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Quantum Zeno Dynamics

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Abstract

The evolution of a quantum system undergoing very frequent measurements takes place in a subspace of the total Hilbert space (quantum Zeno effect). The dynamical properties of this evolution are investigated and several examples are considered.

1 Introduction

A quantum system, prepared in a state that does not belong to an eigenvalue of its total Hamiltonian, starts to evolve quadratically in time [1, 2]. This behavior leads to the so-called quantum Zeno phenomenon: by performing frequent measurements on the system, in order to check whether it is still in its initial state, one can "slow down" its temporal evolution (hindering transitions to states different from the initial one) [3].

This curious feature of the quantal evolution has recently attracted much attention in the physics community. This is mainly due to a nice idea put forward by Cook [4], who proposed to check this effect on a two-level system, and to a related experimental test [5], that motivated an interesting discussion [6]. In turn, this has led to new proposals and experiments [7, 8]. However, it should be emphasized that these studies do not deal with *bona fide* unstable systems, following (approximately) exponential laws, as in the original proposals [1, 2]. The presence of a non-exponential decay at short times has been detected only recently [9].

The aim of the present paper is to investigate an interesting (and often overlooked) feature of what we might call a quantum Zeno dynamics. We shall see that a series of "measurements" (von Neumann's projections [10]) does not necessarily hinder the evolution of the quantum system. On the contrary, the system can evolve away from its initial state, provided it remains in the subspace defined by the "measurement" itself. This interesting feature is readily understandable in terms of a theorem proved by Misra and Sudarshan (MS) [2], but it seems to us that it is worth clarifying it further by analyzing some interesting examples.

2 Misra and Sudarshan's theorem

Consider a quantum system Q, whose states belong to the Hilbert space $\mathcal{H}$ and whose evolution is described by the unitary operator $U(t) = \exp(-iHt)$, where H is a time-independent semi-bounded Hamiltonian. Let E be a projection operator that does not commute with the Hamiltonian, $[E, H] \neq 0$, and $E\mathcal{H}E = \mathcal{H}_E$ the subspace spanned by its eigenstates. The initial density matrix ρ_0 of system Q is taken to belong to $\mathcal{H}_E$. If Q is let to follow its "undisturbed" evolution, under the action of the Hamiltonian H (i.e., no measurements are performed in order to get information about its quantum state), the final state at time T reads

$$\rho(T) = U(T)\rho_0 U^\dagger(T) \tag{2.1}$$

and the probability that the system is still in $\mathcal{H}_E$ at time T is

$$P(T) = \mathrm{Tr}\left[U(T)\rho_0 U^\dagger(T)E\right]. \tag{2.2}$$

We call this a "survival probability:" it is in general smaller than 1, since the Hamiltonian H induces transitions out of $\mathcal{H}_E$. We shall say that the quantum system has "survived" if it is found to be in $\mathcal{H}_E$ by means of a suitable measurement process [11].

We stress that we do not distinguish between one- and many-dimensional projections: in the examples to be considered in this note, E will be infinite-dimensional.

Assume that we perform a measurement at time t, in order to check whether Q has survived. Such a measurement is formally represented by the projection operator E. By definition,

$$\rho_0 = E\rho_0 E, \qquad \mathrm{Tr}[\rho_0 E] = 1. \tag{2.3}$$

After the measurement, the state of Q changes into

$$\rho_0 \to \rho(t) = EU(t)\rho_0 U^\dagger(t)E, \tag{2.4}$$

with probability

$$\begin{aligned} P(t) &= \mathrm{Tr}\left[U(t)\rho_0 U^\dagger(t)E\right] = \mathrm{Tr}\left[EU(t)E\rho_0 EU^\dagger(t)E\right] \\ &= \mathrm{Tr}\left[V(t)\rho_0 V^\dagger(t)\right]. \qquad (V(t) \equiv EU(t)E) \end{aligned} \tag{2.5}$$

This is the probability that the system has "survived" in $\mathcal{H}_E$. There is, of course, a probability $1-P$ that the system has not survived (i.e., it has made a transition outside $\mathcal{H}_E$) and its state has changed into $\rho'(t) = (1-E)U(t)\rho_0 U^\dagger(t)(1-E)$. The states ρ and ρ' together make up a block diagonal matrix: The initial density matrix is reduced to a mixture and any possibility of interference between "survived" and "not survived" states is destroyed (complete decoherence).

