\nu)=\frac{g(n')\phi_\lambda(\lambda'-n')}{\lambda-\lambda'+i\epsilon}+\int d\omega\,\phi_\lambda(\omega)\phi_{\lambda'n'}(\omega)\left[\frac{\gamma(\lambda'-\omega)-\gamma(\lambda-\omega)}{\lambda'-\lambda+i\epsilon}-1\right]\ , \tag{5.23}$$

which together with the other terms in (5.22) yield

$$S(\lambda;\lambda',n')=-2\pi i\alpha(\lambda)\eta_\lambda\eta_{\lambda'n'}\delta(\lambda-\lambda')\ . \tag{5.24}$$

Finally,

$$S(\lambda,n;\lambda',n')=\eta_{\lambda n}\eta_{\lambda'n'}+\int d\omega\left[\phi_{\lambda n}(\omega)\phi_{\lambda'n'}(\omega)+\int d\nu\psi_{\lambda n}(\omega,\nu)\psi_{\lambda'n'}(\omega,\nu)\right]\ . \tag{5.25}$$

But

$$\int d\omega\int d\nu\,\psi_{\lambda n}(\omega\nu)\psi_{\lambda'n'}(\omega\nu)=\delta(n-n')\delta(\lambda-\lambda')+\frac{g(n')\phi_{\lambda n}(\lambda'-n')}{\lambda-\lambda'+i\epsilon}+\frac{g(n)\phi_{\lambda n'}(\lambda-n)}{\lambda'-\lambda+i\epsilon} +\int d\omega\phi_{\lambda n}(\omega)\phi_{\lambda'n'}(\omega)\left[-1+\frac{\gamma(\lambda-\omega)-\gamma(\lambda'-\omega)}{\lambda-\lambda'}\right]\ .$$

Adding the other terms in (5.24) and rearranging, we obtain

$$S(\lambda,n;\lambda'n')=\delta(\lambda-\lambda')\left\{\delta(n-n')-2\pi i\left[\alpha(\lambda)\eta_{\lambda n}\eta_{\lambda' n'}+\frac{g^2(n)}{\gamma(n)}\delta(n-n')\right]\right\}. \tag{5.26}$$

The occurrence of the common factor $\delta(\lambda-\lambda')$ in S was only to be anticipated in view of (5.8).

By writing

$$S(\lambda)=\delta(\lambda-\lambda')[\mathbb{1}+2iT(\lambda)], \tag{5.27}$$

we have the scattering amplitude matrix

$$T(\lambda)=\begin{bmatrix} T(\lambda;\mu,\mu) & T(\lambda;\mu,n') \\ T(\lambda;n,\mu) & T(\lambda;n,n')\end{bmatrix}=-\pi\alpha(\lambda)\begin{bmatrix}\eta_\lambda^2 & \eta_\lambda\eta_{\lambda n'} \\ \eta_\lambda\eta_{\lambda n} & \eta_{\lambda n}\eta_{\lambda n'}+\dfrac{g^2(n)\delta(n-n')}{\alpha(\lambda)\gamma(n)}\end{bmatrix}, \tag{5.28}$$

with the form factor taken to be real. The symmetry of the T matrix is a direct reflection of the implied time-reversal invariance.

VI. UNITARITY OF THE S MATRIX

The unitarity of the S matrix is a consequence of the unitarity of the Möller matrices [see (4.3), (5.6), and (5.7)], and, in turn, implies the unitarity relation of the (dimensionless) scattering amplitude (5.27) in the form

$$T-T^\dagger=2iTT^\dagger . \tag{6.1}$$

Since T is energy diagonal, the λ dependence of the T-matrix elements can be suppressed. It is instructive to verify this relation directly.

Compute the (μ,μ) matrix element of (6.1): the right-hand side of the equation gives the expression

$$\begin{aligned}T(\mu;\mu)T^*(\mu;\mu)+\int dn\, T(\mu;n)T^*(\mu;n)&=\pi^2\alpha(\lambda)\alpha^*(\lambda)\eta_\lambda^*\eta_\lambda\left[\eta_\lambda^*\eta_\lambda+\int dn\, \eta_{\lambda n}\eta_{n\lambda}^*\right]\\&=\pi^2\alpha(\lambda)\alpha^*(\lambda)\left[\frac{1}{2\pi i}\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right]\right]\\&=\frac{\pi\eta_\lambda^*\eta_\lambda}{2i}[\alpha(\lambda)-\alpha^*(\lambda)]=\frac{\pi}{2i}\frac{f(\lambda-\mu)}{\sqrt{\gamma'}}(\eta_\lambda^*-\eta_\lambda),\end{aligned} \tag{6.2}$$

which equals

$$\frac{1}{2i}[-\pi\alpha(\lambda)\eta_\lambda^2+\pi\alpha^*(\lambda)\eta_\lambda^{*2}]=\frac{\pi}{2i}\alpha(\lambda)\eta_\lambda(\eta_\lambda^*-\eta_\lambda)=\frac{\pi}{2i}\frac{f(\lambda-\mu)}{\sqrt{\gamma'}}(\eta_\lambda^*-\eta_\lambda). \tag{6.3}$$

For the $T(\mu,n)$ matrix element of (6.1), the right-hand side is

$$\begin{aligned}T(\mu;\mu)T^*(\mu;n)+\int dl\, T(\mu;l)T^*(n;l)&=\pi^2\alpha(\lambda)\alpha^*(\lambda)\eta_\lambda\eta_{\lambda n}^*\left[\eta_\lambda\eta_\lambda^*+\int dl\,\eta_{\lambda l}\eta_{\lambda l}^*\right]+\pi^2\eta_\lambda\frac{g^2(n)}{\gamma^*(n)\gamma(n)}f(\lambda-n)g(n)\\&=\frac{\pi}{2i}\eta_\lambda\eta_{\lambda n}^*[\alpha(\lambda)-\alpha^*(\lambda)]+\frac{\pi\eta_\lambda}{2i}[\alpha^*(\lambda)\eta_{\lambda n}^*-\alpha(\lambda)\eta_{\lambda n}]\\&=\frac{\pi}{2i}[\alpha^*(\lambda)\eta_\lambda^*\eta_{\lambda n}^*-\alpha(\lambda)\eta_\lambda\eta_{\lambda n}],\end{aligned} \tag{6.4}$$

which coincides with the left-hand side.

