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Can Infinite Multiplets Be Inferred from the Weak Decay Rates?*

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An algebraic model $\mathcal{B}$ describing symmetry breaking, which was considered in a previous paper for the mass spectrum problem, is applied to the weak decays K_{l3}, K_{l2}, π_{l3}, and π_{l2}. It gives constant form factors, $\xi=1$, and a suppression of the strangeness-changing decays (Cabibbo angles) of $S_{l2}=0.28$ ($\tan\theta_A{}^M$) and $S_{l3}=0.18$ ($\tan\theta_V{}^M$). The results also allow the determination of all parameters that characterize the representation of $\mathcal{B}$ and its spectrum-generating group $SL(3,c)$, from which the infinite dimensionality of the multiplets can be inferred.

I. INTRODUCTION

In a previous paper[1] we have introduced an associative albegra $\mathcal{B}$ as a model for the mesons and obtained a mass spectrum [(37) of II] which was in reasonable agreement with experimental data. In the meantime this method of infinite multiplets has become quite popular[2,3] and there exists a variety of approaches to the derivation of dynamical properties. Therefore, it appears worthwhile to investigate further consequences of the algebraic structure and its application to weak interactions.

The $V-A$ theory of weak interactions of nonstrange particles can now be considered to be well established,[4] with the magnitude of the axial-vector coupling constant in β decay having been computed from pion-nucleon scattering data. The treatment of the weak interactions of strange particles, by contrast, appears to suffer from the need to introduce two new smallness parameters called the Cabibbo angles. It would be desirable to relate the apparent suppression of the strangeness-changing weak decays to the marked violation of unitary symmetry. In a successful model of the weak interactions the Cabibbo angles would themselves be deduced in terms of the unitary-symmetry violation as manifested in the very different masses of the pion and the kaon.

From another point of view we should also seek to establish a connection between strong and weak interactions. The $V-A$ structure of weak interactions is reproduced in the existence of both vector and pseudovector couplings in strong interactions. The departure from unitary symmetry of strong interactions would then be expected to imply differences between strangeness-conserving and strangeness-violating weak interactions.[4]

We propose to study the question of weak meson decays within the framework of the algebraic model introduced earlier.

Before we turn to the subject of this paper—the application of $\mathcal{B}$ in the calculation of weak meson decay rates—we want to give a brief description of the properties of $\mathcal{B}$. The algebra $\mathcal{B}$ contains the enveloping algebra of the Poincaré group, $\mathcal{E}(\mathcal{P})$, and the enveloping algebra of $SL(3,c)$. It is generated by the usual P_μ, $L_{\mu\nu}$ and the generators of the spectrum-generating group $SL(3,c)$: H_i, $E_{\pm\alpha}$, G_i, and $F_{\pm\alpha}$, where $i=1, 2, \alpha=1, 2, 3$, and the multiplication is defined by the basic relations (2)–(6) of II. The nontriviality of the combination of $\mathcal{P}$ and $SL(3,c)$ in $\mathcal{B}$ is expressed by the relation (5) of II; in particular, the relation (5g) gives the mass splitting inside an $SU(3)$ multiplet [which is in good agreement with experimental data (Gell-Mann–Okubo formula)], and the relations (5h) and (5i) give the mass splitting between the different $SU(3)$ multiplets in an irreducible representation of $\mathcal{B}$, which is in not as good agreement with experiment and leads in the representation of $\mathcal{B}$, which we want to consider and which contains the pseudoscalar-meson octet, to further difficulties. In the definition of $\mathcal{B}$ appears a constant g, which is a "universal constant" of our model and whose value is determined from the mass spectrum to be $g=0.142$ BeV^2.

The physical system—which contains the various mesons as different states—is described by an irreducible representation of $\mathcal{B}$ which is characterized by four numbers: m_3, ρ_3, s, and z (connected with the eigenvalues of the invariant operators of $\mathcal{B}$). (m_3,ρ_3) characterize the degenerate series representation of $SL(3,c)$ which is contained in $\mathcal{B}$[5], s is the spin, which is an invariant in our model, and z is the invariant appearing in the mass formula. For the particular physical system which contains the pseudoscalar mesons π, K, and η, the four numbers have the following values: $s=0$; $z=-0.111$

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[1] A. Böhm, Phys. Rev. **158**, 1408 (1967), hereafter referred to as II. Relation (5i) of II contains a printing error and an omission; it must read

$$[MP_\mu,F_\alpha M]=4gr_2(\alpha)\{H_2,F_\alpha\}P_\mu +3g\Big(\sum_{\beta=\pm2,\pm3} N_{-\beta\alpha}\{E_\beta,F_{\alpha-\beta}\}-\partial_{\alpha\beta}r^i(\beta)\{E_\beta,G_i\}\Big)P_\mu.$$

In the right-hand side of $(5h) r_1(\alpha)$ is to be replaced by $r_i(\alpha)$.

[2] Y. Nambu, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Interscience Publishers, Inc., New York, 1967), p. 347, and references therein.

[3] A. O. Barut, report, Boulder Lectures, 1967 (unpublished); A. Böhm, Syracuse University Report No. SU-125, Boulder Lectures, 1967 (unpublished).

[4] E. C. G. Sudarshan, Nature **216**, 979 (1967); Proc. Roy. Soc. (London) **305A**, 319 (1968).

[5] The principal series representations of $SL(3,c)$ have been excluded because this would require the introduction of additional quantum numbers.

BeV^2, determined from the mass formula (the dimensionless $\zeta=z/g=-0.782$); $m_3=0$, determined from the fact that only the representations $(0,\rho_3)$ contain an $SU(3)$ octet; and $\rho_3=2b$, where b will be determined later.

In the irreducible representation space $\mathcal{H}(\zeta,s,(0,\rho_3))$ we have the basis of generalized eigenvectors

$$|I,I_3,Y,\lambda;\,p_is_3\rangle=|\alpha;\,p_is_3\rangle\,, \qquad (1.1)$$

where λ characterizes the $SU(3)$ multiplets. In Fig. 1 we show the lowest states of the "weight diagram" of $\mathcal{B}(\zeta,s,(0,\rho_3))$ for a given value of $p/m=v$ [which is the weight diagram of the $SL(3,c)$ representation $(0,\rho_3)$ with the only difference that to each point is also assigned a definite mass given by (37) of II and a momentum $p=mv$, $v=\mathrm{const}$]. The generators $E_{\pm\alpha}$ transform between the states of an $SU(3)$ multiplet, not only changing the intrinsic quantum numbers, but also changing the mass in such a way that (37) of II is always fulfilled; the $F_{\pm\alpha}$ and the G_i transform between states of neighboring $SU(3)$ multiplets, and in general (except for $\rho_3=0$) also between states of the same $SU(3)$ multiplet [as given by (A9) of Appendix A]—also changing simultaneously the mass of the state. Since the mass is determined by the intrinsic quantum numbers, it is not needed for the labeling of the states; by m_K, m_π is meant the value of m given by (37) of II with the values of I, Y, and λ being those for the K, π, $\cdots$. However, as we have remarked above, the λ dependence in (37) of II is probably not correct and the defining relations (5h) and (5i) are not the right ones and will have to be replaced with relations giving a better mass formula. The only consequence of (5h) and (5i) that we need in our present work is the value of m_σ, the mass of the hadron state with the quantum numbers of the vacuum: $\sigma=(I=0,\,I_3=0,\,Y=0,\,\lambda=0)$.[6] The mass formula (37) of II gives an unphysical negative value of $m_\sigma{}^2$; we shall use, instead, the intuitively correct value $m_\sigma=0$.[7]

From Fig. 1 we see that the transition from a state with hadron quantum numbers of the K^+ to a state with hadron quantum numbers of π^0 is performed by the operators E_{-2} and F_{-2}, and that the transition from a K^+ state to the state with hadron quantum numbers of the vacuum σ is performed by F_{-2}. Transitions of this kind occur in nature in the weak decay $K\to\pi^0 l\nu$ and $K\to\mu\nu$, respectively. This suggests a connection between these decays and the transition matrix elements of the corresponding operators in our model. The lepton pair in the weak decay cannot, of course, be treated in the frame of our model for hadrons. Therefore we must split the transition operator into two parts: the hadronic part, of which we assume that it carries the dynamics of the process and which is given by our model, and the kinematical leptonic part, for which we make the assumption that it has the usual well-known form.