We shall concentrate henceforth our attention on the measurement outcome (2.4)-(2.5). We observe that the evolution just described is time-translation invariant and the dynamics is *not* reversible (not only not time-reversal invariant).

The above is the Copenhagen interpretation: the measurement is considered to be instantaneous. The "quantum Zeno paradox" [2] is the following. We prepare Q in the initial state ρ_0 at time 0 and perform a series of E-observations at times $t_j = jT/N$ $(j = 1, \cdots, N)$. The state of Q after the above-mentioned N measurements reads

$$\rho^{(N)}(T) = V_N(T)\rho_0 V_N^\dagger(T), \qquad V_N(T) \equiv [EU(T/N)E]^N \tag{2.6}$$

and the probability to find the system in $\mathcal{H}_E$ ("survival probability") is given by

$$P^{(N)}(T) = \mathrm{Tr}\left[V_N(T)\rho_0 V_N^\dagger(T)\right]. \tag{2.7}$$

Equations (2.6)-(2.7) display the "quantum Zeno effect:" repeated observations in succession modify the dynamics of the quantum system; under general conditions, if N is sufficiently large, all transitions outside $\mathcal{H}_E$ are inhibited. Notice again that the dynamics (2.6)-(2.7) is not reversible.

In order to consider the $N \to \infty$ limit ("continuous observation"), one needs some mathematical requirements: assume that the limit

$$\mathcal{V}(T) \equiv \lim_{N\to\infty} V_N(T) \tag{2.8}$$

exists in the strong sense. The final state of Q is then

$$\rho(T) = \mathcal{V}(T)\rho_0 \mathcal{V}^\dagger(T) \tag{2.9}$$

and the probability to find the system in $\mathcal{H}_E$ is

$$\mathcal{P}(T) \equiv \lim_{N\to\infty} P^{(N)}(T) = \mathrm{Tr}\left[\mathcal{V}(T)\rho_0\mathcal{V}^\dagger(T)\right]. \tag{2.10}$$

One should carefully notice that nothing is said about the final state $\rho(T)$, which depends on the characteristics of the model investigated and on the *very measurement performed* (i.e. on the projection operator E, by means of which V_N is defined). By assuming the strong continuity of $\mathcal{V}(t)$

$$\lim_{t\to 0^+} \mathcal{V}(t) = E, \tag{2.11}$$

one can prove that under general conditions the operators

$$\mathcal{V}(T) \quad \text{exist for all real } T \text{ and form a semigroup.} \tag{2.12}$$

Moreover, by time-reversal invariance

$$\mathcal{V}^\dagger(T) = \mathcal{V}(-T), \tag{2.13}$$

so that $\mathcal{V}^\dagger(T)\mathcal{V}(T) = E$. This implies, by (2.3), that

$$\mathcal{P}(T) = \mathrm{Tr}\left[\rho_0\mathcal{V}^\dagger(T)\mathcal{V}(T)\right] = \mathrm{Tr}\left[\rho_0 E\right] = 1. \tag{2.14}$$

If the particle is "continuously" observed, in order to check whether it has survived inside $\mathcal{H}_E$, it will never make a transition to $\mathcal{H}_E^\perp$. This was named "quantum Zeno paradox" [2]. The expression "quantum Zeno effect" seems more appropriate, nowadays.

Two important remarks are now in order: first, it is not clear whether the dynamics in the $N \to \infty$ limit is time reversible. Although one ends up, in general, with a semigroup, there are concrete elements of reversibility in the above equations. Second, the theorem just summarized *does not* state that the system *remains* in its initial state, after the series of very frequent measurements. Rather, the system is left in the subspace $\mathcal{H}_E$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$. This subtle point, implied by Eqs. (2.9)-(2.14), is not duely stressed in the literature [12].