For the $T(n,n')$ matrix element of (6.1) the right-hand side is

$$\begin{aligned}&\pi^2\alpha(\lambda)\alpha^*(\lambda)\eta_\lambda\eta_\lambda^*\eta_{\lambda n}\eta_{\lambda n'}^*+\pi^2\int dl\left[\frac{g(n)g(n)}{\gamma(n)}\delta(l-n)+\alpha(\lambda)\eta_{\lambda n}\eta_{\lambda l}\right]\left[\frac{g(n')g(n')}{\gamma^*}\delta(n-l)+\alpha^*(\lambda)\eta_{\lambda n'}^*\eta_{\lambda l}^*\gamma(n)\right]\\&\quad=\frac{\pi}{2i}\alpha(\lambda)\alpha^*(\lambda)\eta_{\lambda n}\eta_{\lambda' n'}^*\left[\frac{1}{\alpha^*(\lambda)}-\frac{1}{\alpha(\lambda)}\right]+\frac{\pi}{2i}g(n)g^*(n)\delta(n-n')\left[\frac{1}{\gamma^*(n)}-\frac{1}{\gamma(n)}\right]\\&\quad\quad+\frac{\pi}{2i}g(n)f(\lambda-n)\left[\frac{1}{\gamma^*(n)}-\frac{1}{\gamma(n)}\right]\eta_{\lambda n'}^*+\frac{\pi}{2i}[\alpha^*(\lambda)\eta_{\lambda n}^*-\alpha(\lambda)\eta_{\lambda n}]\eta_{\lambda n'}^*+\frac{\pi}{2i}[\alpha^*\eta_{\lambda n'}^*-\alpha(\lambda)\eta_{\lambda n'}]\eta_{\lambda n}\\&\quad=\frac{\pi}{2i}[(\alpha(\lambda)\eta_{\lambda n}\eta_{\lambda n'})^*-(\alpha(\lambda)\eta_{\lambda n}\eta_{\lambda n'})]+\frac{\pi}{2i}g(n)g(n)\left[\frac{1}{\gamma^*(n)}-\frac{1}{\gamma(n)}\right]\delta(n-n'),\end{aligned}$$

which coincides with the left-hand side.

VII. EIGENCHANNELS AND EIGENPHASES

For energies $\mu<\lambda<0$, the scattering is elastic since only the $B\theta$ channel is open. For this open channel the S matrix has the unimodular eigenvalue

$$S(\lambda)=e^{i[2\arg\alpha^*(\lambda)]}=\frac{\alpha^*(\lambda)}{\alpha(\lambda)} . \tag{7.1}$$

For $\lambda>0$ the $C\theta\phi$ channel is also open and we have a continuum of states $\Psi_{\lambda n}0<n<\lambda$ which are also open. The S matrix now takes the form

$$S(\lambda;n,n')=\begin{bmatrix} 1+2iT_{\mu,\mu} & 2iT_{\mu,n'} \\ 2iT_{n,\mu} & \delta(n-n')+2iT_{n,n'} \end{bmatrix} . \tag{7.2}$$

The eigenchannels of scattering must satisfy

$$S\zeta=\sigma\zeta,\quad |\sigma|^2=1 , \tag{7.3}$$

which is equivalent to

$$\left\{\sigma-1+\frac{2\pi i g(n)g(n)}{\gamma(n)}\right\}\zeta_n = -2\pi i\alpha(\lambda)\eta_{\lambda n}\left[\eta_\lambda\zeta_\mu+\int dl\;\eta_{\lambda l}\zeta_l\right]$$

or

$$\left\{\sigma-\frac{\gamma^*(n)}{\gamma(n)}\right\}\zeta_n=-2\pi i\alpha(\lambda)\eta_{\lambda n}\left[\eta_\lambda\zeta_\mu+\int dl\;\eta_{\lambda l}\zeta_l\right] \tag{7.4}$$

and

$$(\sigma-1)\zeta_\mu=-2\pi i\alpha(\lambda)\eta_\lambda\left[\eta_\lambda\zeta_\mu+\int dl\;\eta_{\lambda l}\zeta_l\right] . \tag{7.5}$$

There is a continuous spectrum for σ and possibly one or two discrete values. If we define

$$\sigma(n)=\frac{\gamma^*(n)}{\gamma(n)}=\exp[2i\theta(n)] , \tag{7.6}$$

then for

$$\sigma=\exp[2i\theta(n_1)] \tag{7.7}$$

we have the solution

$$\zeta_\mu=-\frac{\eta_\lambda\eta_1}{(\sigma-1+i\epsilon)\chi(n_1)}\frac{1}{\sqrt{\sigma'}} ,\qquad \zeta_n=\sqrt{\sigma'}\delta[\sigma-\sigma(n)]-\frac{\eta_{\lambda n}\eta_1}{(\sigma-\sigma_n+i\epsilon)\chi(n_1)}\frac{1}{\sqrt{\sigma'}} , \tag{7.8}$$

where

$$\eta_1=\eta_{\lambda n_1},\quad \epsilon=0^+,\quad \sigma'=\frac{\partial}{\partial n}\exp[2i\theta(n)]|_{n=n_1} ,$$

and the function $\chi(n_1)$ is given by

$$\chi(n_1)=\frac{1}{2\pi i\alpha}+\left\{\frac{\eta_\lambda^2}{\sigma-1}+\int dl\frac{\eta_{\lambda l}^2}{\sigma-\sigma(l)+i\epsilon}\right\} . \tag{7.9}$$

For the case where there is a bound state, $n=\mu<0$, $\mathrm{Re}\gamma(0)>0$ and $\mathrm{Im}\gamma(0)=0$. Beyond the threshold $\mathrm{Im}\gamma$ is positive. As n goes from 0 to λ, $\theta(n)$ decreases from 0 to $\theta(\lambda)$ and hence $\sigma(n)=\exp[2i\theta(n)]$ moves along a circle of unit radius from $\theta=0$ to $\theta(\lambda)$ in the complex plane; as long as $\gamma(n)$ has no resonance up to energy λ, the unimodular quantity

$$\sigma(n)=\gamma^*(n)/\gamma(n)=\exp[2i\theta(n)]$$

will remain in the fourth quadrant, see Fig. 2.

