f-oscillators deformation for Moyal algebras

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Abstract

Using general construction of star-product the q-deformed Wigner-Weyl-Moyal quantization procedure is elaborated. The q-deformed Groenewold kernel determining the product of quantum observables is given in explicit form for small nonlinearities corresponding to nonlinear vibrations of classical and quantum q-oscillators. The deformation of Groenewold kernel related to general kinds of nonlinear vibrations described by f-oscillators are considered. *Key words*
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1 Introduction

By using the vector space of functions on phase-space one can formulate both classical and quantum mechanics on the same carrier space. What accounts for the difference between the two theories is the product structure: it is associative, commutative and pointwise for classical mechanics while it is associative, non-commutative and nonlocal for quantum mechanics. This formulation of quantum mechanics on phase space seems the most appropriate to take care of the annoyance of Heisenberg as expressed in a letter to Pauli "...the worst thing is that I am quite unable to clarify the transition to the classical theory"[1].

It took almost twenty years before Moyal and Groenewold independently arrived at a formulation of quantum mechanics on phase space [2, 3].

By using the Weyl correspondence associating operators to functions on phase space [4] and the inverse one proposed by Wigner associating functions to operators acting on some Hilbert space, many people have been able to induce a star-product on functions by using the operator product (see, e.g., [5, 6, 7, 8, 9]). To be precise, what Wigner did was to associate functions on phase space with rank-one projectors (pure states). However by taking appropriate linear combinations of these pure states it is not difficult to extend the association to more general operators. In the same spirit one may consider the diagonal representation introduced by Sudarshan [10] or the one introduced by Husimi and Kano separately [11, 12].

The time evolution as given by the Moyal bracket is fully equivalent to the one generated by the von Neumann equation for density states [13]. When the associative product is written in terms of integral kernels, the commutative and associative product for classical mechanics has a kernel expressed in terms of Dirac delta functions. The kernel appropriate for the product of Weyl symbols corresponding to operators acting on some Hilbert space was given by Groenewold and appear to be a twisted version of the classical one: the twisting factor uses the symplectic structure available on phase space. This circumstance makes clear that if we change the symplectic structure by adding an electromagnetic field, the product of function will change and therefore in general a deformation of the product may be associated with the presence of an external field, i.e. it is a way to incorporate such specific interactions. As the kernel is expressed in terms of an exponent which uses the symplectic area and contains Planck's constant as a parameter, when this parameter is set to zero one recovers the kernel appropriate for classical mechanics [14, 15].

This approach is often considered also to introduce twisted products on space-time in the framework of noncommutative geometry [16]. From what we have said, it should be clear that any one-to-one correspondence between operators and functions will induce a product on the space of functions, in particular one may use symplectic tomographic probability distributions. The general mathematical structure for the description of quantum states in terms of probability distribution was clarified recently [17, 18, 19]. The operator-symbol correspondence was also presented recently from a unified pq anharmonic point of view by Klauder and Skagerstam [20].

Thus as the star-products are derived from the operator products, it is clear that any deformation of the operator product, i.e. an alternative product on the space of operators, will induce an alternative product in the space of symbols. Alternative products in the space of operators were considered by us in connection with a problem raised by Wigner [21] and analyzed to search bi-Hamiltonian descriptions at the quantum level [22]. An interpretation in terms of nonlinear phenomena also with several physical consequences was given in [23].

The q-oscillators [24, 25] were shown [26] to correspond to nonlinear vibrations of classical (and quantum) oscillators with specific nonlinearity. The phase of such q-vibrations depends on the amplitude of the vibrations quasi exponentially. In [23] the notion of f-oscillator was introduced to describe more general nonlinear vibrations in the analogy of q-oscillators. The f-oscillators are deforming commutation relations in very generic form which contains also the particular case of the q-commutation relations corresponding to quantum q-oscillators.