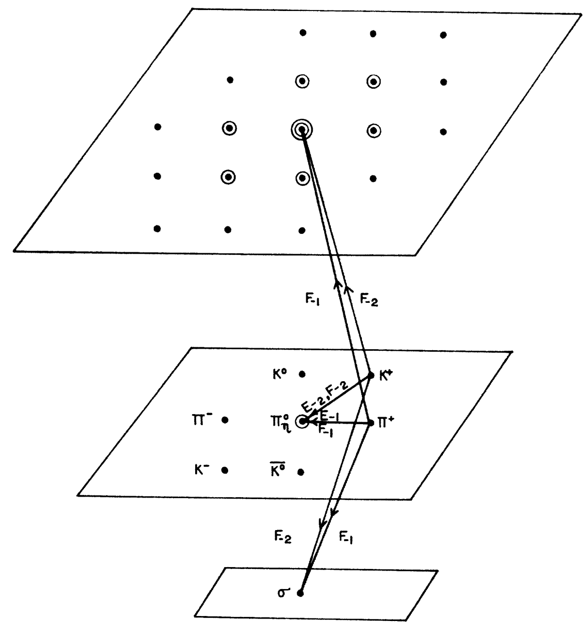

FIG. 1. Weight diagram of $\mathcal{B}(\zeta,s,(0,\rho_3))$ or weight diagram of the degenerate series representation of $SL(3,c)$ which contains an octet.

II. PREREQUISITES

We want to apply our scheme to the following weak meson decays

$$\begin{aligned} &K^+\to\mu\bar\nu, \quad \pi^+\to\mu\bar\nu\,,\\ &K^+\to\pi^0 l\nu, \quad \pi^+\to\pi^0 l\nu\,. \end{aligned} \qquad (2.1)$$

We assume that the decaying K^+ system (similarly for the π^+ system) is described by a physical state ϕ_K. As a physically preparable state, $\phi_K\in\phi$, where

$$\phi\subset\mathcal{H}(z,s,(0,\rho_3))\subset\phi^\times$$

is the rigged Hilbert space of II.

We use the abbreviations

$$|\mathfrak{p}_K,K^+\rangle = |I=\tfrac12,I_3=\tfrac12,\,Y=1,\lambda=1,\,\mathfrak{p},s_3;\,(0,\rho_3),\,s=0,\zeta\rangle \quad (2.2)$$

(and similarly for π^+, π^0, σ, or generally α); assuming the normalization (B3), (B6) of Appendix B, we can write

$$\begin{aligned}\phi_K&=\int\frac{d^3\mathfrak{p}_K}{2E_K(\mathfrak{p}_K)c_K}|\mathfrak{p}_K,K^+\rangle\langle K^+,\mathfrak{p}_K|\phi\rangle\\ &=\int\frac{d^3\mathfrak{p}_K}{2c_KE_K(\mathfrak{p}_K)}|\mathfrak{p}_K,K^+\rangle\phi(\mathfrak{p}_K)\,. \end{aligned}\qquad (2.3)$$

[6] This model does not predict the existence of a "σ meson" and the σ state might have to be thought of in the same manner as the vacuum state in quantum field theory.

[7] We retain $m_\sigma\neq0$ in the calculation and let $m_\sigma\to0$ in the final formula (4.2), which is the only place where we need the value of m_σ.

$\phi_K(\mathfrak{p}_K)$ is, as a (vector) function of $\mathfrak{p}_K$, an element of the test-function space which is a realization of the space ϕ, and

$$1=(\phi_K,\phi_K)=\int\frac{d^3\mathfrak{p}_K}{2c_KE_K(\mathfrak{p}_K)}|\phi_K(\mathfrak{p}_K)|^2. \tag{2.4}$$

The reflection of the physical situation that a state with exact momentum cannot exist with our mathematical description is that $|\phi_K(\mathfrak{p}_K)|^2$ cannot be a δ distribution but must be a test function. Under this assumption the following remains mathematically correct. However, to make connection with the usual calculation, we shall in the final stage replace the assumption that ϕ_K is a state of almost exact momentum $\mathfrak{q}$ [i.e., that $\phi_K(\mathfrak{p}_K)$ is a test function which is sharply peaked at $\mathfrak{p}_K=\mathfrak{q}$] with the assumption that ϕ_K is a "state with exact momentum $\mathfrak{q}$." Then $\phi_K(\mathfrak{p}_K)$ has to be chosen as

$$|\phi_K(\mathfrak{p})|^2=2E_Kc_K\delta^3(\mathfrak{p}-\mathfrak{q}) \tag{2.5}$$

in order that ϕ_K fulfill the "normalization" (2.4):

$$\text{``}\int\frac{d^3\mathfrak{p}_K}{2c_KE_{(K)}(\mathfrak{p}_{(K)})}2c_KE_K(\mathfrak{p})\delta^3(\mathfrak{p}-\mathfrak{q})\text{''}=1. \tag{2.4$'$}$$

The initial decay rate[8] or transition probability per unit time for the decay of the state ϕ_K into lepton pairs with all possible momenta and polarizations is given by

$$P=\int\int\frac{d^3\mathfrak{p}_\mu}{2E_\mu}\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\delta(E_K-E_\mu-E_\nu)\times\sum_{\text{pol}}|\langle\mu,\nu,\mathfrak{p}_{(\mu)},\mathfrak{p}_{(\nu)}|T|\phi_K\rangle|^2, \tag{2.6}$$

where T is the transition operator ($T\sim H^{\text{interaction}}$) performing the transition from ϕ_K into the states of lepton pairs (μ,ν). If we insert (2.3), this may be written

$$P=\sum_{\text{pol}}\int\int\frac{d^3\mathfrak{p}_\mu}{2E_\mu}\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\delta(E_K-E_\mu-E_\nu)\times\left|\int\frac{d^3\mathfrak{p}_K}{2E_K(\mathfrak{p}_K)c_K}\phi_K(\mathfrak{p}_K)\langle\mu,\nu,\mathfrak{p}_\mu,\mathfrak{p}_\nu|T|\mathfrak{p}_K,K^+\rangle\right|^2. \tag{2.7}$$

For the $\pi\to\mu\nu$ decay we just replace the K^+ with π^+ in (2.7).

Similarly, one obtains the transition probability per unit time for the $K^+\to\pi^0l\nu$ decay

$$P=\sum_{\text{pol}}\int\int\int\frac{d^3\mathfrak{p}_l}{2E_l}\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\frac{d^3\mathfrak{p}_{\pi^0}}{2c_\pi E_{\pi^0}}\times\delta(E_K-E_\pi-E_l-E_\nu)\left|\int\frac{d^3\mathfrak{p}_K}{2c_KE_K(\mathfrak{p}_K)}\phi_K(\mathfrak{p}_K)\times\langle\mu,\nu,\pi^0,\mathfrak{p}_\nu,\mathfrak{p}_\mu,\mathfrak{p}_{\pi^0}|T|\mathfrak{p}_K,K^+\rangle\right|^2, \tag{2.8}$$

and analogously for $\pi^+\to\pi^0l\nu$.

[8] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

III. CALCULATION OF THE WEAK DECAY RATES AND MATRIX ELEMENTS

To calculate these transition probabilities and related quantities, we must postulate the expression for the transition operator T from the general ideas laid down at the end of Sec. I. Guided by the usual current-current structure of the weak interaction, we split T into a leptonic part L^λ and a hadronic part H_λ

$$T=L^\lambda H_\lambda+\text{H. c.}, \tag{3.1}$$

such that H_λ acts between one-particle hadron states only and L^λ does not change hadron quantum numbers. Since our algebraic model $\mathfrak{B}$ describes only hadrons, only the hadronic part H_λ can be formulated in the frame of $\mathfrak{B}$ and only H_λ is an operator in the representation space $\mathcal{H}(\zeta,s,(0,\rho_3))$,[9] so that the leptonic part L^λ cannot be expressed in operator form and its matrix elements between a one-hadron state and a state containing the lepton pair and hadrons must be postulated separately. The explicit form of this matrix element will be written later. It is to contain the usual assumptions, i.e., (i) $L^\lambda\sim\gamma^\lambda(1+\gamma_5)$ and (ii) the lepton states are to be replaced with the usual free wave functions, and it is to serve its usual purpose, namely, to transfer the mass difference between the decaying hadron state and the final hadron state to the lepton pair.