The solutions (7.8) are continuum normalized:

$$\zeta_\mu(\sigma+i\epsilon)\zeta_\mu(\sigma'-i\epsilon)+\int dl\;\zeta_l(\sigma+i\epsilon)\zeta_l(\sigma'-i\epsilon)=\delta(\sigma-\sigma') . \tag{7.10}$$

This will not be complete if there are discrete zeros of $\chi(\theta)$ for

$$\frac{1}{2i}\ln\sigma<0\quad\text{or}\quad\frac{1}{2i}\ln\sigma>\theta(\lambda) . \tag{7.11}$$

The equation $\chi[(1/2i)\ln\sigma]=0$ implies

$$-\frac{\alpha(\lambda)}{\pi}=\frac{f(\lambda-\mu)f(\lambda-\mu)}{\gamma'}\frac{2i}{\sigma-1}+\int\frac{f(\lambda-\mu)f(\lambda-\mu)g(l)g(l)}{\gamma^*(l)\gamma(l)}\times\frac{2i}{\sigma e^{-2i\theta(l)}-1}dl . \tag{7.12}$$

But the imaginary part of this equation is automatically satisfied by virtue of the relation

$$\frac{\mathrm{Im}\alpha(\lambda)}{\pi}=\frac{f(\lambda-\mu)f(\lambda-\mu)}{\gamma'}+\int\frac{f(\lambda-l)f(\lambda-l)g(l)g(l)}{\gamma^*(l)\gamma(l)}dl . \tag{7.13}$$

Hence

$$F(\cot\delta)\equiv\frac{\mathrm{Re}\alpha(\lambda)}{\pi}+\frac{[f(\lambda-\mu)]^2}{\gamma'}\cot\delta+\int\frac{[g(l)f(\lambda-l)]^2}{\gamma^*(l)\gamma(l)}\cot[\delta-\theta(l)]dl=0 , \tag{7.14}$$

where

$$e^{2i\delta}=\sigma . \tag{7.15}$$

We note that

$$\frac{\partial F(\cot\delta)}{\partial(\cot\delta)}=\left\{[f(\lambda-\mu)]^2\frac{1}{\gamma'}+\int[g(l)f(\lambda-l)]^2\times\frac{\csc^2[\delta-\theta(l)]}{\csc^2\delta}dl\right\}>0 . \tag{7.16}$$

The discrete eigenvalue should be outside the range of $\theta(l)$, $0<l<\lambda$. The two possibilities are δ values less than $\theta(\lambda)$ or positive [more than $\theta(0)$]. This question is discussed in the Appendix.

Usually when we have an S matrix with more than one channel open, the S-matrix elements contain not only a phase but also an inelasticity factor [16] familiar from the case of nucleon-nucleon scattering with tensor forces [17] in the channels with parity $(-)^{J+1}$. In that case the diagonalization of the eigenchannels is energy and potential dependent. In the present case the problem can be completely solved and hence the eigenphases and the (ideal) eigenchannels of scattering can be explicitly determined.

VIII. ANALYTIC CONTINUATIONS OF THE STATE SPACE

In the presentation so far we have considered the state vectors $\Psi_{\lambda n}$, Ψ_λ, and Ψ_M expanded in terms of wave functions $\psi(\omega,\nu)$, $\phi(\omega)$, and η, where ω and ν range over real values $0<\omega$ and $\nu<\infty$. These wave functions are boundary values of analytic functions of ω and ν provided the form factors $f(\omega)$ and $g(\nu)$ are themselves analytic functions. The question arises as to the analytic continuation of these wave functions into the complex plane [18]. This would entail a continuation of the vector space $\mathcal{H}$ of the vectors Ψ into a collection of vector spaces $\mathcal{G}$ [19]. The Hamiltonian operator is defined by the same form with $f^*(\omega)\to f^*(\omega^*)$ and $g^*(\nu)\to g^*(\nu^*)$.

The vectors $\Psi_{\lambda n}$, Ψ_λ, and Ψ_M have wave functions which have analytic forms which have natural extensions to ω,ν becoming complex provided we make an extension of the δ function for complex arguments [20]. This can be done by choosing a deformation of the real spectrum of H to a complex contour. The space of *analytic* continuations of the complete orthonormal set of (ideal) states $\Psi_{\lambda n}$, Ψ_λ, and Ψ_M can be analytically continued so that the spectra are along complex contours. Each of these choices of the contours provide an analytic continuation of the vector space $\mathcal{H}$ to a new space $\mathcal{G}$.

Before presenting the analytic continuation of the vector space of the cascade model in this sector, it would be instructive to deal with the lower sector $B\rightleftarrows C\varphi$ and its analytic continuation [18].

The states in this sector can be written in terms of wave functions

$$\Phi=\begin{bmatrix}\varphi(\mu)\\ \varphi(\nu)\end{bmatrix} \tag{8.1}$$

with an inner product

$$\Phi^\dagger\Phi=\varphi^*(\mu)\varphi(\mu)+\int_0^\infty d\nu\,\varphi^*(\nu)\varphi(\nu) \tag{8.2}$$

and may be completed to form a Hilbert space $\mathcal{H}$. For the effective Hamiltonian in this sector,

$$H=\begin{bmatrix}\mu_0 & g^*(\nu')\\ g(\nu) & \nu\delta(\nu-\nu')\end{bmatrix}, \tag{8.3}$$

we have the (ideal) eigenvectors

$$\Phi_n=\begin{bmatrix}\varphi_n(\mu)\\ \varphi_n(\nu)\end{bmatrix}=\begin{bmatrix}\dfrac{g^*(n)}{\gamma(n)}\\ \delta(n-\nu)+\dfrac{g^*(n)g(\nu)}{\gamma(n)(n-\nu+i\epsilon)}\end{bmatrix} \tag{8.4}$$

with possibly an additional state

$$\Phi_\mu=\begin{bmatrix}\varphi_\mu(\mu)\\ \varphi_\mu(\nu)\end{bmatrix}=\begin{bmatrix}(\gamma')^{-1/2}\\ (\gamma')^{-1/2}\dfrac{g(\nu)}{\mu-\nu}\end{bmatrix}. \tag{8.5}$$

These states are (ideal) orthonormal and complete:

$$\Phi_n^\dagger\Phi_{n'}=\delta(n-n'),\quad \Phi_\mu^\dagger\Phi_\mu=1,\quad \Phi_\mu^\dagger\Phi_n=0\,, \tag{8.6}$$

$$\int dn\,\Phi_n\Phi_n^\dagger+\Phi_\mu\Phi_\mu^\dagger=\mathbb{1}\,. \tag{8.7}$$

There are also left eigenvectors which are the adjoints of the right eigenvectors:

$$\Phi_n^\dagger H=n\Phi_n^\dagger\,. \tag{8.8}$$

The spectrum of n is the positive real axis with μ negative:

$$\mu<0<n<\infty\,.$$

Depending on the value of μ_0, the discrete state may or may not exist.