The f-oscillators describe the generic nonlinear relations in which the frequency of vibrations depends on the energy of vibrations and this dependence is determined by the function f. The aim of the present work is to incorporate the f-nonlinearity of the vibrations and to consider its influence on possible deformations of the Moyal-Groenewold star-product. This product is actually unique provided translational invariance is imposed as shown in [27]. Thus we study two kinds of deformations of point-wise product of functions on classical phase space: the first deformation associated with Planck constant providing the Moyal representation of quantum mechanics while the next deformation uses extra parameters (we will focus on q-deformation with extra q-parameter). These deformations are associated with nonlinear vibrations of f-oscillator and the nonlinear function f dependent on energy of vibrations is exploited in our work as a specific K-operator providing the K-deformation of associative product of matrices in infinite dimensional Hilbert space. Another goal of our work is to consider Lie algebra deformation based on the deformed star-product of functions.

The paper is organized as follows. In section 2 we review the generic star-product scheme and Moyal-Weyl-Wigner star-product scheme. In sect. 3 we consider the f-oscillator formalism. In sect. 4 we construct the q-deformed Moyal star-product and Moyal brackets. In sect. 5 the f-deformed Lie algebras will be discussed. The conclusions are given in sect. 6.

2 Weyl symbols and Moyal brackets

In this section we review the general scheme of construction of star-products on functions associated with operators [28, 14, 15, 29].

This construction contains two main ingredients and we consider them at a purely formal and algebraic level; in specific cases one has to check when the appropriate conditions are satisfied.

Given a space S, one defines two families of operators thought of as elements of a vector space V and its dual space V', say for $x \in S$, $\hat{U}(x) \in V'$ and $\hat{D}(x) \in V$. We require that these two families allow the construction of a partition of the unity for the space of operators we are interested in, say

$$\hat{\mathbf{1}} = \int dx \hat{D}(x) \otimes \hat{U}(x) \tag{1}$$

or

$$\hat{\mathbf{1}} = \sum_{n} \hat{D}(n) \otimes \hat{U}(n) \tag{2}$$

when S is a discrete space.

With the help of these two families of operators we construct functions on S, the symbols associated with operators by setting

$$f_{\hat{A}}(x) = \hat{U}(x)(\hat{A}) \tag{3}$$

and vice versa for any f in the range of the previous map

$$\hat{A}_f = \int dx f(x) \hat{D}(x) \ . \tag{4}$$

Clearly

$$\hat{\mathbf{1}}\cdot\hat{A}=\hat{A}=\int dx\hat{D}\left(x\right)\hat{U}\left(x\right)(\hat{A})\ . \tag{5}$$

Very often, when the space of operators we are considering contains inner product, with abuse of notation we write

$$\hat{U}\left(x\right)\left(\hat{A}\right)=Tr\left(\hat{U}^{\dagger}\left(x\right)\hat{A}\right)\ . \tag{6}$$

The second ingredient consists of introducing a product structure on the symbols in correspondence with any product on the vector space V. We set

$$\left(f_{\hat{A}}*f_{\hat{B}}\right)(x)=\int dx_1dx_2f_{\hat{A}}\left(x_1\right)f_{\hat{B}}\left(x_2\right)K\left(x_1.x_2;x\right) \tag{7}$$

where the integral kernel reads

$$K\left(x_1.x_2;x\right)=\hat{U}\left(x\right)\left(\hat{D}\left(x_1\right)\hat{D}\left(x_2\right)\right)=Tr\left(\hat{U}^{\dagger}\left(x\right)\hat{D}\left(x_1\right)\hat{D}\left(x_2\right)\right)\ . \tag{8}$$

This kernel is a reproducing kernel and we have

$$\left(f_{\hat{A}}*f_{\hat{B}}\right)(x)=f_{\hat{A}\hat{B}}\left(x\right)\ . \tag{9}$$