Thus, the dynamical property is essentially given by the hadronic part H_λ. It therefore should be proportional to P_λ [note that P_λ is not the usual "free" momentum operator describing the motion of noninteracting particles, but owing to the additional relations (5) of II it describes properties of the interacting system]. As already remarked in the Introduction, the generators of $\mathfrak{B}$ that perform the transitions between the hadron states which appear in the weak decay process are the $E_{\pm\alpha}$ and $F_{\pm\alpha}$, so that we are led to the following assumption for H_λ

$$H_\lambda=G\sum_{\alpha=1,2,3}\{P_\lambda,\,E_\alpha+F_\alpha\}, \tag{3.2}$$

where G is a dimensionless constant describing the strength of the interaction.[10] For a specific reaction, only one or two terms of the sum in (3.2) contribute, because the matrix elements of the other F_α and E_α are zero.

We calculate first the transition matrix element for $K^+\to\mu\bar{\nu}$

$$A=\langle\mu,\bar{\nu},\mathfrak{p}_\mu,\mathfrak{p}_\nu|T|\mathfrak{p}_K,K^+\rangle. \tag{3.3}$$

[9] From (3.2) one can see that H_λ is a continuous self-adjoint operator in the rigged Hilbert space $\phi\subset\mathcal{H}\subset\phi^\times$ (i.e., its closure in the completion of ϕ with respect to the scalar product of $\mathcal{H}$ is self-adjoint).

[10] It is obvious that the numerical value of G depends upon the normalization that we use for E_α and F_α and the normalization in (3.11). The numerical value of G for the normalization that we use in II will be given later.

Using the above postulated assumptions, we write

$$A=\sum_{\alpha}\int \frac{d^3\mathfrak{p}_\alpha}{2c_\alpha E_\alpha(\mathfrak{p}_\alpha)}\langle\bar{\nu},\mu,\mathfrak{p}_\nu,\mathfrak{p}_\mu|L^\lambda|\mathfrak{p}_\alpha,\alpha\rangle \times\langle\alpha,\mathfrak{p}_\alpha|H_\lambda|\mathfrak{p}_K,K^+\rangle, \quad (3.4)$$

where the summation runs over all generalized basis vectors (1.1) of the irreducible representation space $\mathfrak{H}(z,s,(0,b))$ and $|\mathfrak{p}_\nu,\mathfrak{p}_\mu,\nu,\mu\rangle$ are the pure lepton states with all hadron quantum numbers equal to zero, i.e., $(I,I_3,Y,\lambda)=(0,0,0,0)=\sigma$. Since L^λ does not change the hadron quantum numbers, $\langle\mu,\nu,\mathfrak{p}_\mu,\mathfrak{p}_\nu|L^\lambda|\mathfrak{p}_\alpha,\alpha\rangle=0$ unless $\alpha=\sigma$, so that

$$A=\int \frac{d^3\mathfrak{p}_\sigma'}{2c_\sigma E_\sigma(\mathfrak{p}_\sigma)}\langle\bar{\nu},\mu,\mathfrak{p}_\nu,\mathfrak{p}_\mu|L^\lambda|\mathfrak{p}_\sigma',\sigma\rangle \times\langle\sigma,\mathfrak{p}_\sigma'|H_\lambda|\mathfrak{p}_K,K^+\rangle. \quad (3.5)$$

The second factor in (3.5) will be calculated in the frame of the model $\mathfrak{B}$

$$\langle\sigma,\mathfrak{p}_\sigma|H_\lambda|\mathfrak{p}_K,K^+\rangle=G\langle\sigma\langle\mathfrak{p}_\sigma|\{P_\lambda,F_{-2}\}|\mathfrak{p}_K,K^+\rangle =G(p_\lambda^{(K)}+p_\lambda^{(\sigma)})\langle\sigma,\mathfrak{p}_\sigma|E_2|\mathfrak{p}_K,K^+\rangle.$$

We can either apply F_{-2} to the state $|\mathfrak{p}_K,K^+\rangle$ or F_{+2} to the state $|\mathfrak{p}_\sigma,\sigma\rangle$, and obtain

$$\langle\sigma,\mathfrak{p}_\sigma|H_\lambda|\mathfrak{p}_K,K^+\rangle=G(p_\lambda^{(K)}+p_\lambda^{(\sigma)}) \times C_{02}\langle\sigma,\mathfrak{p}_\sigma|(m_\sigma/m_K)\mathfrak{p}_K,\sigma\rangle \quad (3.6a)$$

or

$$\langle\sigma,\mathfrak{p}_\sigma|H_\lambda|\mathfrak{p}_K,K^+\rangle=G(p_\lambda^{(K)}+p_\lambda^{(\sigma)}) C_{02}\langle K^+,(m_K/m_\sigma)\mathfrak{p}_\sigma|\mathfrak{p}_K,K^+\rangle, \quad (3.6b)$$

respectively. C_{02} is the $SL(3,c)$ transition coefficient given in Appendix A, Eqs. (A10) and (A12), from which we see that it depends upon b, i.e., upon the irreducible representation $\mathfrak{B}(\zeta=0.782,\ s=0,\ m_3=0,\ \rho_3=2b)$. Using the "normalization" of the generalized eigenvectors given in Appendix B, Eqs. (B3) and (B6),

$$\left\langle K^+,\frac{m_K}{m_\sigma}\mathfrak{p}_\sigma\middle|\mathfrak{p}_K,K\right\rangle=2c_K E_K(\mathfrak{p}_K)\delta^3\left(\frac{m_K}{m_\sigma}\mathfrak{p}_\sigma-\mathfrak{p}_K\right), \quad (3.7a)$$

$$\left\langle\sigma,\mathfrak{p}_\sigma\middle|\frac{m_\sigma}{m_K}\mathfrak{p}_K,\sigma\right\rangle=2c_\sigma E_\sigma(\mathfrak{p}_\sigma)\delta^3\left(\mathfrak{p}_\sigma-\frac{m_\sigma}{m_K}\mathfrak{p}_K\right), \quad (3.7b)$$

we see that the right-hand sides of (3.6a) and (3.6b) are the same, as they should be, because

$$\delta^3\left(\frac{m_K}{m_\sigma}\mathfrak{p}_\sigma-\mathfrak{p}_K\right)=\left(\frac{m_\sigma}{m_K}\right)^3\delta^3\left(\mathfrak{p}_\sigma-\frac{m_\sigma}{m_K}\mathfrak{p}_K\right), \quad (3.8a)$$

$$\mathfrak{p}_K/m_K=\mathfrak{p}_\sigma/m_\sigma, \quad (3.8b)$$

$$E_K(\mathfrak{p}_K)=E_K\left(\frac{m_K}{m_\sigma}\mathfrak{p}_\sigma\right)=\left(m_K^2+\frac{m_K^2}{m_\sigma^2}\mathfrak{p}_\sigma^2\right)^{1/2} =\frac{m_K}{m_\sigma}(m_\sigma^2+\mathfrak{p}_\sigma^2)^{1/2}=\frac{m_K}{m_\sigma}E_\sigma(\mathfrak{p}_\sigma), \quad (3.8c)$$

$$c_K=c_\sigma(m_K/m_\sigma)^2. \quad (3.8d)$$

Equations (3.8b) and (3.8c) are a consequence of relation (6) of II. Inserting (3.7) into (3.6) and using (3.8b) again, we obtain

$$\langle\sigma,\mathfrak{p}_\sigma|H_\lambda|\mathfrak{p}_K,K^+\rangle =GC_{02}p_\lambda^{(K)}\left(1+\frac{m_\sigma}{m_K}\right)2c_\sigma E_\sigma(\mathfrak{p}_\sigma)\delta^3\left(\mathfrak{p}_\sigma-\frac{m_\sigma}{m_K}\mathfrak{p}_K\right), \quad (3.9)$$

and therewith (3.5) becomes

$$A=\left\langle\bar{\nu},\mu,\mathfrak{p}_\nu,\mathfrak{p}_\mu\middle|L^\lambda\middle|\frac{m_\sigma}{m_K}\mathfrak{p}_K.\sigma\right\rangle GC_{02}p_\lambda^{(K)}\left(1+\frac{m_\sigma}{m_K}\right). \quad (3.10)$$