Incidentally, we stress that there is a conceptual gap between Eqs. (2.7) and (2.10): to perform an experiment with N finite is only a practical problem, from the physical point of view. On the other hand, the $N \to \infty$ case is physically unattainable, and is rather to be regarded as a mathematical limit (although a very interesting one). In this paper, we shall not be concerned with this problem (investigated in [13]; see also [14], where an interesting perspective is advocated) and shall consider the $N \to \infty$ limit for simplicity. This will make the analysis more transparent.

3 Evolution in the "Zeno" subspace

We start off by looking at some explicit examples. Consider a free particle of mass m on the real line. The Hamiltonian and the corresponding evolution operator are

$$H = \frac{p^2}{2m}, \qquad U(t) = \exp(-itH). \tag{3.1}$$

Observe that H is a positive-definite self-adjoint operator in $L^2(\mathbb{R})$ and $U(t)$ is unitary. We shall study the quantum Zeno effect when the system undergoes a measurement defined by the projector

$$E_A = \int dx\ \chi_A(x)|x\rangle\langle x|, \tag{3.2}$$

where χ_A is the characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \subset \mathbb{R} \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

and A an interval of $\mathbb{R}$. In a few words, we check whether a particle, initially prepared in a state with support in A and free to move on the real line, is still found in A at a later time T. Our objective is to understand how the system evolves in the "Zeno" subspace $\mathcal{H}_{E_A} = E_A \mathcal{H} E_A$. We call this a "quantum Zeno dynamics with a nonholonomic constraint."

We shall work with the Euclidean Feynman integral. Let the particle be initially $(t = 0)$ at position $y \in E_A$. The propagator at time $t = T/N$, when a measurement is carried out, reads

$$G(x, t; y) \equiv \langle x|E_A\, U(t)|y\rangle = \chi_A(x)\langle x|U(t)|y\rangle. \tag{3.4}$$

For imaginary time $t = -i\tau$, we get the Green function of the heat equation

$$\begin{aligned} \langle x|U(-i\tau)|y\rangle &= \langle x|\exp(-\tau H)|y\rangle = \int dp\ \langle x|p\rangle e^{-\tau p^2/2m}\langle p|y\rangle \\ &= \int \frac{dp}{2\pi}\ e^{-\tau p^2/2m + ip(x-y)} = \sqrt{\frac{m}{2\pi\tau}} \exp\left[-\frac{m(x-y)^2}{2\tau}\right], \end{aligned} \tag{3.5}$$

so that the Euclidean propagator for a single "step" reads

$$W(x, \tau; y) \equiv G(x, -i\tau; y) = \chi_A(x)\sqrt{\frac{m}{2\pi\tau}} \exp\left[-\frac{m(x-y)^2}{2\tau}\right]. \tag{3.6}$$

The evolution operator after N measurements, see (2.6), can be written as

$$V_N(T) \equiv [E_A\, U(T/N)]^N E_A \tag{3.7}$$

and the resulting propagator is

$$G_N(x_\mathrm{f}, T; x_\mathrm{i}) = \langle x_\mathrm{f}|V_N(T)|x_\mathrm{i}\rangle. \tag{3.8}$$

For imaginary $\mathcal{T} = iT$ this becomes

$$\begin{aligned} W_N(x_\mathrm{f}, \mathcal{T}; x_\mathrm{i}) &\equiv G_N(x_\mathrm{f}, -i\mathcal{T}; x_\mathrm{i}) \\ &= \int dx_1 \cdots dx_{N-1} W\left(x_\mathrm{f}, \tau; x_{N-1}\right) \cdots W\left(x_1, \tau; x_\mathrm{i}\right) \chi_A(x_i), \end{aligned} \tag{3.9}$$

