Entanglement structure of adjoint representation of unitary group and tomography of quantum states

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Submitted in English 1 August 2003

Abstract

The density matrix of composite spin system is discussed in relation to the adjoint representation of unitary group $U(n)$. The entanglement structure is introduced as an additional ingredient to the description of the linear space carrying the adjoint representation. Positive maps of density operator are related to random matrices. The tomographic probability description of quantum states is used to formulate the problem of separability and entanglement as the condition for joint probability distribution of several random variables represented as the convex sum of products of probabilities of random variables describing the subsystems. The property is discussed as a possible criterion for separability or entanglement. The convenient criterion of positivity of finite and infinite matrix is obtained. The $U(n)$-tomogram of a multiparticle spin state is introduced. The entanglement measure is considered in terms of this tomogram.

KEY WORDS: unitary group, entanglement, adjoint representation, tomogram, operator symbol, random matrix.

1 Introductiion

The notion of entanglement [1] is related to the quantum composition principle of the states of subsystems for a given multipartite system. For pure states, the notion of entanglement and separability can be given as follows.

If the wave function of a state of a bipartite system is represented as the product of two wave functions depending on coordinates of the subsystems, the state is simply separable; correspondingly, in other cases, the state is entangled. An intrinsic approach to the entanglement measure was suggested

in [2]. The measure was introduced as the distance between the system density matrix and the tensor product of the subsystem states. There are several other different characteristics and measures of entanglement considered by several authors [3–9]. Each of the entanglement measures describes a degree of correlations between the subsystems' properties. The notion of entanglement is not an absolute notion for a given system but depends on the decomposition into subsystems. The same quantum state can be considered as entangled, if one kind of division of the system into subsystems is given, or as completely disentangled, if another decomposition of the system into subsystems is considered.

For instance, the state of two continuous quadratures can be entangled in Cartesian coordinates and disentangled in polar coordinates. Coordinates are considered as measurable observables labeling the subsystems of the given system. The choice of different subsystems mathematically implies the existence of two different sets of the subsystems' characteristics (we focus on bipartite case). We may consider the Hilbert space of states $H(1,2)$ or $H(1',2')$. The Hilbert space for the total system is, of course, the same but the index $(1,2)$ means that there are two sets of operators P_1 and P_2, which select subsystem states 1 and 2. The index $(1',2')$ means that there are other two sets of operators P_1' and P_2', which select subsystem states $1'$ and $2'$. The operators $P_{1,2}$ and $P_{1',2'}'$ have specific properties. They are represented as tensor products of operators acting in the space of states of the subsystem 1 (or 2) and unit operators acting in the subsystem 2 (or 1). In other words, we consider the space H, which can be treated as tensor product of spaces $H(1)$ and $H(2)$ or $H(1')$ and $H(2')$. In the subsystems 1 and 2, there are basis vectors $\mid n_1\rangle$ and $\mid m_2\rangle$, as well as in the subsystems $1'$ and $2'$ there are basis vectors $\mid n_1'\rangle$ and $\mid m_2'\rangle$. The vectors $\mid n_1\rangle \mid m_2\rangle$ and the vectors $\mid n_1'\rangle \mid m_2'\rangle$ form the sets of basis vectors in the composite Hilbert space, respectively. These two sets are related by means of unitary transformation. An example of such a composite system is a bipartite spin system.

If one has spin-j_1 [the space $H(1)$] and spin-j_2 [the space $H(2)$] systems, the combined system can be treated as having basis $\mid j_1m_1\rangle \mid j_2m_2\rangle$.

Another basis in the composite-system-state space can be considered in the form $\mid jm\rangle$, where j is one of the numbers $|j_1 - j_2|$, $|j_1 - j_2| + 1, \ldots,$ $j_1 + j_2$ and $m = m_1 + m_2$. The basis $\mid jm\rangle$ is related to the basis $\mid j_1m_1\rangle \mid j_2m_2\rangle$ by means of unitary transform given by Clebsch–Gordon coefficients $C\ (j_1m_1j_2m_2|jm)$. From the viewpoint of given definition, the states $\mid jm\rangle$ are entangled states. For example, if $j_1 = j_2 = 1/2$, there are entangled spin

states of the composite system, which nowadays are called Bell states

$$| \Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}\frac{1}{2} \right\rangle_1 \left| \frac{1}{2}\frac{1}{2} \right\rangle_2 \pm \left| \frac{1}{2}\frac{-1}{2} \right\rangle_1 \left| \frac{1}{2}\frac{-1}{2} \right\rangle_2 \right),$$

$$|\Psi^{\pm}\rangle = -\frac{1}{\sqrt{2}} \left(\left| \frac{1}{2}\frac{1}{2} \right\rangle_1 \left| \frac{1}{2}\frac{-1}{2} \right\rangle_2 \pm \left| \frac{1}{2}\frac{-1}{2} \right\rangle_1 \left| \frac{1}{2}\frac{1}{2} \right\rangle_2 \right).$$

These states are maximally entangled states. In terms of spin, the states $| \Phi^{\pm}\rangle$ are the superpositions of $j = 1$, $m = \pm 1$ states and the states $| \Psi^{\pm}\rangle$ are the superpositions of $j = 0, 1$, $m = 0$ states.