We need now the explicit form of the matrix element for the leptonic part L^λ. The requirements on these matrix elements formulated at the beginning of Sec. III, dimensional considerations and relativistic covariance of the matrix element of T, lead to the following general expression[11]:

$$\langle\bar{\nu},l,\mathfrak{p}_\nu,\mathfrak{p}_l,\alpha_1,\cdots.\alpha_N,\mathfrak{p}_{\alpha_1},\cdots\mathfrak{p}_{\alpha_N}|L^\lambda|(m_\beta/m_\gamma)\mathfrak{p}_\gamma=\mathfrak{p}_\beta,\beta\rangle =\frac{a\sqrt{c_{\alpha_1}}}{2m_{\alpha_1}}\cdots\frac{a\sqrt{c_{\alpha_N}}}{2m_{\alpha_N}}\frac{a\sqrt{c_\beta}}{2m_\beta} \times\delta^3(\mathfrak{p}_\nu+\mathfrak{p}_\mu+\mathfrak{p}_{\alpha_1}+\cdots+\mathfrak{p}_{\alpha_N}-\mathfrak{p}_\gamma) \times\bar{u}^{(l)}(\mathfrak{p}_l)\gamma^\lambda(1+\gamma_5)u^{(\nu)}(-\mathfrak{p}_\nu), \quad (3.11a)$$

if the hadronic quantum numbers of $(\alpha_1\cdots\alpha_N)$ and β are the same (i.e., $\alpha_1\times\alpha_2\times\alpha_3\times\cdots\times\alpha_N=\beta$), and $=0$ otherwise.

It is easy to see that (3.11a) fulfills the above requirements for every $N=0,1,2,\cdots$. Since L^λ has the dimension 1, the matrix element has the dimension MeV^{-3-N}; since we exclude the possibility of introducing a new constant of the dimension of MeV (or length), the right-hand side can only have this form, because replacing the $m_{\alpha_1}, m_{\alpha_2},\cdots,m_\beta$ with Lorentz-invariant expressions formed out of the momenta is equivalent to the introduction of dimensionless functions of these expressions on the right-hand side, which we excluded with the requirement that dynamical properties are not to be carried by L^λ.[12] As $c_\alpha/m_\alpha^2=c_\beta/m_\beta^2$ is an arbitrary constant we could have chosen instead of it any arbitrary constant of dimension MeV^{-2}, e.g., g^{-1}. It is easy to see that this introduces in the result only a new constant, which can be absorbed in a. The constant a will appear in the branching ratios of decay processes which have in the final state different numbers of hadrons

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[11] A discussion of (3.11a) in the usual field-theoretic formulation is given in a different place; A. Böhm and E. C. G. Sundarshan, Syracuse University Report No. SU-158 (unpublished).

[12] The introduction of a dimensionless factor $f(q^2)$, $q=p_\alpha-p_\gamma$ for $N=1$, e.g., will change the constancy of the form factor in (4.5a) and (3.23a) into q dependence, which is favored by the experimental data; however, such a q dependence of the form factor should come from the hadron matrix element.

[cf. $\Gamma(K\to\pi l\nu)/\Gamma(K\to l\nu)$]. However, we will see in the following that this constant a is uniquely determined from the fact that in the process $K\to\pi l\nu$ the hadron parity does not change whereas it has to change in the process $K\to l\nu$.

For our special case $K^+\to\mu\bar\nu$, (3.11a) gives

$$\langle\bar\nu,\mu;\,\mathfrak{p}_\nu,\mathfrak{p}_\mu|L^\lambda\left|\frac{m_\sigma}{m_K}\mathfrak{p}_K,\sigma\right\rangle=\frac{a\sqrt{c_\sigma}}{2m_\sigma}\delta^3(\mathfrak{p}_\nu+\mathfrak{p}_\mu-\mathfrak{p}_K)\times\bar u^{(\mu)}(\mathfrak{p}_\mu)(1-\gamma_5)\gamma^\lambda u^{(\nu)}(-\mathfrak{p}_\nu).\quad(3.11\text{b})$$

Inserting this into (3.10) and using (3.8d), we obtain

$$\langle\mu,\bar\nu,\mathfrak{p}_\mu,\mathfrak{p}_\nu|T|\mathfrak{p}_K,K^+\rangle=Gc_{02}p_\lambda{}^{(K)}\frac{1}{2m_K}\left(1+\frac{m_\sigma}{m_K}\right)(a\sqrt{c_K})\delta^3(\mathfrak{p}_\nu+\mathfrak{p}_\mu-\mathfrak{p}_K)\times\bar u^{(\mu)}(\mathfrak{p}_\mu)(1-\gamma_5)\gamma^\lambda u^{(\nu)}(-\mathfrak{p}_\nu).\quad(3.12)$$

To calculate the decay rate, we insert this into (2.7) and obtain

$$\dot P=\int\int\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\frac{d^3\mathfrak{p}_\mu}{2E_\mu}\delta(E_K-E_\mu-E_\nu)|\phi_K(\mathfrak{p}_\nu+\mathfrak{p}_\mu)|^2\times(2c_KE_K(\mathfrak{p}_\nu+\mathfrak{p}_\mu))^{-2}8m_\mu{}^2(E_\mu E_\nu-\mathfrak{p}_\mu\mathfrak{p}_\nu)\times\left[Gc_{02}\frac{1}{m_K}\left(1+\frac{m_\sigma}{m_K}\right)a\sqrt{c_K}\right]^2,\quad(3.13)$$

where we have used

$$\sum_{\text{pol}}|\bar u^{(\mu)}(\mathfrak{p}_\mu)\gamma^\lambda(1+\gamma_5)(p_\lambda{}^{(\nu)}+p_\lambda{}^{(\mu)})u^{(\nu)}(-\mathfrak{p}_\nu)|^2=8m_\mu{}^2p_\lambda{}^{(\mu)}p^{(\nu)}.$$

If we make now the incorrect but usual assumption that ϕ_K is a state of exact momentum $\mathfrak{q}=(0,0,0)$ and insert (2.5) into (3.13), we obtain

$$P=\left[GC_{02}\frac{a}{2m_K}\left(1+\frac{m_\sigma}{m_K}\right)\right]^2\frac{8m_\mu{}^2}{2E_K(\mathfrak{q})}\times\int\int\frac{d^3\mathfrak{p}_\mu}{2E_\mu}\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\delta^4(q-p^{(\mu)}-p^{(\nu)})p^{(\mu)}\cdot p^{(\nu)},\quad(3.14)$$

so that the decay width for $K^+\to\mu\nu$ is

$$\Gamma K(^+\to\mu\nu)=\left(\frac{G}{m_K}\right)C_{02}{}^2\left(1+\frac{m_\sigma}{m_K}\right)^2a^2\times\tfrac14\pi m_\mu{}^2m_K\left(1-\frac{m_\mu{}^2}{m_K{}^2}\right)^2.\quad(3.15\text{a})$$

We obtain the decay rate for $\pi^+\to\mu\nu$ in exactly the same way, replacing everywhere in the previous calculation the index K^+ with π^+ and F_{-2} with F_{-1}; the result is

$$\Gamma(\pi^+\to\mu\nu)=\left(\frac{G}{m_\pi}\right)^2C_{01}{}^2\left(1+\frac{m_\sigma}{m_\pi}\right)^2a^2\times\tfrac14\pi m_\mu{}^2m_\pi\left(1-\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2.\quad(3.15\text{b})$$

We remark that the "normalization" factors c_K, c_σ, $\cdots$ have completely disappeared as we would have expected, because the physical result can not depend upon normalization.