whose relation with Wiener integration is manifest. Notice that if we could drop the characteristic function χ_A in the propagator (3.6), then (3.9) would be a sequence of nested Gaussian integrals, that could be evaluated exactly for every N by applying Feynman's recipe [15]. In (3.9) the characteristic functions restrict at every step the set of possible paths, modifying the structure of the functional integral [16]. Let us therefore try to reduce the integral (3.9) to a Gaussian form. To this end we apply a trick that is often used when one endeavours to relate probability and potential theory [17]. We first rewrite the characteristic function in terms of a potential, which is infinite outside A [18], so that the Brownian paths of the Wiener process (3.9) can never leak out of A:

$$\chi_A(x) = \exp\left(-\tau V_A(x)\right), \qquad \text{with} \qquad V_A(x) = \begin{cases} 0 & \text{for } x \in A \\ +\infty & \text{otherwise} \end{cases}, \tag{3.10}$$

Hence, by using (3.10), the Euclidean one-step propagator (3.6) becomes

$$W(x, \tau; y) = \sqrt{\frac{m}{2\pi\tau}} \exp\left[-\frac{m(x-y)^2}{2\tau} - \tau V_A(x)\right] = \langle x| e^{-\tau V_A} e^{-\tau H} |y\rangle \tag{3.11}$$

and returning to real time

$$G(x, t; y) = W(x, it; y) = \langle x| e^{-itV_A} e^{-itH} |y\rangle. \tag{3.12}$$

Consider now the limit of continuous observation $N \to \infty$. The limiting propagator reads

$$\mathcal{G}(x_{\mathrm{f}}, T; x_{\mathrm{i}}) = \lim_{N\to\infty} G_N(x_{\mathrm{f}}, T; x_{\mathrm{i}}) = \lim_{N\to\infty} \langle x_{\mathrm{f}}| \left(e^{-iTV_A/N} e^{-iTH/N}\right)^N E_A |x_{\mathrm{i}}\rangle, \tag{3.13}$$

which, by using the Trotter product formula, yields

$$\mathcal{G}(x_{\mathrm{f}}, T; x_{\mathrm{i}}) = \langle x_{\mathrm{f}}| e^{-iT(H+V_A)} E_A |x_{\mathrm{i}}\rangle = \langle x_{\mathrm{f}}| \mathcal{V}(T) |x_{\mathrm{i}}\rangle, \tag{3.14}$$

where the evolution operator is

$$\mathcal{V}(T) = \exp(-iTH_{\mathrm{Z}})\, E_A, \tag{3.15}$$

$$\text{with} \qquad H_{\mathrm{Z}} \equiv \frac{p^2}{2m} + V_A(x). \tag{3.16}$$

The above formula is of general validity: the dynamics *within* the Zeno subspace $\mathcal{H}_{E_A}$ is governed by the operators (3.15)-(3.16).

It is worth stressing that the previous calculation only makes use of the properties of the kinetic energy operator p^2: we have *not* considered the momentum operator p. It goes without saying that p can be symmetric, maximally symmetric or self-adjoint, according to the structure of A and the boundary conditions. This will be thoroughly discussed in the following. However, we emphasize that any requirement on p would be a *physical* requirement: the mathematical properties of the "Zeno" evolution *only* involve the Hamiltonian (which is defined in terms of the kinetic energy).

Before we proceed further, let us look at two particular cases:

$$A_1 = [0,1], \tag{3.17}$$
$$A_2 = [0,+\infty). \tag{3.18}$$

In the first case, the free Hamiltonian

$$H_{\rm Z}^0 = \frac{-\partial^2}{2m} \quad \left(\partial \equiv \frac{d}{dx}\right) \tag{3.19}$$

is a self-adjoint operator on the space

$$D_{[0,1]}(H_{\rm Z}^0) = \left\{\phi \in {\rm AC}^2[0,1] | \phi(0) = \phi(1) = 0\right\}, \tag{3.20}$$