In the particular case of $S=\mathbb{R}^{2n}$ or $S=\mathbb{C}^n$ a possible association is given by the Weyl correspondence

$$\hat{U}^{\dagger}(q,p)=2\pi\hat{D}(q,p)=2e^{\sqrt{2}[(q+ip)a^{\dagger}-h.c.]}e^{i\pi a^{\dagger}a} \tag{10}$$

where $a^{\dagger}$ and a are bosonic creation and annihilation operators (harmonic oscillator amplitudes) satisfying the commutation relation

$$aa^{\dagger}-a^{\dagger}a=\mathbf{1}\ . \tag{11}$$

One finds that, introducing complex coordinates, the operator $\hat{D}(\alpha)$, $\alpha\in\mathbb{C}$, $\alpha=\frac{1}{\sqrt{2}}(q+ip)$, becomes the displacement operator

$$\hat{T}(\alpha)=e^{\alpha a^{\dagger}-\alpha^*a} \tag{12}$$

giving rise to the unitary ray representation of $\mathbb{C}$

$$\hat{T}(\alpha)\hat{T}(\beta)=\hat{T}(\alpha+\beta)e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)} \tag{13}$$

along with the trace formula

$$Tr\hat{D}(q,p)=\pi\delta(q)\delta(p)\ . \tag{14}$$

It should be remarked that all bounded operators can be expressed as a linear combination of such Weyl operators. The partition of unity requirement is satisfied in virtue of

$$Tr\hat{U}^{\dagger}(q,p)\hat{D}(q',p')=\delta(q-q')\delta(p-p')\ . \tag{15}$$

The Weyl symbol associated with the operator $\hat{A}$ is

$$f_{\hat{A}}(q,p) = 2Tr[\hat{A}\hat{T}(2\alpha)e^{i\pi a^\dagger a}] \ . \tag{16}$$

In the case of density state $\hat{\rho}$ eqs. 15 and 16 give the expression for Wigner function of quantum state

$$W_\rho\left(q,p\right) = 2Tr\left[\hat{\rho}e^{\sqrt{2}(q+ip)a^\dagger - h.c.}e^{i\pi a^\dagger a}\right] \ . \tag{17}$$

The operator $e^{i\pi a^\dagger a}$ is the parity operator and using its properties, e.g.

$$\hat{T}\left(\alpha\right)e^{i\pi a^\dagger a} = e^{i\pi a^\dagger a}\hat{T}\left(-\alpha\right) \tag{18}$$

and eqs. 11 and 12 one can calculate the Groenewold kernel [3] for Weyl symbols

$$\begin{aligned} K_G\left(q_1,p_1,q_2,p_2,q_3,p_3\right) &= Tr\left[\hat{D}\left(q_1,p_1\right)\hat{D}\left(q_2,p_2\right)\hat{U}^\dagger\left(q_3,p_3\right)\right] \\ &= \pi^{-2}e^{2i\left(q_3p_1-q_1p_3+q_1p_2-q_2p_1+q_2p_3-q_3p_2\right)} \ . \end{aligned} \tag{19}$$

3 f- and q-oscillators

The standard bosonic commutation relation, eq.(11), can be deformed, e.g. providing q-commutation relation [24, 25]. Such a deformation is described by the generic formalism of $f-$oscillators [23]. Considering classical vibrations of nonlinear oscillators [26] it was clarified that the dependence of phase of vibration on its energy provides a generic deformation of the bosonic commutation relations. The deformation is described by a function $f\left(a^\dagger a\right)$ determining new annihilation operator

$$A = af\left(a^\dagger a\right) \ , \quad A^\dagger = f\left(a^\dagger a\right)a^\dagger \ . \tag{20}$$

For $f = 1$ one has the standard annihilation operator. This deformation replaces the commutator (11) by the commutator

$$AA^\dagger - A^\dagger A = af^2\left(a^\dagger a\right)a^\dagger - a^\dagger af^2\left(a^\dagger a\right) \ . \tag{21}$$

This commutator can be rewritten in the form

$$\left[A, A^\dagger\right] = F\left(A^\dagger A\right) \tag{22}$$

where the function F is determined by the nonlinearity function $f\left(a^\dagger a\right)$. For $f = 1$ we have $F = 1$.