We shall calculate now the decay rates for $K^+\to\pi^0\bar l\nu$ and $\pi^+\to\pi^0\bar l\nu$ according to the same principles. For

$$A=\langle\bar l,\nu,\pi^0,\mathfrak{p}_l,\mathfrak{p}_\nu,\mathfrak{p}_{\pi^0}|T|\mathfrak{p}_K,K^+\rangle\quad(3.16\text{a})$$

we obtain

$$A=\sum_\alpha\int\frac{d^3\mathfrak{p}_\alpha'}{2c_\alpha E_\alpha}\langle\bar l,\nu,\pi^0,\mathfrak{p}_i,\mathfrak{p}_\nu,\mathfrak{p}_{\pi^0}|L^\lambda|\mathfrak{p}_\alpha',\alpha\rangle\times\langle\alpha,\mathfrak{p}_\alpha'|H_\lambda|\mathfrak{p}_K,K^+\rangle=\int\frac{d^3\mathfrak{p}_\pi'}{2c_{\pi^0}E_{\pi^0}(\mathfrak{p}_\pi')}\langle\bar l,\nu,\pi^0,\mathfrak{p}_l,\mathfrak{p}_\nu,\mathfrak{p}_{\pi^0}|L^\lambda|\mathfrak{p}_\alpha',\pi^0\rangle\times\langle\pi^0,\mathfrak{p}_\alpha'|H_\lambda|\mathfrak{p}_K,K^+\rangle,\quad(3.16\text{b})$$

because of the property of L^λ not to change the hadron quantum number. In this case two terms contribute to the hadron matrix element, because, in general, E_{-2} and F_{-2} transform between the K^+ and π^0 state

$$\langle\pi^{0\prime}\mathfrak{p}_{\pi^0}'|H_\lambda|\mathfrak{p}_K,K^+\rangle=G\langle\pi^{0\prime},\mathfrak{p}_{\pi^0}'|\{P_\lambda,\,F_{-2}+E_{-2}\}|\mathfrak{p}_K,K^+\rangle=GC(p_\lambda{}^{(K)}+p_\lambda{}'^{(\pi^0)})\langle\pi^{0\prime}\mathfrak{p}_{\pi^0}'|(m_{\pi^0}/m_K)\mathfrak{p}_K,\pi^0\rangle,\quad(3.17)$$

where

$$C=C_{12}+\langle I=1,\,I_3=0,\,Y=0,\,\lambda=1|\hat{E}_{-2}\times|\lambda=1,\,Y=1,\,I_3=\tfrac12,\,I=\tfrac12\rangle=C_{12}-\sqrt{\tfrac{1}{12}},\quad(3.18)$$

and C_{12} is given in Appendix A, Eqs. (A10) and (A15). So we obtain for the hadron matrix element, using (3.8), (B3), and (B6),

$$\langle\pi^{0\prime},\mathfrak{p}_{\pi^0}'|H_\lambda|\mathfrak{p}_K,K^+\rangle=GCp_\lambda{}^{(K)}\left(1+\frac{m_{\pi^0}}{m_K}\right)\times2c_{\pi^0}E_{\pi^0}(\mathfrak{p}_{\pi^0}')\delta^3\left(\mathfrak{p}_{\pi^0}'-\frac{m_{\pi^0}}{m_K}\mathfrak{p}_K\right).\quad(3.19)$$

Inserting (3.19) into (3.16a) gives

$$A=\langle\bar l,\nu,\pi^0,\mathfrak{p}_l,\mathfrak{p}_\nu,\mathfrak{p}_{\pi^0}|L^\lambda\left|\frac{m_\pi}{m_K}\mathfrak{p}_K,\pi^{0\prime}\right\rangle\times GCp_\lambda{}^{(K)}\left(1+\frac{m_{\pi^0}}{m_K}\right).\quad(3.20)$$

For the leptonic matrix element we obtain from (3.11)

$$\langle \nu,\bar{l},\pi^0,\mathfrak{p}_\nu,\mathfrak{p}_l,\mathfrak{p}_{\pi^0}|L^\lambda\left|\frac{m_{\pi^0}}{m_K}\mathfrak{p}_K,\pi^{0\prime}\right\rangle$$

$$=\frac{a\sqrt{c_{\pi^0}}}{2m_{\pi^0}}\frac{a\sqrt{c_{\pi^{0\prime}}}}{2m_{\pi^{0\prime}}}\delta^3(\mathfrak{p}_\nu+\mathfrak{p}_\mu+\mathfrak{p}_{\pi^0}-\mathfrak{p}_K)$$

$$\times\bar{u}^{(\nu)}(\mathfrak{p}_\nu)(1-\gamma_5)\gamma^\lambda u^l(-\mathfrak{p}_l)\,,\quad(3.11'')$$

and insert this into (3.20) and use $(\sqrt{c_{\pi^{0\prime}}})/m_{\pi^{0\prime}}=(\sqrt{c_K})/m_K$

$$A=GC\left(1+\frac{m_\pi}{m_K}\right)p_\lambda{}^{(K)}\frac{\sqrt{c_{\pi^{0\prime}}}}{2m_{\pi^0}}\frac{\sqrt{c_K}}{2m_K}a^2$$

$$\times\delta^3(\mathfrak{p}_\nu+\mathfrak{p}_\mu+\mathfrak{p}_{\pi^0}-\mathfrak{p}_K)\bar{u}^{(\nu)}(\mathfrak{p}_\nu)(1-\gamma_5)\gamma^\lambda u^l(-\mathfrak{p}_l)\,.\quad(3.21)$$

To obtain the decay rate we insert (3.21) into (2.8) and make again the assumption that we have initially a K-meson state with sharp momentum, so that we can use (2.5) and obtain

$$P=\left[a^2GC\frac{1}{2m_K}\left(1+\frac{m_{\pi^0}}{m_K}\right)\right]^2\frac{1}{(2m_{\pi^0})^2}\frac{1}{2E_K(\mathfrak{p}_l+\mathfrak{p}_\nu+\mathfrak{p}_{\pi^0})}$$

$$\times\int\int\int\frac{d^3\mathfrak{p}_l}{2E_l}\frac{d^3\mathfrak{p}_\nu}{2E_\nu}\frac{d^3\mathfrak{p}_{\pi^0}}{2E_{\pi^0}}\delta(E_K-E_{\pi^0}-E_\nu-E_l)$$

$$\times\delta^3(\mathfrak{p}_\pi+\mathfrak{p}_\nu+\mathfrak{p}_l-\mathfrak{p}_K)\sum_{\text{pol}}|\bar{u}^{(\nu)}(\mathfrak{p}_\nu)$$

$$\times(1-\gamma_5)\gamma^\lambda u^{(l)}(-\mathfrak{p}_l)p_\lambda{}^{(K)}|^2\,.\quad(3.22\text{a})$$

Again the "normalization" factor for the (unphysical) generalized states disappears in the final (physical) result. Comparing (3.22a) with the usual expression for the decay width[13]

$$\Gamma=\frac{1}{16m_K}\int\int\int\frac{d^3\mathfrak{p}_l}{E_l(2\pi)^3}\frac{d^3\mathfrak{p}_\nu}{E_\nu(2\pi)^3}\frac{d^3\mathfrak{p}_{\pi^0}}{E_\pi(2\pi)^3}(2\pi)^4$$

$$\times\delta^4(p_l+p_\nu+p_{\pi^0}-p_K)\sum_{\text{pol}}|\mathfrak{M}|^2\,,\quad(3.22\text{b})$$

we obtain for the invariant amplitude $\mathfrak{M}$

$$\mathfrak{M}=(2\pi)^{3/2}\tfrac{1}{2}\pi\frac{a^2G}{m_K}\left(1+\frac{m_{\pi^0}}{m_K}\right)C\frac{1}{m_{\pi^0}}$$

$$\times p_\lambda{}^{(K)}\bar{u}^{(\nu)}(1-\gamma_5)\gamma^\lambda u^{(l)}\,.\quad(3.23\text{a})$$

Repeating the same calculation with everywhere K^+ replaced with π^+ and $E_{-2}+F_{-2}$ with $E_{-1}+F_{-1}$, we obtain the invariant amplitude for the process $\pi^+\to\pi^0 e\nu$

$$\mathfrak{M}=(2\pi)^{3/2}\tfrac{1}{2}\pi\frac{Ga^2}{m_{\pi^+}}\left(1+\frac{m_{\pi^0}}{m_{\pi^+}}\right)C'\frac{1}{m_{\pi^0}}$$

$$\times p_\lambda{}^{(\pi^+)}\bar{u}^{(\nu)}(1-\gamma_5)\gamma^\lambda u^{(e)}\,,\quad(3.23\text{b})$$

[13] See, e.g., J. S. Bell, in *High Energy Physics*, edited by C. DeWitt and M. Jacob (Gordon and Breach, Science Publishers, Inc., New York, 1965), p. 401.

where C' is the sum of $SL(3,c)$ matrix elements,

$$C'=\langle\lambda=1,\,I=1,\,I_3=0,\,Y=0|\hat{F}_{-1}+\hat{E}_{-1}$$
$$\times|I=1,\,I_3=1,\,Y=0,\,\lambda=1\rangle=C_{11}+\sqrt{\tfrac{1}{3}}\,,\quad(3.24)$$

and C_{11} is given in Appendix A [(A16)].