where $\mathrm{AC}^2[S]$ is the set of functions in $L^2[S]$ whose weak derivatives are in $\mathrm{AC}[S]$. ($\mathrm{AC}[S]$ is the set of absolutely continuous functions whose weak derivatives are in $L^2[S]$.) Notice that these are the "correct" boundary condition for the potential (3.10). For this reason, the evolution operators $\mathcal{V}(T)$ in (3.15) form a one-parameter *group*. We notice, incidentally, that MS's mathematical hypotheses (2.8) and (2.11) are satisfied and acquire in this example an appealing physical meaning. We also stress that the theorem (2.12) appears in this case too restrictive: indeed the operators $\mathcal{V}(T)$ form a group and not simply a semigroup. One might say that in the example considered, the quantum Zeno effect (engendered by the projection operators) automatically yields the "natural" dynamics in the Zeno subspace, with the correct boundary conditions for the "new" Hamiltonian $H_{\rm Z}$. This is an interesting observation in itself. We also notice that in this example the momentum operator $-i\partial$ is symmetric, but *not* self-adjoint: its deficiency indices in (3.20) are (1,1). Therefore, a self-adjoint extension of $-i\partial$ is possible. It is important to stress that the Hamiltonian $H_{\rm Z}$ is self-adjoint because it involves only ∂^2 [which is self-adjoint in (3.20)]. There is here an interesting classical analogy: when a classical particle elastically bounces between two rigid walls, any trajectory is characterized by a definite value of energy ($p^2/2m$), although momentum changes periodically between $\pm p$. This is reflected in the symmetry (rather than self-adjointness) of the quantum mechanical p operator.

Let us now look at the example A_2 in (3.18). The free Hamiltonian (3.19) is self-adjoint on the space

$$D_{[0,\infty)}(H_{\rm Z}^0) = \left\{\phi \in {\rm AC}^2[0,\infty) | \phi(0) = 0\right\}, \tag{3.21}$$

Once again, this is just the "correct" boundary condition for the potential (3.10), so that the evolution operators $\mathcal{V}(T)$ form a one-parameter group. One can draw the same conclusions as in the previous example. There is only one difference: the momentum operator $-i\partial$ is again symmetric, but its deficiency indices are (0,1). This is irrelevant as far as one's attention is restricted to the Hamiltonian and the Zeno dynamics; however, if one is motivated (on physical grounds) to consider the properties of momentum, the best one can do in this case is to obtain the most appropriate maximally symmetric momentum operator. (We wonder whether this has spin-offs at a fundamental quantum mechanical level.)

4 The problem of the lower-boundedness of the Hamiltonian

Let us consider now the model Hamiltonian $H_{\mathrm{Z}}^0 = p = -i\partial$ in $A_1 = [0,1]$, describing an ultrarelativistic particle in an interval. The mathematical features of this example are very interesting and deserve careful investigation. A similar example was considered in [2], although in a different perspective. The Zeno dynamics yields

$$\mathcal{V}(T) = \exp(-iTH_{\mathrm{Z}})\, E_{A_1}, \tag{4.1}$$

$$\text{with} \qquad H_{\mathrm{Z}} \equiv p + V_{A_1}(x), \tag{4.2}$$

where V_A is defined in (3.10). The "natural" boundary conditions imposed by the Zeno dynamics are

$$D_{\mathrm{Z}}(p) = \{\phi \in \mathrm{AC}[0,1] | \phi(0) = 0 = \phi(1)\}\,. \tag{4.3}$$

In this domain the Hamiltonian p is symmetric but *not* self-adjoint: its deficiency indices are $(1,1)$. Therefore, by Stone's theorem, the Zeno dynamics is not governed by a group and is certainly not time-reversal invariant. More to this, this Hamiltonian is not lower bounded and therefore violates one of the premises of the MS theorem. In order to understand what happens during a Zeno dynamics, look at the first row in Figure 1, where an arbitrary wave packet evolves under the action of the free Hamiltonian p (incidentally, notice that the wave packet does not disperse, due to the form of the Hamiltonian). The probability of "surviving" inside A_1 decreases with time: in other words, even though a "continuous" measurement is performed, in order to check whether the particle is outside A_1, the particle *does* leak out of A_1 and *no* quantum Zeno effect takes place.