For analytic continuation we choose a contour Γ which begins at 0 and makes an excursion into the lower half-plane and choose λ along this contour. For z along Γ we denote by $z\pm i\epsilon$ points to one side of Γ or the other and close to it. So ϵ really is a complex infinitesimal quantity normal to Γ and pointing to the left of the contour Γ when we traverse it from 0 to ∞. The δ function $\delta(\lambda-z)$ gives, for integration along the contour Γ,

$$\delta(\lambda-z)=0,\quad \lambda\neq z,\quad \int dz\,\delta(\lambda-z)=1\,. \tag{8.9}$$

These are illustrated in Fig. 3 where both the possibilities of having a discrete state or not having one are displayed.

The continuous spectrum in $\mathcal{G}$ is along Γ with a discrete point at μ which may be real and negative or complex. The states in $\mathcal{G}$ corresponding to (8.4) and (8.5) in $\mathcal{H}$ are

$$\Phi_\lambda=\begin{bmatrix}\dfrac{\bar g(\lambda)}{\gamma(\lambda+i\epsilon)}\\ \delta(\lambda-z)+\dfrac{g(z)\bar g(\lambda)}{\gamma(\lambda+i\epsilon)(\lambda-z+i\epsilon)}\end{bmatrix}, \tag{8.10}$$

$$\Phi_\mu=\begin{bmatrix}(\gamma')^{-1/2}\\ \dfrac{(\gamma')^{-1/2}g(z)}{\mu-z}\end{bmatrix}, \tag{8.11}$$

where

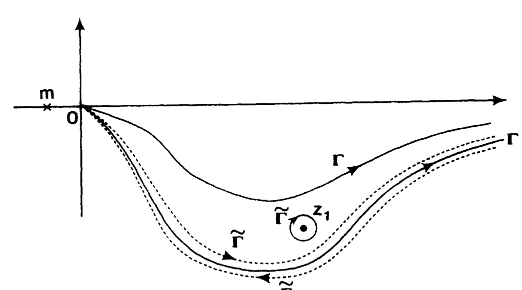

FIG. 3. Spectra of H in $\mathcal{H}$ and in $\mathcal{G}$.

$$\bar{g}(\lambda)\equiv g^*(\lambda^*) \ . \tag{8.12}$$

These are also left eigenvectors [19] in a dual space $\bar{\mathcal{G}}$:

$$\bar{\Phi}_\lambda=\left[\frac{g(\lambda)}{\gamma(\lambda-i\epsilon)},\delta(\lambda-z)+\frac{\bar{g}(z)g(\lambda)}{\gamma(\lambda-i\epsilon)(\lambda-z-i\epsilon)}\right] , \tag{8.13}$$

$$\bar{\Phi}_\mu=\left[(\gamma')^{-1/2}\frac{(\gamma')^{-1/2}\bar{g}(z)}{\mu-z}\right] . \tag{8.14}$$

The orthogonality and completeness relations now take the form

$$\bar{\Phi}_\lambda\Phi_{\lambda'}=\delta(\lambda-\lambda'),\ \ \bar{\Phi}_\lambda\Phi_\mu=0,\ \ \bar{\Phi}_\mu\Phi_\mu=1 \ , \tag{8.15}$$

$$\int_\Gamma \Phi_\lambda\bar{\Phi}_\lambda d\lambda+\Phi_\mu\bar{\Phi}_\mu=\mathbb{1} \ . \tag{8.16}$$

The effective Hamiltonian acting from the left in $\mathcal{G}$ (and acting from the right in $\bar{\mathcal{H}}$) is

$$H=\begin{bmatrix}\mu_0 & \bar{g}(z') \\ g(z) & z\delta(z-z')\end{bmatrix} . \tag{8.17}$$

The verification of the orthogonality relations is straightforward. Completeness may be verified block by block:

$$\phi_\mu(\mu)\bar{\phi}_\mu(\mu)+\int_\Gamma \frac{g(\lambda)\bar{g}(\lambda)d\lambda}{\gamma(\lambda+i\epsilon)\bar{\gamma}(\lambda-i\epsilon)}=\frac{1}{2\pi i}\int_{C_D}\frac{d\lambda}{\gamma(\lambda+i\epsilon)}+\frac{1}{2\pi i}\int_{C_\Gamma}\frac{d\lambda}{\gamma(\lambda+i\epsilon)}=\frac{1}{2\pi i}\int_{C_\infty}\frac{dz}{\gamma(z)}=1 \ , \tag{8.18}$$

where the discrete integration contour C_D, the continuous spectrum C_Γ, and the contour C_∞ are illustrated in Fig. 4. Then we have

$$\begin{aligned}\phi_\mu(\mu)\bar{\phi}_\mu(\mu)\int_\Gamma \phi_\lambda(\mu)\bar{\phi}_\lambda(z)d\lambda&=\frac{\bar{g}(z)}{\gamma'(\mu-z+i\epsilon)}+\int d\lambda\frac{\bar{g}(\lambda)}{\gamma(\lambda+i\epsilon)}\left[\delta(\lambda-z)+\frac{\bar{g}(z)g(\lambda)}{(\lambda-z+i\epsilon)\gamma(\lambda-i\epsilon)}\right] \\ &=\frac{\bar{g}(z)}{2\pi i}\int_{C_D}\frac{d\lambda}{(\lambda-z)\gamma(\lambda)}+\frac{\bar{g}(z)}{2\pi i}\int_{C_\Gamma}\frac{d\lambda}{\gamma(\lambda)(\lambda-z)} \\ &=\frac{\bar{g}(z)}{2\pi i}\int_{C_\infty}\frac{d\lambda}{\gamma(\lambda)(\lambda-z)}=0 \ .\end{aligned} \tag{8.19}$$