IV. PREDICTION OF THE MODEL

We have considered four processes:

$$K^+\to\pi^0 l\nu,\quad \pi^+\to\pi^0 e\nu\,,$$
$$K^+\to\mu\nu,\quad \pi^+\to\mu\nu\,,$$

and we have the unknown parameters G, a, and b. However a and b are not independent but connected by parity considerations. After b has been fixed by parity properties, a is determined by the ratio $\Gamma(K_{l3})/\Gamma(K_{l2})$ *or* the ratio $\Gamma(\pi_{l3}/\Gamma(\pi_{l2})$. After a and b are known, G can be determined from any of the partial decay widths. Our model predicts then the ratio between the strange and nonstrange decay widths, i.e., essentially the Cabibbo angles θ_V and θ_A. In addition, our model gives expressions for the weak form factor, which can also be compared with experimental data.

We first calculate those properties which are independent of the values a and $b=(\frac{1}{2}\rho_3)$ which characterizes the representation $\mathfrak{B}(\zeta=-0.782,\,s=0,\,(0,b))$; then we evaluate a, b, and G.

We define the suppression of the strangeness-changing leptonic decay by the definition

$$S_{l2}{}^2\equiv\frac{\Gamma(K^+\to\mu\nu)}{\Gamma(\pi^+\to\mu\nu)}\frac{m_\pi}{m_K}\frac{(1-m_\mu{}^2/m_\pi{}^2)^2}{(1-m_\mu{}^2/m_K{}^2)^2}\,,\quad(4.1)$$

which is identical in the Cabibbo theory[14] with $\tan^2\theta_A{}^M$. Inserting (3.15a) and (3.15b) into (4.1), we obtain as the result of our model

$$S_{l2}=\frac{m_\pi}{m_K}\frac{(1+m_\sigma/m_K)}{(1+m_\sigma/m_\pi)}=0.28\,,\quad(4.2)$$

which is in very good agreement with the experimental value.[14,15] Here we have used the fact that $C_{02}=C_{01}$, which can be seen from their values given in Appendix A, but which is generally true for these matrix elements of $SU(3)$ octet operators. For m_σ, the mass of the hadron state with the quantum numbers of the vacuum, we have chosen, in accordance with our considerations in Sec. I, $m_\sigma=0$.

The invariant amplitude for the decay $K^+\to\pi^0 l\nu$ is expressed by the weak form factors $f_+(q^2)$ and $f_-(q^2)$ $[q^2=(p_\pi-p_K)^2]$:

$$\mathfrak{M}=(G_K/\sqrt{2})[(f_++f_-)p_\lambda{}^{(K)}+(f_+-f_-)p_\lambda{}^{(\pi^0)}]$$
$$\bar{u}^{(\nu)}(1-\gamma_5)\gamma^\lambda u^l\,,\quad(4.3)$$

[14] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963); R. H. Dalitz, *Varenna Lectures, 1964* (Academic Press Inc., New York, 1966).
[15] N. Cabibbo, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 29.

where G_K is the weak coupling constant for the strange-particle decay.

Comparing this with the result of our model (3.23a), we obtain

$$f_+(q^2)=f_-(q^2)=f_K \tag{4.4}$$

and

$$G_K f_K=\pi^{5/2}\frac{a^2G}{m_K}\left(1+\frac{m_{\pi^0}}{m_K}\right)C\frac{1}{m_{\pi^0}}\,. \tag{4.5a}$$

In the same way we can write for the decay $\pi^+\to\pi^0 e\nu$ from (3.23b)

$$G_\pi f_\pi=\pi^{5/2}\frac{a^2G}{m_{\pi^+}}\left(1+\frac{m_{\pi^0}}{m_K}\right)C'\frac{1}{m_{\pi^0}}\,. \tag{4.5b}$$

Here G_π is the usual weak coupling constant (for the nonstrange-particle decay[13])

$$G_\pi=(1.023\pm0.002)\times(10^{-5}/m_p{}^2)\times0.980\,, \tag{4.6}$$

and

$$f_\pi=\sqrt{2}\pm(10\%)\,. \tag{4.7}$$

Thus, our model gives a constant form factor, whereas latest experimental results[16] seem to indicate a weak q^2 dependence of the form factor

$$f_+(q^2)=f_+(0)(1+\lambda q^2/m_\pi{}^2)\,, \tag{4.8}$$

with average value of[16] $\lambda=0.023\pm0.008$. For $\xi=f_-/f_+$ our model predicts $\xi=1$, consistent with the latest expermental results,[16] and [17]

$$\xi=1.0\pm0.3\,. \tag{4.9}$$

We define the suppression of the strangeness-changing semileptonic decay by

$$S_{l3}=\frac{G_K f_K(1/C)}{G_\pi f_\pi(1/C')}=2\frac{G_K f_K}{G_\pi f_\pi}\,, \tag{4.10}$$

which is to be compared with $\tan\theta_V{}^M$ in the Cabibbo theory.[14] Here $C'/C=2$ is the ratio of the relevant $SU(3)$ Clebsch-Gordan coefficients (in particular, this value is obtained also from the values of C and C' in Appendix A). Inserting (4.5a) and (4.5b) [(3.23a) and (3.23b)] into (4.10), we obtain

$$S_{l3}=\frac{m_\pi}{m_K}\frac{(1+m_{\pi^0}/m_{K^+})}{(1+m_{\pi^0}/m_{\pi^+})}=0.183\,, \tag{4.11}$$

as compared with the experimental value[14,15] of 0.21.

These results of our model may indicate that the Cabibbo suppression is connected with the breaking of unitary symmetry.[18]

We now proceed to the determination of the constants a and b.

It is clear that transitions between different $SU(3)$ multiplets in an irreducible representation $\mathcal{B}(\zeta,s,m_3=0,b)$ can only be performed by the noncompact operators F_α, G_i of $SL(3,C)$ because the compact operators E_α do not transform out of an $SU(3)$ multiplet. Therefore transitions such as $K\to\mu\nu$, $\pi\to\mu\nu$, and also $K\to\pi\pi l\nu$ can only be performed by the F_α. However transitions which do not lead out of an $SU(3)$ multiplet such as $K\to\pi l\nu$, $\pi\to\pi l\nu$ are in general performed by E_α as well as F_α, as seen from (3.18) and (3.19). For the general case, we can see from (A3) and (A6) of Appendix A or for the special case under consideration from (A15) and (A16) of Appendix A, that only in the representation $\mathcal{B}(\zeta,s,m_3=0,b)$ with $b=0$ the noncompact generators F_α do not contribute to transitions such as $K\to\pi l\nu$, $\pi\to\pi e\nu$.

Since the transitions $K\to\mu\nu$, $\pi\to\mu\nu$ are $0^-\to0^+$ for the hadrons, the F_α must change the parity

$$\Pi F_\alpha\Pi=-F_\alpha\,,$$

i.e., must be pseudoscalars and therefore $J_\lambda{}^{\alpha5}=G\{P_\lambda,F_\alpha\}$ be axial vectors.

Since the transitions $K\to\pi l\nu$, $\pi\to\pi l\nu$ are $0^-\to0^-$ for the hadrons, the operator that performs this transition should be a scalar. As mentioned above, the transition matrix element for this process has a contribution coming from E_α and a contribution coming from the F_α matrix elements. Since we already have found that F_α changes the parity, we conclude that the F_α matrix element must be zero, i.e., that $b=0$ and E_α is a scalar $\Pi E_\alpha\Pi^{-1}=E_\alpha$. Only for the representations with $b=0$ do the compact and noncompact terms have definite parity properties, one of them generating the vector transitions and the other the axial vector transitions for leptonic decays of hadrons.