Let us now assume, on physical grounds, the validity of periodic boundary conditions:

$$D^\alpha(p) = \left\{\phi \in \mathrm{AC}[0,1] | \phi(0) = \phi(1) e^{i\alpha}\right\}, \tag{4.4}$$

where the phase α determines the specific self-adjoint extension. Notice that this is a physical requirement: it is not a consequence of the Zeno dynamics. The Hamiltonian is now self-adjoint and the dynamics is governed by a unitary group (Stone's theorem). Obviously, the physical picture given by this self-adjoint extension is completely different from the previous case. See the second row in Figure 1: a quantum Zeno effect takes place.

We also stress that the dependence of the Hamiltonian on the p operator is not a sufficient condition to yield the behavior described above. In order to clarify this point, let us consider an additional example. Let (we set $m = 1/2$)

$$H = p^2 + p \quad \Longleftrightarrow \quad H_{\mathrm{Z}} = p^2 + p + V_A(x). \tag{4.5}$$

We first observe that H is lower bounded [$p^2 + p = (p + 1/2)^2 - 1/4$; notice also that this Hamiltonian can be tranformed into the usual form by adding a phase $x/2$ to the wave function.] Consider again the quantum Zeno dynamics on the sets A_1 and A_2.

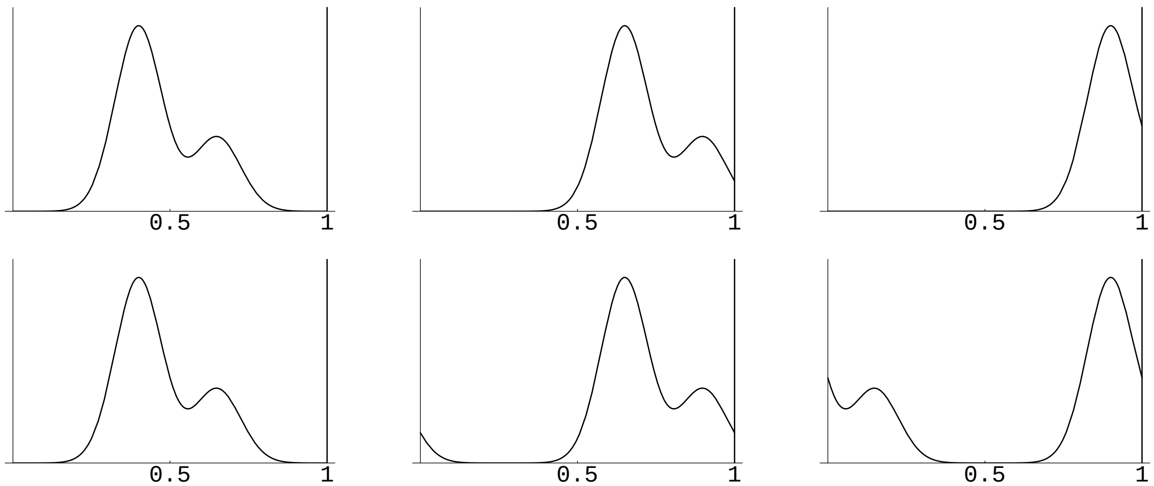

Figure 1: "Natural" (Zeno) vs. periodic (self-adjoint) boundary conditions for the Hamiltonian $H = p$. Above: quantum evolution of a wave packet of arbitrary shape with the boundary conditions (4.3), "naturally" arising in a Zeno dynamics: there is no quantum Zeno effect (increasing time from left to right). Below: evolution of the same wave packet with the additional requirement that the unbounded Hamiltonian operator p be self-adjoint in $[0, 1]$: a quantum Zeno effect occurs (increasing time from left to right).