Finally,

$$\begin{aligned}\phi_\mu(z)\bar{\phi}_\mu(z')+\int_\Gamma d\lambda\,\phi_\lambda(z)\bar{\phi}_\lambda(z')&=\frac{1}{2\pi i}g(z)\bar{g}(z')\int_{C_D}\frac{d\lambda}{\gamma(\lambda)}\frac{1}{\lambda-z}\frac{1}{\lambda-z'}+\delta(z-z') \\ &\quad+\frac{g(z)\bar{g}(z')}{2\pi i}\int_\Gamma d\lambda\frac{1}{\lambda-z'+i\epsilon}\frac{1}{\gamma(\lambda+i\epsilon)}\left[\frac{1}{\lambda-z-i\epsilon}-\frac{1}{\lambda-z+i\epsilon}\right] \\ &\quad+\frac{g(z)\bar{g}(z')}{2\pi i}\int_\Gamma d\lambda\left[\frac{1}{\lambda-z-i\epsilon}-\frac{1}{\lambda-z'+i\epsilon}\right]\frac{1}{\gamma(\lambda+i\epsilon)}\frac{1}{\lambda-z-i\epsilon} \\ &\quad+\frac{g(z)\bar{g}(z')}{2\pi i}\int_\Gamma d\lambda\frac{1}{\lambda-z'+i\epsilon}\left[\frac{1}{\gamma(\lambda-i\epsilon)}-\frac{1}{\gamma(\lambda+i\epsilon)}\right]\frac{1}{\lambda-z-i\epsilon} \\ &=\delta(z-z')+\frac{g(z)\bar{g}(z')}{2\pi i}\int_{C_\infty}\frac{d\lambda}{(\lambda-z)\gamma(\lambda)(\lambda-z')}=\delta(z-z') \ .\end{aligned} \tag{8.20}$$

Thus, the discrete solution at the (real or complex) value μ is an essential part of the spectrum and together with the continuum $0<\lambda<\infty$ constitute a complete set [18] in the space $\mathcal{G}$.

The spectrum of the Hamiltonian in the A-$B\theta$-$C\theta\varphi$ sector in $\mathcal{H}$ was determined in Sec. II. It consists of an infinitely degenerate continuum $\Psi_{n\lambda}, 0<n<\lambda<\infty$ with a nondegenerate continuum $\Psi_\lambda, \mu<\lambda<\infty$ if there is a real zero for $\gamma(\zeta)$ at $\zeta=\mu<0$; and a discrete state Ψ_M if there is a real zero for $\alpha(z)$ at $z=M$. We have also demonstrated that they are mutually orthogonal (ideal) states and complete in $\mathcal{H}$. When we continue $\Psi_{\lambda n}$ solutions and the corresponding spectrum falls along the contour Γ, we have to look for complex zeros of $\gamma(\zeta)$ and $\alpha(z)$. We know that $\gamma(\zeta)$ would develop a complex zero if there is no real zero, which would be uncovered if Γ sweeps over it. With regard to $\alpha(z)$ there is the possibility that once μ becomes complex there may be more than one complex zero. We denote this collection by M_j, $j=1,2,\ldots$. Then the completeness that we need to verify in $\mathcal{G}$ is

$$\int_{\Gamma_3}d\lambda\int dn\ \Psi_{\lambda n}\bar{\Psi}_{\lambda n}+\int_{\Gamma_2}d\lambda\,\Psi_\lambda\bar{\Psi}_\lambda+\sum_n \Psi_{Mj}\bar{\Psi}_{Mj}=\mathbb{1} \ . \tag{8.21}$$

The set M_j is studied later in this section.

Along the real axis the functions $\gamma(\zeta)$ and $\alpha(z)$ are given by

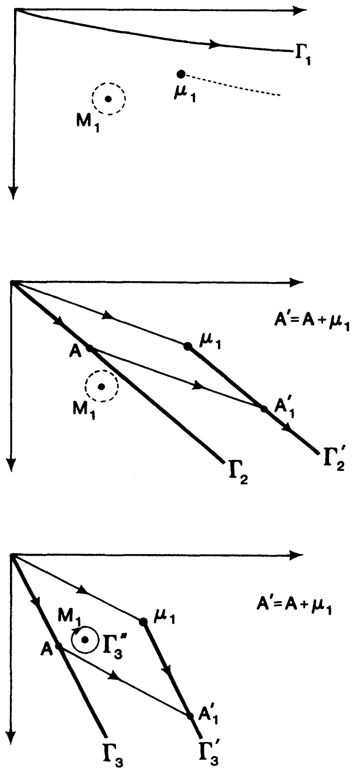

FIG. 4. Spectra and contours of integration in $\mathcal{G}$.

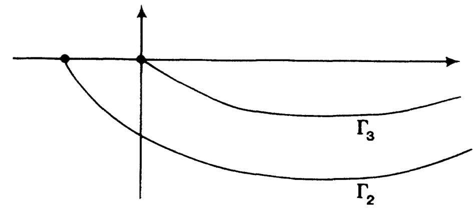

FIG. 5. Spectra in $\mathcal{G}$.

$$\gamma(x)\equiv\gamma(x+i\epsilon)$$
$$=x-\mu_0-\int_0^\infty \frac{g^*(\nu')g(\nu')d\nu'}{x-\nu'+i\epsilon}\,, \tag{8.22}$$
$$\alpha(\lambda)\equiv\alpha(\lambda+i\epsilon)$$
$$=\lambda-M_0-\int_0^\infty \frac{f^*(\omega')f(\omega')d\omega'}{\gamma(\lambda-\omega'+i\epsilon)}\,, \tag{8.23}$$

which, on making analytic continuations along K and Γ, become (see Figs. 4–6)

$$\gamma(\zeta+i\epsilon)=\zeta-\mu_0-\int_K \frac{\bar{g}(\nu)g(\nu)}{\zeta-\nu+i\epsilon}\,, \tag{8.24}$$

$$\alpha(z+i\epsilon)=z-M_0-\int_\Gamma \frac{\bar{f}(\omega)f(\omega)d\omega}{\gamma(z-\omega+i\epsilon)}\,, \tag{8.25}$$

with the understanding that ϵ are complex numbers along the *positive normal* to the contours K and Γ and ν and ω are *complex* numbers along K and Γ. The complex "denominator functions" γ and α have discontinuities across the contours K and Γ. So does $1/\gamma(z)$. We then have

$$\text{Disc}\frac{1}{\gamma(z)}\equiv\frac{1}{\gamma(z)}-\frac{1}{\bar{\gamma}(z)}=-2\pi i\left[\frac{\delta(z-\mu)}{\gamma'}+\frac{\bar{g}(z)g(z)}{\gamma(z)\bar{\gamma}(z)}\right]\,, \tag{8.26}$$

$$\text{Disc}\alpha(z)\equiv\alpha(z)-\bar{\alpha}(z)=-\int d\omega\,\bar{f}(\omega)f(\omega)\text{Disc}\left[\frac{1}{\gamma(z-\omega)}\right] \tag{8.27}$$

$$=2\pi i\left[\frac{\bar{f}(z-\mu)f(z-\mu)}{\gamma'}+\int dn\frac{\bar{f}(\lambda-n)f(\lambda-n)}{\gamma(n)\bar{\gamma}(n)}\bar{g}(n)g(n)\right]\,. \tag{8.28}$$