Having found $b=0$, we determine the value of a from the ratio of the experimental values of $\Gamma(\pi^+\to\pi^0 l\nu)$ and $\Gamma(\pi^+\to\mu\nu)$. Comparing (3.15b) with Eq. (17) of Ref. 13, we obtain

$$|C_{01}|=\left|\frac{m_\pi}{Ga}\frac{G_\pi f}{\sqrt{2}\pi}\right|\,, \tag{4.12}$$

where the value of the weak coupling constant G_π is given in (4.6) and f is determined from the experimental value of $\Gamma(\pi^+\to\mu\nu)$ to be

$$|f|=(\sqrt{1.92}\times10^{-1})m_p\pm5\%\,. \tag{4.13}$$

[16] G. E. Kalmus and A. Kernan, University of California Radiation Laboratory Report No. UCRL-183351, 1967 (unpublished); E. Berlotti, E. Fiorini, and A. Pullia, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Interscience Publishers, Inc., New York, 1968); J. W. Cronin, in *Proceedings of the International Conference on Particles and Fields, Rochester, 1967* (Interscience Publishers, Inc., New York, 1967), p. 3; W. Willis, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Interscience Publishers, Inc., New York, 1968).

[17] R. W. Garland, Kosta Tsipis, S. Devons, J. Rosen, D. Tycko, L. G. Poudrom, and S. L. Meyer, Phys. Rev. **167**, 1225 (1968).

[18] For earlier attempts of a theoretical explanation of the Cabibbo suppression, cf. M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); R. Oehme, Ann. Phys. (N. Y.) **33**, 108 (1965); T. Pradhan and M. Patnaik, Saha Institute (Calculta, India) Report, 1967 (unpublished); E. C. G. Sudarshan, Syracuse University Report No. SU-137 (unpublished).

From (4.5b) we obtain

$$C'=\frac{1}{2}\frac{1}{\pi^{5/2}}\frac{G_\pi f_\pi}{Ga^2}m_\pi^2\,, \qquad (4.14)$$

We define

$$r\equiv\left|\frac{f2^{1/2}\pi^{3/2}}{f_\pi m_\pi}\right|^2 \qquad (4.15)$$

which gives, with the numerical values (4.7) and (4.13),

$$r=27.0\pm(30\%)\,.$$

Then from (4.12) and (4.14) we obtain

$$r=\frac{1}{a^2}\frac{|C_{01}|^2}{|C_{11}+1/\sqrt{3}|^2}=\frac{3}{8}\frac{1}{a^2}\,,$$

where we have used (A13) and (A16) with $b=0$. With the above numerical value for r we obtain

$$a^2=1/72\pm30\%.$$

From either (4.12) or (4.14) we can then calculate the value of the constant G. From (4.12) we obtain

$$Ga=1.316\times10^{-7}\pm5\%$$

and

$$G=0.155\times10^{-7}\pm20\%.$$

We shall now investigate whether the assumption that there is only a finite number of meson resonances with the same spin can be correct. For that purpose we assume that we have a finite multiplet or that not $SL(3,c)$ but $SW_3=SU(3)\times SU(3)$ is the noninvariance group. Then we must replace in the foregoing the noncompact generators G_i and $F_{\pm\alpha}$ with iG_i and $iF_{\pm\alpha}$. Because of the physical interpretation, again the representations must be unitary, i.e., $(iG_i)^+=iG_i$ and $(iF_{\pm\alpha})^+=iF_{\mp\alpha}$. Therefore b must be real and have one of the values $|b|=3, 5, 7, \cdots, 2n+3$. As noted before this is not possible if E_α is to be a scalar and F_α a pseudoscalar. So we see that parity considerations rule out the possibility that SW_3 is the spectrum generating group and the multiplets are finite.

It is clear that the previous calculations can immediately be carried over to K^-, K^0, and K^0 decay. The only change is in the Clebsch-Gordan coefficients. So our model also gives the value of $\Gamma(K_2{}^0\to\pi l\nu)$, or the suppression of the strangeness-changing K^0 decay:

$$S_{l3}{}^0=\frac{m_{\pi^-}}{m_{K^0}}\frac{(1+m_{\pi^-}/m_{K^0})}{(1+m_{\pi^0}/m_{\pi^-})}=0.180.$$

[We have here used the experimental values of the masses and not the predictions of our mass formula, therefore, this value differs slightly from (4.11).] This is to be compared with the experimental value of the phenomenological Cabibbo angle determined from $(K_2{}^0\to\pi e\nu)$ which is 0.20.

APPENDIX A

We describe here the calculation of the matrix elements of the noncompact generators of $SL(3,c)$ in the degenerate series representation ($m_3=0$, $\rho_3=\pm2b$).[19]

The degenerate series representations

$$(m_3,\rho_3)\ (m_3=\text{integer},\ -\infty<i\rho_3<+\infty)$$

contain every representation (ν,μ) of its maximal compact subgroup $SU(3)$ at most once; only the representations ($m_3=0$, ρ_3) contain an octet, and their reduction with respect to $SU(3)$ is[19]

$$(0,\rho_3)\rightleftharpoons(\nu=0,\ \mu=0)\oplus(1,1)\oplus(2,2)\oplus\cdots. \qquad (A1)$$

Because of this reduction property the usual $SU(3)$ eigenstates $|I,I_3Y;\nu\rangle$, $\nu=0, 1, 2, \cdots$, form a basis of the representation space of $(0,\rho_3)$.

For the calculation of the matrix elements of the noncompact generators, we use the results on these representations given in Ref. 20 and the tabulation of the Clebsch-Gordan coefficients of $SU(3)$ in Ref. 21. In analogy to (2.6) of Ref. 21, we introduce the notation $N_{I,I_3,Y}$ for the noncompact generators:

$$N_{1,\pm1,0}=\mp\hat{F}_{\pm1},\ N_{\frac12,\pm\frac12,\pm1}=\hat{F}_{\pm2},\ N_{\frac12,\mp\frac12,\pm1}=\pm\hat{F}_{\pm3}\,,$$
$$N_{1,0,0}=\hat{G}_1,\quad N_{0,0,0}=\hat{G}_2. \qquad (A2)$$

[$\hat{F}_\alpha$, $\hat{G}_i$, $\hat{E}_\alpha$, and $\hat{H}_i$ have the same $SL(3,c)$ properties as the generators of the associative algebra F_α, G_i, E_α, and H_i; however, they act on the $SL(3,c)$ quantum numbers only and do not change the mass.]

Using the fact that $N_{I,I_3,Y}$ are properly normalized octet operators of $SU(3)$, their matrix elements can be expressed in the form

$$\langle\nu',Y',I_3',I'|N_{I,I_3,Y}|\hat{I},\hat{I}_3,\hat{Y},\hat{\nu}\rangle$$
$$=\sum_{\gamma=1}^{2}C(\nu,1,\nu',\gamma,\hat{I},\hat{I}_3,\hat{Y};I,I_3,Y;I',I_3',Y')\times\langle\nu'\|N\|\nu\rangle_\gamma\,, \qquad (A3)$$

where the reduced matrix elements $\langle\nu'\|N\|\nu\rangle_\gamma$ depend upon the representation of $SL(3,c)$ (i.e., upon b) only.

$$C(\nu,1,\nu';\gamma;\hat{I},\hat{I}_3,\hat{Y};I,I_3,Y;I',I_3',Y')$$
$$=c(\hat{I}II',\hat{I}_3I_3I_3')U(\nu,1,\nu';\gamma,\hat{I},\hat{Y};I,Y;I'Y') \qquad (A4)$$

are the Clebsch-Gordan coefficients of $SU(3)$; the relevant $SU(2)$ Clebsch-Gordan coefficients $c(\hat{I}II';\hat{I}_3I_3I_3')$ are tabulated in Tables I.1 and I.2 of Ref. 22, and the

[19] I. M. Gelfand and M. A. Neimark, *Unitäre Darstellungen der klassischen Gruppen* (Berlin, Akademie Verlag, 1957).

[20] S. K. Bose, Phys. Rev. **150**, 1231 (1966); S. K. Bose and E. C. G. Sudarshan, *ibid.* **162**, 1396 (1967).

[21] J. G. Kuriyan, D. Lurie, and A. J. Macfarlane, J. Math. Phys. **6**, 722 (1965).

isoscalar factors $U(\cdots)$ are tabulated in Table II of Ref. 21.