The q-oscillators are described by the specific function

$$f_q\left(a^\dagger a\right) = \sqrt{\frac{\sinh(\lambda\ a^\dagger a)}{\lambda\ a^\dagger a}} \ ; \ q = e^\lambda \ . \tag{23}$$

For $\lambda = 0$ the q-parameter equals 1. For small parameter λ which corresponds to weak nonlinearity of vibrations one has

$$f_q\left(a^\dagger a\right) \simeq 1 + \frac{\lambda^2}{12}\left(a^\dagger a\right)^2 \ . \tag{24}$$

The explicit form of nonlinearity of vibration follows from the equation of motion for the amplitude of the oscillator. For standard oscillator with unit frequency the equation reads

$$\dot{a}\left(t\right) = -ia\left(t\right) \tag{25}$$

which has the solution

$$a\left(t\right) = ae^{-it} \ . \tag{26}$$

One sees that the phase of vibrations does not depend on the amplitude. For the nonlinear oscillator the equation of motion reads

$$\dot{a}\left(t\right) = -i\chi\left(a^\dagger a\right)a\left(t\right) \ . \tag{27}$$

Here the function $\chi\left(a^\dagger a\right)$ is a constant of the motion and the solution to this equation reads

$$a\left(t\right) = ae^{-i\chi\left(a^\dagger a\right)t} \ . \tag{28}$$

Thus the phase of the vibration depends on the energy of nonlinear vibrations. Below we address the question how the nonlinearity of the vibrations of the f-oscillator can be implemented to deform the Moyal star-product. As we discussed the product is determined by quantizer and dequantizer, which depend on the annihilation and creation operators a and $a^\dagger$ of the standard harmonic oscillator. One can incorporate the nonlinear vibrations using the nonlinearity function $f\left(a^\dagger a\right)$. But if one simply replaces the oscillator operators $\left(a \rightarrow A = af\left(a^\dagger a\right)\right)$, in the expressions for quantizer and dequantizer 8 the compatibility condition 13 will be violated. We will then apply the function $f\left(a^\dagger a\right)$ to deform the Moyal star-product using the scheme of K-product which we below review.

4 K-product of matrices

In this section we review an approach to deform the product of matrices keeping the associative property[30]. We consider such K-product in order to apply it to deform the Weyl-Moyal-Wigner product by means of the nonlinearity function $f\left(a^\dagger a\right)$, used in the previous section to construct f-oscillators.

The K-product is introduced by replacing the formula for associative matrix product

$$c = a \cdot b \ \text{ or } \ c_{jk} = \sum\nolimits_s a_{js}b_{sk} \ ; \ j,k,s = 1,2,\ldots,N \tag{29}$$

by the product

$$c = a \cdot_K b \ \text{ or } \ c_{jk} = \sum\nolimits_{s,m} a_{js}K_{sm}b_{mk} \ ; \ j,k,s,m = 1,2,\ldots,N \tag{30}$$

where the matrix K is used to define the K-product. Hereafter, K may depend on a parameter λ in such a way that K becomes the identity when λ goes to zero.

One can check that

$$((a \cdot_K b) \cdot_K c) = (a \cdot_K (b \cdot_K c)) \ . \tag{31}$$

It means that the K-product is associative. For matrix $K = \mathbf{1}$ the K-product becomes the usual product of matrices.

However the new product has a unit only if K is invertible. Moreover if we consider the one-to-one correspondence $A \to \sqrt{K} A \sqrt{K} \doteq \tilde{A}$ we find that there is a homomorphism $A \cdot B \to \tilde{A} \cdot \tilde{B}$ which maps the identity to K. Of course the square root exists only if K is positive. For instance if A_1, A_2, A_3 close on the Lie algebra of the rotation group, the use of a positive K will give rise to a new realization of the rotation group. The matrix K may be thought of as being responsible for replacing the sphere, orbits of the standard realization of the rotation group, with ellipsoids, orbits of the K-realization of the rotation group.