In Fig. 4, three different configurations of the analytic continuation defined through the deformation of the contour Γ are displayed. The first configuration is defined along $\Gamma=\Gamma_1$, where there is only the three-particle ($C\theta\phi$)-branch cut on the "continued physical plane." The second configuration of Γ is along Γ_2, and also along the quasi-two particle ($\mu\theta$)-branch cut Γ_2'. The third configuration is along Γ_3, Γ_3', the $\mu\theta$ cut, and the contour Γ_3'', which encircles the point M_1, where $\alpha(M_1)=0$. The completeness relation along Γ can be worked out following the same procedure as the computation of the relation in $\mathcal{H}$ carried out in Sec. III. For any given, analytically continued configuration, it is the contour, which encircles all the branch cuts and the discrete poles exposed, that contributes to the generalized spectrum of the theory.

In general, it is also possible to have two zeros of $\alpha(z)$ exposed, which would occur on the two sides of the quasi-two-particle cut. We start afresh with *new notations* here. Let the quasi-two-particle cut be along Γ_2 and the three-particle cut be along Γ_3 (see Fig. 5). The two second sheet discrete zeros of $\alpha(z)$ are at M_1 and M_2 (see Fig. 6). The C_∞ in Fig. 7 would be obtained, when and only when all the exposed cuts and the contour D_1 and D_2 around the two discrete zeros of $\alpha(z)$ are included.

The additional discrete state is the "image state." Let us have a closer look at the situation. Using (8.27) and the asymptotic form of $\alpha(z)$, we can write the integral representation

$$\alpha(z+i\epsilon)=z-M_0-\int d\lambda'\frac{\bar{f}(\lambda'-\mu)f(\lambda'-\mu)}{\gamma'(\lambda-\lambda'+i\epsilon)}-\int d\lambda'\int dn'\frac{\bar{f}(\lambda'-n)\bar{g}(n)f(\lambda'-n)g(n)}{(\lambda-\lambda'+i\epsilon)\bar{\gamma}(n)\gamma(n)}\,. \tag{8.29}$$

Therefore,

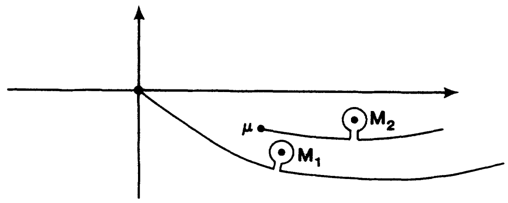

FIG. 6. Spectra in $\mathcal{G}$.

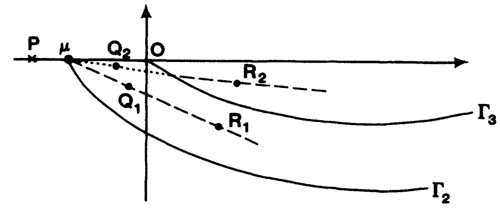

FIG. 8. Movement of the resonance when $\mu<0$.

$$\alpha'(z)=1+\int\frac{\bar{f}(\lambda'-\mu)f(\lambda'-\mu)}{\gamma'(z-\lambda)^2}d\lambda'+\int d\lambda'\int dn'\frac{\bar{f}(\lambda'-n)\bar{g}(n)f(\lambda-n)g(n)}{\bar{\gamma}(n)\gamma(n)(z-\lambda')^2}\gamma'$$
$$=1+\int\frac{\bar{g}(\nu)g(\nu)d\nu}{(\mu-\nu)^2}\;. \tag{8.30}$$

There is a "bound state" at M if

$$\alpha(0)>0,\quad \alpha(M)=0,\quad \alpha(\infty)=-\infty\;. \tag{8.31}$$

If $\alpha(0)<0$ there is no real zero of $\alpha(z)$ but a complex zero appears when the contour Γ sweeps sufficiently down in the complex plane. We distinguish two cases.

Case 1. $\mu<0$. The trajectory of the zero of $\alpha(z)$ is given by the line P,Q,R in Fig. 8. As M_0 increases the real zero disappears at the threshold $\mu<0$ and then moves into the lower half-plane first to positions such as Q_1 where Re$M<0$ and eventually to places such as R_1 where Re$M>0$. To the extent that the discontinuity along Γ_3 is weak to start with, there should be a nearby zero on the sheet reached by analytic continuation through winding around the branch point of Γ_3, i.e., the origin in a clockwise manner. The zero trajectory as a function of M_0 on this sheet is indicated in Fig. 8 by the dotted line where Q_2 is a typical point. As M_0 further increases, this zero trajectory emerges from the contour Γ_3 and extends into the exposed region. It appears as the dashed trajectory in the figure with R_2 being a typical point. The completeness integrals would include the combination of the poles exposed by the contours (see Fig. 9).

Case 2. Re$\mu>0$, μ complex. In this case the real continuous spectrum in $\mathcal{H}$ consists of only the three-particle infinitely continuum $0<\lambda<\infty$. When Γ_3 sweeps down the continuous spectrum, Γ_2 in $\mathcal{G}$ appears. The zero of $\alpha(z)$ now moves along P to Q and this trajectory bifurcates at the threshold μ of Γ_2. A zero appears at R_1 and an image zero at R_2. Depending upon the location of Γ_2, R_2 may or may not enter the complete set of states in $\mathcal{G}$.

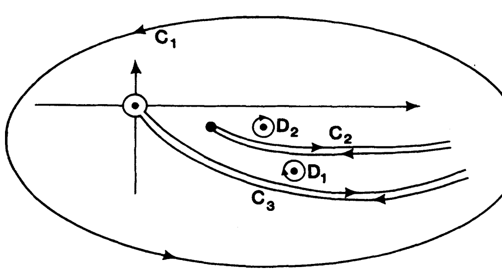

FIG. 7. Contours of intergration in $\mathcal{G}$.