Since this is not a classical textbook example, we explicitly derive the deficiencies. In the first case (A_1) one gets

$$\begin{aligned}(H_{\mathrm{Z}}\phi, \psi) - (\phi, H_{\mathrm{Z}}^* \psi) &= -i\overline{\phi(0)}\psi(0) + \overline{\phi'(0)}\psi(0) - \overline{\phi(0)}\psi'(0) \\ &\quad + i\overline{\phi(1)}\psi(1) - \overline{\phi'(1)}\psi(1) + \overline{\phi(1)}\psi'(1). \end{aligned} \tag{4.6}$$

It is easy to check that H_{Z} is lower bounded and self adjoint on the space (3.20). The Zeno evolution is therefore unitary.

In the second case (A_2) one gets

$$(H_{\mathrm{Z}}\phi, \psi) - (\phi, H_{\mathrm{Z}}^* \psi) = -i\overline{\phi(0)}\psi(0) + \overline{\phi'(0)}\psi(0) - \overline{\phi(0)}\psi'(0). \tag{4.7}$$

It is straightforward to check that the Hamiltonian is lower bounded and self-adjoint on the space (3.21). Once again, the Zeno evolution is unitary.

5 Discussion

One is led to the following question: is it possible to find an example in which the Zeno dynamics is governed by a dynamical *semigroup*? The answer to this question would be positive if one could find a quantum Zeno dynamics yielding a symmetric, but not self-adjoint, Hamiltonian operator. Indeed, in such a case, by Stone's theorem one cannot have a group, and by MS's theorem one must have a semigroup.

It would be incorrect to think that the model Hamiltonian $H = p = -i\partial$ in $A_1 = [0,1]$ (or even more in $A_2 = [0,\infty)$) provides us with the counterexample we seek. Indeed, such a Hamiltonian is not a satisfactory example, because it violates one of the premises of the MS theorem, that requires a lower-bounded Hamiltonian from the outset (see beginning of Section 2).

We are unable, at the present stage, to give a clear-cut answer to this problem. However, some comments are in order. If, for some reason, the quantum Zeno dynamics yields a symmetric Hamiltonian operator, the search for its self-adjoint extensions seems to us a very important one, on physical grounds. Suppose then that one is willing to consider a self-adjoint extension of the Zeno Hamiltonian $H_{\rm Z}$. If this is the case, close inspection shows that a quantum Zeno dynamics always yields a group, at least in the class of systems considered in this note. Indeed, a theorem due to von Neumann, Stone and Friedrichs [19] states that "every semi-bounded symmetric transformation S can be extended to a semi-bounded self-adjoint transformation S' in such a way that S' has the same (greatest lower or least upper) bound as S." Therefore, if the Hamiltonian is lower-bounded on the real line, as for instance in (3.1) (one could even add a non-pathological potential to the kinetic energy), the Zeno dynamics in an interval of $\mathbb{R}$ will also be engendered by a lower-bounded Hamiltonian, like in (3.16); this would always admit a self-adjoint extension (due to the above-mentioned theorem), which in turn would yield a *group* of evolution operators. Therefore, in order to avoid the consequences of von Neumann's theorem, the operators arising from the Zeno dynamics must *not* be lower bounded. Only in such a case the Zeno Hamiltonian might not admit self-adjoint extensions.

In conclusion, we have seen that in the situations considered in this paper the quantum Zeno effect yields a *unitary* dynamics, governed by groups, not by semigroups. We are therefore left with two possible options: i) The MS theorem can be made stronger and the Zeno dynamics is *always* governed by a group; ii) Different projections, more general than (3.2)-(3.3), and/or different Hamiltonian operators may yield symmetric Zeno Hamiltonian operators that are not self-adjoint (or, even more, maximally symmetric operators with no self-adjoint extensions) and therefore (due to the MS theorem) a *semi*group of evolution operators.

The answer to the above alternative would clarify whether a quantum Zeno dynamics introduces some elements of irreversibility in the evolution of a quantum system. This is an interesting open problem.

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References

[1] A. Beskow and J. Nilsson, Arkiv für Fysik **34**, 561 (1967); L.A. Khalfin, Zh. Eksp. Teor. Fiz. Pis. Red. **8**, 106 (1968) [JETP Letters **8**, 65 (1968)]; L. Fonda,

G.C. Ghirardi, A. Rimini and T. Weber, Nuovo Cim. **A15**, 689 (1973); **A18**, 805 (1973).