If one has matrices a and b corresponding to operators $\hat{a}$ and $\hat{b}$ acting in infinite dimensional Hilbert space the K-product of the matrices is determined by the kernel of the product, i.e.

$$(a \cdot_K b)(x, x') = \int a(x, y) K(y, z) b(z, x') \, dy dz \ . \tag{32}$$

The kernel $K(y, z)$ determines the new K-product of the operators acting on the Hilbert space.

5 f-deformed Moyal product

In this section we use the machinery of the K-product to deform the Moyal product as was suggested in [31] and calculate in explicit form the q-deformed Groenewold kernel for the deformation parameter q close to unity. By definition the deformation of associative product determined by the nonlinear operator function $f\left(a^\dagger a\right)$ is given by relation

$$f_{AB}(q, p) = Tr\left[\hat{U}(q, p) \hat{A} f\left(a^\dagger a\right) \hat{B}\right] \ . \tag{33}$$

Using this definition and rewriting the symbol of product symbol in integral form we get for the kernel of the product the relation

$$K_f(q_1, p_1, q_2, p_2, q, p) = Tr\left[\hat{D}(q_1, p_1) f\left(a^\dagger a\right) \hat{D}(q_2, p_2) \hat{U}(q, p)\right] \ . \tag{34}$$

The above kernel satisfies the properties of the solutions of associative equation. To determine the kernel of q-deformed Groenewold kernel one needs to calculate the following trace

$$
\begin{aligned}
& K_\lambda\left(q_1, p_1, q_2, p_2, q, p\right) && (35)\\
= \quad & \frac{2}{\pi^2} Tr\left[\left(1+\frac{\lambda^2}{12}\left(a^{\dagger} a\right)^2\right) e^{2\left(\alpha_2 a^{\dagger}-\alpha_2^* a\right)} e^{-2\left(\alpha a^{\dagger}-\alpha^* a\right)} e^{2\left(\alpha_1 a^{\dagger}-\alpha_1^* a\right)}(-1)^{a^{\dagger} a}\right]
\end{aligned}
$$

where

$$
\alpha_{1,2}=\frac{1}{\sqrt{2}}\left(q_{1,2}+i p_{1,2}\right), \quad \alpha=\frac{1}{\sqrt{2}}(q+i p) . \tag{36}
$$

The first term which does not contain the nonlinearity parameter λ coincides with Gronewold kernel

$$
K_G\left(q_1, p_1, q_2, p_2, q, p\right)=\pi^{-2} e^{2 i\left(q p_1-q_1 p+q_1 p_2-q_2 p_1+q_2 p-q p_2\right)} . \tag{37}
$$

One can calculate the nonlinear correction to the Groenewold kernel using generating function for the deformed kernel. It means that we first calculate the f-deformed kernel with the nonlinearity function of the form

$$
f_\tau\left(a^{\dagger} a\right)=e^{i \tau a^{\dagger} a} . \tag{38}
$$

Having the τ-deformed kernel we can get the correction in the q-deformed kernel by taking the second derivative of the τ-deformed kernel since

$$
e^{-i \tau a^{\dagger} a}=1-i \tau a^{\dagger} a-\frac{1}{2} \tau^2\left(a^{\dagger} a\right)^2+\ldots . \tag{39}
$$