We have not considered other possible image zeros for "deep" analytic continuation when the contours sweep past the third quadrant. For the kind of contour configurations that we have considered they do not enter in the determination of the complete set of states.

IX. CONCLUDING REMARKS: FUTURE PROSPECTS

In this paper we have explored the spectra, the complete set of (ideal) state vectors in a three-particle sector, and their analytic continuations. Because of the structure of the effective Hamiltonian in this sector the solution can be carried out explicitly. We have traced the evolution of the two-particle $B\theta$ spectrum coupled to the three-particle $C\theta\varphi$ spectrum and a possible discrete A state.

It is gratifying that the spectra are such as one would expect from a B particle of energy $\mu<0$ coupled to a θ particle of energy ω, $0<\omega<\infty$; and a C particle of energy 0 coupled to θ,φ particles of energy ω,ν; $0<\omega$, $\nu<\infty$. It is reassuring that the *interacting* field theory has a *particle interpretation*.

The "mass" renormalization of the B particle alters the continuum threshold of the $B\theta$ states. While we are familiar with the "mass" renormalization for a discrete

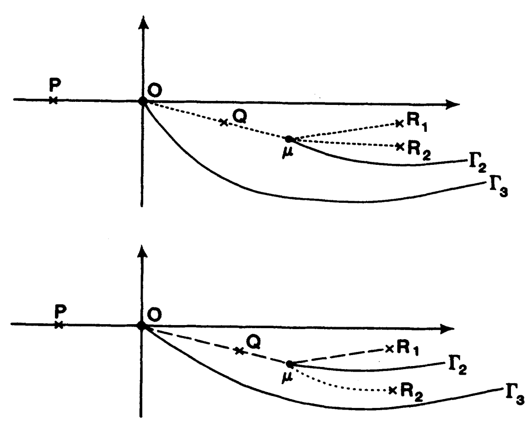

FIG. 9. Movement of resonance position for complex μ.

state, its propagation to alter the threshold of the $B\theta$ continuum is not so familiar. True, it can be seen in the higher sector of the Lee model but the analytic structure of the solution is formidable. Not only the "mass" renormalization but also the "wave-function" renormalization of the B particle is seen explicitly in the plane-wave component of the solution (2.24) and (2.25).

The diagonalized form of the (effective) Hamiltonian is what we have called the comparison Hamiltonian. The comparison Hamiltonian is isospectral with the total Hamiltonian. Thus, its spectrum differs from that of the free Hamiltonian in the displacement of the two-particle continuum in addition to the discrete values. This is a nonperturbative effect and can be handled only by a renormalized perturbation scheme in which the comparison Hamiltonian is taken as the starting point.

We have also considered the analytic continuation of the vector space $\mathcal{H}$ into a generalized vector space $\mathcal{G}$. The correspondence is between the dense subset of analytic vectors in $\mathcal{H}$ to those in $\mathcal{G}$ [19]. The explicit solutions for the (ideal) eigenvectors can be analytically continued into a corresponding set of (ideal) eigenvectors of the analytically continued Hamiltonian in $\mathcal{G}$. The adjoint left eigenvectors are continued into left eigenvectors in $\overline{\mathcal{G}}$, the dual to $\mathcal{G}$. Despite the continuous spectrum changing from being along the real axis in $\mathcal{H}$ to being along complex contours Γ_2 and Γ_3 in $\mathcal{G}$, we can get a complete set of (ideal) eigenvectors in $\mathcal{G}$. The spectra in $\mathcal{H}$ and in $\mathcal{G}$ may differ in the set of discrete states that need be included or even in the continuous spectrum. But these are implicit in the analytic form of the solutions in $\mathcal{H}$. The relatively simple analytic form makes the continuations rather straightforward. This enabled us to explore a range of circumstances in which either the A particle or the B particle (or both) became unstable.

We have, in this paper, used the Dirac formalism [21] of ideal eigenstates corresponding to points in the continuous spectrum. These "eigenvectors" are not in the space $\mathcal{H}$ or $\mathcal{G}$ since they do not have a finite norm. Yet they are very convenient to use. A more rigorous way of handling the continuous spectrum is to use the formalism of Gelfand triplets [22] of these linear spaces $\mathcal{H}_1$, $\mathcal{H}$, and $\mathcal{H}_2$ such that

$$\mathcal{H}_1 \supset \mathcal{H} \supset \mathcal{H}_2 , \qquad (9.1)$$

where $\mathcal{H}_2$ is the dual of $\mathcal{H}_1$ with $\mathcal{H}$ remaining its own dual. Böhm has advocated the use of this rigged Hilbert space [23] formalism for quantum mechanics. In the problem of analytic continuation of the vector space $\mathcal{H}$, this formalism can be extended. It has already been carried out for the Lee model by Gorini, Parravicini, and Sudarshan [24], and by Böhm [24]. The cascade model may also be handled using this more rigorous formalism, though no essentially new results would be obtained. Then we will have to introduce spaces $\mathcal{G}_1, \supset \mathcal{G}$ and $\overline{\mathcal{G}}_2 \subset \overline{\mathcal{G}}$. The continuum eigenvectors would be in $\mathcal{G}_1$, but not $\mathcal{G}$. We shall carry out this refinement elsewhere.

We found that when we explore the $B\theta/C\theta\varphi$ sector it can be thought of as the addition of the θ particle to the $B/C\varphi$ sector. This kind of particle interpretation continues to be true under analytic continuation. This would justify our introducing creation and annihilation operators for complex energy particles and thus have a field theory which is analytically continued. For this purpose we define annihilation operators with complex energies,

$$A, B, C, \theta(z), \varphi(\zeta) ,$$

and their adjoints

$$\overline{A}, \overline{B}, \overline{C}, \overline{\theta}(z), \overline{\varphi}(\zeta)$$

with the commutation relations

$$[\varphi(\nu), \overline{\varphi}(\nu')] = \delta(\nu - \nu'), \qquad (9.2)$$

etc., where $\delta(\nu-\nu')$ is defined on a complex contour Γ. The Hamiltonian is

$$H = H_0 + V ,$$

$$H_0 = M_0 \overline{A} A + \mu_0 \overline{B} B + \int_{\Gamma_2} dz\, z \overline{\theta}(z) \theta(z) \qquad (9.3)$$

$$+ \int_{\Gamma_1} d\zeta\, \zeta \overline{\varphi}(\zeta) \varphi(\zeta) , \qquad (9.4)$$

$$V = \int_{\Gamma_2} dz \{ f(z) \overline{B} \overline{\theta}(z) A + \overline{f}(z) \overline{A} B \theta(z) \}$$

$$+ \int_{\Gamma_1} d\zeta \{ g(\zeta) \overline{C} \overline{\varphi}(\zeta) B + \overline{g}(\zeta) \overline{B} C \varphi(\zeta) \} , \qquad (9.5)$$

where the contours Γ_1 and Γ_2 may be chosen independently. The parameters m_0 and μ_0 are real. Γ_1 and Γ_2 both begin at 0. The conserved quantities (2.6) are appropriately modified and so are the denominator functions $\gamma(\zeta)$ and $\alpha(z)$.