The reduced matrix elements can be calculated, using the results of Ref. 20, to be

$$\langle\nu\|N\|\nu\rangle_{\gamma=1}=0\,, \tag{A5}$$

$$\langle\nu\|N\|\nu\rangle_{\gamma=2}=-\tfrac{1}{3}ib\left[\frac{\nu(\nu+2)}{(2\nu+1)(2\nu+3)}\right]^{1/2}, \tag{A6}$$

$$\langle\nu-1\|N\|\nu\rangle=-\frac{i}{\sqrt{12}}(\nu+1)^{3/2}\left[\frac{(2\nu+1)^2-b^2}{\nu(\nu+1)(2\nu+1)}\right]^{1/2}, \tag{A7}$$

$$\langle\nu+1\|N\|\nu\rangle=-\frac{i}{\sqrt{12}}(\nu+1)^{3/2}\times\left[\frac{(2\nu+3)^2-b^2}{(\nu+1)(\nu+2)(2\nu+3)}\right]^{1/2}. \tag{A8}$$

The number b characterizes the representation ($m_3=0$, $\rho_3=\pm 2b$) of $SL(3,c)$ and has for unitary representations the values $-\infty<ib<+\infty$. For an arbitrary linear representation of $SL(3,c)$ with the $SU(3)$ contents (A1), b can be any number, e.g., for $b=2n+3$, $n=0, 1, 2, 3,\cdots$, the representation is finite-dimensional.

Summarizing, the matrix elements of the noncompact generators $N_{I,I_3,Y}$ of the $SL(3,c)$ representation $(0,b)$ are

$$\langle\nu',Y',I_3',I'|N_{I,I_3,Y}|\hat{I},\hat{I}_3,\hat{Y},\hat{\nu}\rangle =c(\hat{I},I,I';\,\hat{I}_3,I_3,I_3')U(\nu,1,\nu';\,(\gamma=2);\,\hat{I},\hat{Y},I,Y,I'Y')\times\langle\nu'\|N\|\nu\rangle, \tag{A9}$$

where the $SU(2)$ Clebsch-Gordan coefficients $c(\cdots)$ and the isoscalar factors are taken from the tables and the reduced matrix elements are given in (A6)–(A8).

With the aid of (A9) one easily calculates the required matrix elements

$$\begin{aligned}
C_{02}&=\langle\nu=0,\,I=0,\,I_3=0,\,Y=0|\hat{F}_{-2}\times|I=\tfrac{1}{2},\,I_3=\tfrac{1}{2},\,Y=1,\,\nu=1\rangle,\\
C_{01}&=\langle\nu=0,\,I=0,\,I_3=0,\,Y=0|\hat{F}_{-1}\times|I=1,\,Y=0,\,\nu=1\rangle,\\
C_{12}&=\langle\nu=1,\,I=1,\,I_3=0,\,Y=0|\hat{F}_{-2}\times|I=\tfrac{1}{2},\,I_3=\tfrac{1}{2},\,Y=1,\,\nu=1\rangle,\\
C_{11}&=\langle\nu=1,\,I=1,\,I_3=0,\,Y=0|\hat{F}_{-1}\times|I=1,\,I_3=1,\,Y=0,\,\nu=1\rangle.
\end{aligned} \tag{A10}$$

For example, for C_{02} one calculates

$$C_{02}=c(\tfrac{1}{2}\,\tfrac{1}{2},0;\,\tfrac{1}{2},\,-\tfrac{1}{2},\,0)\times U(\nu=1,\,1,\,\nu=0,\,I=\tfrac{1}{2},\,Y=1,\,\tfrac{1}{2},\,-1,\,0,\,0)\times\langle\nu=0\|N\|\nu=1\rangle.$$

From Table I.1 of Ref. 21 and Table II, part 3, of Ref. 22 one obtains

$$c(\tfrac{1}{2},\tfrac{1}{2},0;\,\tfrac{1}{2},\,-\tfrac{1}{2},\,0)=\sqrt{\tfrac{1}{2}},\quad U=-\tfrac{1}{2},$$

and from (A5) one obtains

$$\langle\nu=0\|N\|\nu=1\rangle=-\tfrac{1}{3}i(9-b^2)^{1/2}, \tag{A11}$$

so that one obtains

$$C_{02}=(9-b^2)^{1/2}/6\sqrt{2}\,. \tag{A12}$$

In the same way, one obtains

$$C_{01}=-(9-b^2)^{1/2}/6\sqrt{2}\,, \tag{A13}$$

and with

$$\langle\nu=1\|N\|\nu=1\rangle=-ib/3\sqrt{5} \tag{A14}$$

one calculates

$$C_{12}=ib/15\sqrt{3}\,, \tag{A15}$$

$$C_{11}=-2ib/15\sqrt{3}\,. \tag{A16}$$

APPENDIX B

Usually the generalized eigenvectors

$$|p_i,\alpha\rangle$$

[α is an abbreviation for the set $(I,I_3,Y,\lambda,\cdots)$] are "normalized" in the following way:

$$\langle p_i',\alpha|\alpha,p_i\rangle=\delta_{\alpha\alpha'}2E_\alpha(\mathfrak{p})\delta^3(\mathfrak{p}-\mathfrak{p}')\,,\qquad E_\alpha(\mathfrak{p})=(m_\alpha^2+\mathfrak{p}^2)^{1/2}. \tag{B1}$$

As long as the mass is an invariant, the "norm" given by (B1) is independent of the states. However, in the cases where an irreducible representation space contains different masses, the norms (B1) are different, and if one wants to compare expectation values of an operator between these generalized states with different masses, one must always divide the matrix element by the norm. Therefore, it is advantageous to use a normalization of these generalized eigenvectors which is the same for any basis vector in the whole irreducible representation space.

To introduce this normalization we restrict ourselves to the case where the mass-changing operator Γ $(=F_\alpha,E_\alpha,G_i)$ fulfills the following relation [Eq. (13) of II]:

$$[p_\mu/M,\Gamma]=0\,. \tag{B2}$$

For the two states $|p_i,\alpha\rangle$ and $|\bar{p}_i,\bar{\alpha}\rangle$ we have

$$\langle\alpha,p_i|p_i',\alpha'\rangle=\delta_{\alpha\alpha'}c_\alpha 2(m_\alpha^2+\mathfrak{p}^2)^{1/2}\delta^3(\mathfrak{p}'-\mathfrak{p})\,, \tag{B3}$$

$$\langle\bar{\alpha},\bar{p}_i|\bar{p}_i'\bar{\alpha}'\rangle=\delta_{\bar{\alpha}\bar{\alpha}'}c_{\bar{\alpha}}2(m_{\bar{\alpha}}^2+\mathfrak{p}^2)^{1/2}\delta^3(\mathfrak{p}'-\mathfrak{p})\,, \tag{B4}$$

where the normalization constants c_α must be determined such that the norm does not depend upon the mass. Because of (B2) we have $\mathfrak{p}/m_\alpha=\mathfrak{p}/m_{\bar{\alpha}}$, so that

$$\langle\bar{\alpha},\bar{p}_i|\bar{p}_i'\bar{\alpha}'\rangle=c_{\bar{\alpha}}(m_\alpha/m_{\bar{\alpha}})^2(m_\alpha^2+\mathfrak{p}^2)^{1/2}\delta^3(\mathfrak{p}'-\mathfrak{p})\,, \tag{B5}$$

where we have used

$$\delta^3\left(\frac{m_{\bar{\alpha}}}{m_\alpha}(\mathfrak{p}'-\mathfrak{p})\right)=\left(\frac{m_{\bar{\alpha}}}{\mathrm{m}_\alpha}\right)^{-3}\delta^3(\mathfrak{p}'-\mathfrak{p})\,.$$

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[22] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[23] In Table II, Pt. 7, of Ref. 21, the last (−) has to be replaced with (+).

So we see [comparing (B3) and (B5)] that if we want the norms to be the same, i.e.,

$$\langle \alpha' p_i' | p_i \alpha \rangle = \langle \bar{\alpha}' \bar{p}_i' | \bar{p}_i \bar{\alpha} \rangle ,$$

then the normalization constants c_α (B3) must be chosen such that

$$c_\alpha / c_{\bar{\alpha}} = (m_\alpha / m_{\bar{\alpha}})^2 . \tag{B6}$$