Using the generating function we get the result

$$
K_\lambda\left(q_1, p_1, q_2, p_2, q, p\right)=K_G\left(q_1, p_1, q_2, p_2, q, p\right)\left[1+\frac{\lambda^2}{192}(\mu-1)^2\right] \tag{40}
$$

where

$$
\mu=\left(q-q_2-q_1\right)^2+\left(p-p_2-p_1\right)^2 . \tag{41}
$$

The correction (41) to the Groenewold kernel is symmetric with respect to replacement

$$
q_1 \longleftrightarrow q_2, \quad p_1 \longleftrightarrow p_2 .
$$

It means that the Lie algebra structure constants related to Groenewold kernel

$$
C_G\left(q_1, p_1, q_2, p_2, q, p\right) \doteq K_G\left(q_1, p_1, q_2, p_2, q, p\right)-K_G\left(q_2, p_2, q_1, p_1, q, p\right) \tag{42}
$$

has the analogous deformed form

$$
C_\lambda\left(q_1, p_1, q_2, p_2, q, p\right)=C_G\left(q_1, p_1, q_2, p_2, q, p\right)\left(1+\frac{\lambda^2}{192}(\mu-1)^2\right) . \tag{43}
$$

The structure constants in eq.(42) define infinite Lie algebra of Weyl algebra. The structure constants in eq.(43) define q-deformed on finite Weyl algebra.

6 Conclusions

To conclude we point out some relevant aspects of the present work.

We realize that equations of motion on the algebra of operators may start with a linear equation of the type $\frac{d}{dt}A = L(A)$ where L is any linear map which does not need to respect the product structure on the algebra. This is the case when we consider generic Markovian evolution for open systems. When the linear map turns out to be a derivation of the specific product we are using, and the algebra is irreducible, the derivation associated with L will be an inner derivation and the dynamical equation of motion becomes $\frac{d}{dt}A = [H_L, A]$, i.e. acquires the Heisenberg form. Thus a given linear map L may be represented by alternative Hamiltonians according to the alternative products it preserves: therefore the correspondence "Hamiltonian $\longrightarrow$ equation of motion" depends on the particular product we use. Therefore these three ingredients appearing in the description of evolution may be used in different manners. If we use a fixed Hamiltonian but change the product, we get a different set of equations of motion (this would be the case when the electromagnetic field is inserted in the Poisson Brackets.) If we change the Hamiltonian *along* with the operator product we may *compensate* the changes so that the equations of motion would be the same (this occurs in the description of biHamiltonian systems when dealing with complete integrability). The particular instance we have considered, changing the product while preserving the Hamiltonian has provided us with an harmonic motion where the frequency depends on the energy. It is thanks to this circumstance that the hydrogen atom may be described as a reduction of harmonic oscillators [33]. For the Harmonic oscillator the frequency and phase are independent of amplitude (the action variable). So if there are several harmonic oscillators with different amplitudes they will keep their relative phases and all of them can be brought to rest by a rotating phase space. On the other hand, for the Kepler problem, Kepler's third law states that larger radii means lesser frequency: $T^2 \sim R^3$. The orbits of the nonlinear oscillators can all be obtained by the central force which varies in specified fashion on the action.

Thus, it is possible to consider 'effective interactions' by changing the product we use to multiply operators.

This point of view was the one taken, for instance, when q-deformed oscillators were considered. Here we would like to stress that one may go beyond these deformations and therefore go beyond Hopf algebras and their deformations.

The other point of view we would like to stress is that by changing the product structure on the space of operators we may obtain alternative structures of $\mathbb{C}^*$-algebras on the same vector space of operators; therefore the GNS construction appropriate for each $\mathbb{C}^*$-algebra structure would provide us with different Hilbert spaces, which, due to the nonlinear change in the product structure, would be connected in a one-to-one correspondence only if the nonlinear transformations would be allowed. This becomes immediately clear if the operators $A, A^\dagger$expressed by 20 are used to construct the Fock space out of the same vacuum as the one used by the usual $a, a^\dagger$.

In conclusion, while this paper exhibits an explicit integral kernel for a nonlinear deformation of the Moyal brackets, we seem to be suggested that nonlinear transformations may find their way in the description of quantum systems on Hilbert spaces by means of $\mathbb{C}^*$-algebras.

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