In the treatment of the Hamiltonian (2.3) with conventional real frequency creation and annihilation operators, we could obtain time-reversal invariance if the form factors $f(\omega)$ and $g(\nu)$ are real [25]. If they are not real the phases would have to be absorbed into a redefinition of $\theta(\omega)$ and $\varphi(\nu)$ before we could display time-reversal invariance.

A corresponding situation is obtained for the cascade model field theory with complex contours if

$$\overline{g}(\zeta) = g(\zeta), \quad \overline{f}(z) = f(z), \quad M_0^* = M_, \quad \mu_0^* = \mu_0 , \qquad (9.6)$$

then the system is time-reversal invariant.

Using an extension of the techniques developed by Bolsterli and by Nelson, the still higher sectors of the cascade model can be worked out explicitly; we defer these solutions to a later investigation.

With a suitable choice of the parameters here the $B-\theta$ channel disappears in $\mathcal{H}$ but can be recovered by analytic continuation. We have thus studied scattering of an unstable particle [26]. Throughout we have recognized the importance of identifying the complete set of states which could differ from what may be deduced from the singularities of the scattering amplitude. In this paper only the barest outlines of analytically continued quantum field theory are given; yet it shows the unreliability of both conventional perturbation theory and of the analytic properties of scattering amplitudes by themselves.

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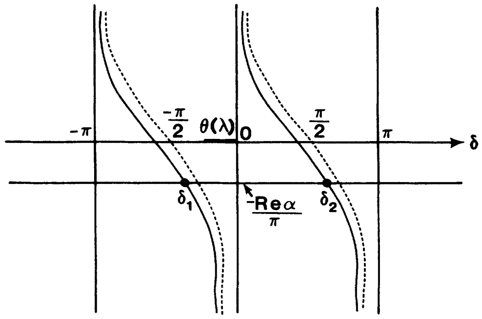

FIG. 10. Spectrum of the eigenphase shift δ.

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APPENDIX: THE DISCRETE EIGENVALUES FOR THE EIGENPHASE SHIFT

We explore the spectrum of the scattering matrix starting with the defining equations

$$(\sigma-1)\zeta_0=-2\pi i\alpha(\lambda)\eta_0(\lambda)\{\eta_0(\lambda)\zeta_0+\langle\eta_n(\lambda)\zeta_n\rangle\}\ , \tag{A1}$$

$$\langle(\sigma-\tau_n\rangle=-2\pi i\alpha(\lambda)\eta_n(\lambda)\{\eta_0(\lambda)\zeta_0+\langle\eta_n(\lambda)\zeta_n\rangle\}\ , \tag{A2}$$

where $\tau_n=\gamma_n^\dagger(\lambda)/\gamma_n(\lambda)$ is unimodular. Writing

$$Z=-2\pi i\alpha[\eta_0(\lambda)\zeta_0+\langle\eta_n(\lambda)\zeta_n\rangle]\ , \tag{A3}$$

we rewrite these equations in the form

$$\zeta_0=Z\eta_0(\sigma-1)^{-1},\quad \zeta_n=Z\eta_n(\sigma-\tau_n)^{-1}\ . \tag{A4}$$

This implies

$$Z=\zeta_0\eta_0+\langle\zeta_n\eta_n\rangle=Z\left\{\frac{\eta_0^2}{\sigma-1}+\left\langle\frac{\eta_n^2}{\sigma-\tau_n}\right\rangle\right\} \tag{A5}$$

or

$$\eta_0^2(\sigma-1)^{-1}+\langle\eta_n^2(\sigma-\tau_n)^{-1}\rangle=1\ . \tag{A6}$$

As mentioned in the text (Sec. VII) the imaginary part of this equation is an identity. Therefore, the defining equation can be rewritten as the real part of this above relation. Introducing

$$e^{2i\delta}=\sigma,\quad e^{2i\theta(n)}=\tau(n)=\frac{\gamma^\dagger(n)}{\gamma(n)}\ , \tag{A7}$$

we obtain

$$0=F(\cot\delta)\equiv\frac{1}{\pi}\,\mathrm{Re}\alpha(\lambda)+\frac{1}{\gamma'}f^*(\lambda-\mu)f(\lambda-\mu)\cot\delta + \int dn\frac{g^*(n)g(n)}{\gamma^*(n)\gamma(n)}f^*(\lambda-n)f(\lambda-n) \times\cot[\delta-\theta(n)]\ . \tag{A8}$$

From their definition, $\sin\theta(n)<0$.

The function $F(\cot\delta)-(1/\pi)\mathrm{Re}\alpha(\lambda)$ is sketched as a function of δ in Fig. 10. The term with cotδ is the familiar dashed curve with asymptotes at $\delta=0,\pm\pi,\ldots$ and intersecting the δ axis at $\pm\pi/2,\pm3\pi/2$, etc. The integrand in the *second term* would be similar but displaced towards the left (since $\theta<0$) by $|\theta(n)|$ and suitably weighted.

The line $F(\cot\delta)=0$, or

$$F(\cot\delta)-\frac{1}{\pi}\,\mathrm{Re}\alpha(\lambda)=-\frac{1}{\pi}\,\mathrm{Re}\alpha(\lambda)$$

intersects the curve in two points δ_1 and δ_2 which are the discrete values of the eigenphase shifts. In addition, there is a continuous spectrum along $\delta=\theta(n)$, where $0<n<\lambda$, as deduced in the text [see discussion after (7.16)].

In an earlier work based on two-body local potential models, Newton [27] showed that the three-particle S matrix divided by the two-particle S matrices has only discrete eigenvalues. Our model gives this result in an even more restricted form: the continuous eigenvalues of the three-particle amplitude is the same as the two-particle amplitude.

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