[2] B. Misra and E.C.G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[3] For a review on the temporal behavior of quantum systems and the quantum Zeno effect, see H. Nakazato, M. Namiki and S. Pascazio, Int. J. Mod. Phys. **B10**, 247 (1996); D. Home and M.A.B. Whitaker, Ann. Phys. **258**, 237 (1997).

[4] R.J. Cook, Phys. Scr. **T21**, 49 (1988).

[5] W.H. Itano, D.J. Heinzen, J.J. Bollinger and D.J. Wineland, Phys. Rev. **A41**, 2295 (1990).

[6] T. Petrosky, S. Tasaki and I. Prigogine, Phys. Lett. **A151**, 109 (1990); Physica **A170**, 306 (1991); A. Peres and A. Ron, Phys. Rev. **A42**, 5720 (1990); T.P. Altenmüller and A. Schenzle, Phys. Rev. **A49**, 2016 (1994); H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Phys. Lett. **A217**, 203 (1996); L.S. A. Beige and G. Hegerfeldt, Phys. Rev. **A53**, 53 (1996); A.G. Kofman and G. Kurizki, Phys. Rev. **A54**, R3750 (1996); A. Luis and J. Periña, Phys. Rev. Lett. **76**, 4340 (1996). L.S. Schulman, Phys. Rev. **A57**, 1509 (1998).

[7] S. Pascazio, M. Namiki, G. Badurek and H. Rauch, Phys. Lett. **A179** (1993) 155; S. Pascazio and M. Namiki, Phys. Rev. **A50** (1994) 4582.

[8] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger and M. Kasevich, Phys. Rev. Lett. **74** (1995) 4763.

[9] The first experimental observation of non-exponential decay at short times for an unstable system was performed three years ago: S.R. Wilkinson *et al.*, Nature **387** (1997) 575. No attempt has yet been made to suppress decay by repeated measurements (quantum Zeno effect).

[10] J. von Neumann, *Die Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932). [English translation by E.T. Beyer: *Mathematical Foundation of Quantum Mechanics* (Princeton University Press, Princeton, 1955)]. For the quantum Zeno effect , see in particular p. 195 of the German edition (p. 366 of the English translation).

[11] Misra and Sudarshan in [2] considered an initial “unstable” state ρ_0 and took E to be the subspace of the undecayed states. The expression “survived” means in their case “undecayed.”

[12] A concrete example is considered by K. Machida, H. Nakazato, S. Pascazio, H. Rauch and S. Yu, Phys. Rev. **A60**, 3448 (1999).

[13] G.C. Ghirardi, C. Omero, T. Weber and A. Rimini, Nuovo Cim. **A52**, 421 (1979); H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Phys. Lett. **A199**, 27 (1995); A. Venugopalan and R. Ghosh, Phys. Lett. **A204**, 11 (1995); A.K. Pati, Phys. Lett. **A215**, 7 (1996); Z. Hradil, H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Phys. Lett. **A239**, 333 (1998).

[14] M.V. Berry and S. Klein, J. Mod. Opt. **43**, 165 (1996).

[15] R.P. Feynman and G. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[16] A similar situation was considered by Yamada and Takagi in the attempt to define the passage time through a barrier. See N. Yamada and S. Takagi, Prog. Theor. Phys. **85**, 985 (1991); **86**, 599 (1991); **87**, 77 (1992); N. Yamada, Phys. Rev. **A54**, 182 (1996).

[17] See for example, L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley, New York, 1981) and references therein.

[18] It is only at this stage that we must require that A is a well-behaved set like, for example, an interval of $\mathbb{R}$ [see our starting point (3.3)]. For instance, the potential (3.10) would be physically meaningless for the Cantor set in $[0,1]$.

[19] F. Riesz and B.S. Nagy, *Functional Analysis* (Dover, New York, 1990), p. 330; M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975), p. 176.