The spin states can be described by means of the tomographic map [10–12].For bipartite spin systems, the states were described by the tomographic probabilities in [13, 14]. Some properties of the tomographic spin description were studied in [15]. In the tomographic approach, the problems of the quantum state entanglement can be cast into the form of some relations among the probability distribution functions. On the other hand, to have a clear picture of entanglement, one needs mathematical formulation of properties of the density matrix of the composite system, a description of the linear space of the composite system states. Since the density matrix is hermitian, the space of states is a subset of linear space of adjoint representation of the group $U(n^2)$, where $n = 2j + 1$ is the dimension of the spin states of two spinning particles. Thus one needs to characterize the connection of the entanglement phenomena with the structures in the space of adjoint representation of the $U(n^2)$ group.

The aim of this paper is to connect entanglement problems with the properties of tomographic probability distributions and discuss the properties of the convex set of positive states for composite system with taking into account the subsystem structures. We used Hilbert–Schmidt distance to calculate the measure of entanglement as the distance between a given state and the tensor product of the partial traces of the density matrix of the given state. In [16] another measure of entanglement as a characteristic of subsystem correlations was introduced. This measure is determined via covariance matrix of some observables. Review of different approaches to the entanglement notion and entanglement measures is given in [17], where the approach to describe entanglement and separability of composite systems is based on entropy methods.

Due to variety of approaches to the entanglement problem, one needs to understand better what in reality this word ‘entanglement’ describes. Is

it a synonym of the word 'correlation' between two subsystems or it has to capture some specific correlations attributed completely and only to the quantum domain?

The paper is organized as follows.

In Sec. 2 we study the subsystem structure of a given linear space. In Sec. 3 we consider the relation of the group $U(n^2)$ to the set of density matrices. In Sec. 4 we discuss positive maps. In Sec. 5 we investigate local transforms. In Sec. 6 we treat the probability distributions as vectors. In. Sec. 7 we prove the invariance of the intrinsic entanglement measure. In Sec. 8 we define the separable states. In Sec. 9 generic symbols of operators are presented. In Sec. 10 an example of Weyl symbols is considered and in Sec. 11 an example of quadrature tomogram is done. In Sec. 12 symbols of density operators for multipartite system are discussed. Spin tomography is reviewed in Sec. 13. Two qubits are considered in the tomographic representation in Sec. 14. The relation of dynamical map to purification procedure is described in Sec. 15. Some properties of quadratic forms are reviewed in Sec. 16. The tomogram for the group $U(N)$ is introduced in Sec. 17. Conclusions and results are listed in Sec. 18.

2 Linear Space of a Composite System, Its Structure, and Its Convex Subset of Positive States

In this section, we review the meaning and notion of composite system in terms of additional structures on the linear space of state for the composite system.

2.1 States and Observables

In quantum mechanics, there are two principal ingredients, which are associated with linear operators acting in a Hilbert space. The first ingredient is related to the concept of quantum state and the second one, to the concept of observable. The state is associated to Hermitian nonnegative, trace-class, linear operator. The observables are associated to Hermitian operators. Though the both states and observables are identified with the Hermitian operators, there is an essential difference between these two objects. The observables

have additional product structure. Thus we consider product of two linear Hermitian operators corresponding to the observables. First measuring an observable and (after measuring the first one) measuring another observable just correspond to the product of two operators.

For the states, the notion of product is redundant. The product of two states is not a state. For states, one keeps only the linear structure of vector space. For finite n-dimensional system, the Hermitian states and the Hermitian observables live in Lie algebra of the unitary group $U(n)$. But the states correspond to nonnegative Hermitian operator. The observables can be associated with both types of the operators including nonnegative and nonpositive ones. Space of states is linear space which, in principle, is not equipped by a product structure. Due to this, if one considers transformations in linear space of states, one does not need to preserve any product structure. In the set of observables, one needs to care what is happening with product of operators provided some transformations are applied.

2.2 Vectors

Let us first introduce some extra constructions of the map of a matrix onto a vector. Given a rectangular matrix M with elements M_{id}, where $i = 1, 2, \ldots, n$ and $d = 1, 2, \ldots, m$. Then one can consider the matrix as a vector $\vec{\mathcal{M}}$ with $N = nm$ components constructed by the following rule:

$$\mathcal{M}_1 = M_{11}, \quad \mathcal{M}_2 = M_{12}, \quad \mathcal{M}_m = M_{1m}, \quad \mathcal{M}_{m+1} = M_{21}, \ldots \mathcal{M}_N = M_{nm}. \tag{1}$$

Thus we construct the map $M \to \vec{\mathcal{M}} = \hat{t}_{\vec{\mathcal{M}}M} M$.

We have introduced the linear operator $\hat{t}_{\vec{\mathcal{M}}M}$ which maps the matrix M on a vector $\vec{\mathcal{M}}$. Now we introduce the inverse operator $\hat{p}_{\vec{\mathcal{M}}M}$ which maps a given vector column in the space with dimension $N = mn$ onto the rectangular matrix. This means that given a vector $\vec{\mathcal{M}} = \mathcal{M}_1, \ldots, \mathcal{M}_N$, we use the rule of relabeling the components of the vector introducing two indices $i = 1, \ldots, n$ and $d = 1, \ldots, m$. The relabeling is accomplished according to (1). Then we collect the relabeled components into matrix table. Thus we get the map

$$\hat{p}_{\vec{\mathcal{M}}M} \vec{\mathcal{M}} = M. \tag{2}$$

One can see that their composition

$$\hat{t}_{\vec{\mathcal{M}}M} \hat{p}_{\vec{\mathcal{M}}M} \vec{\mathcal{M}} = 1 \cdot \vec{\mathcal{M}} \tag{3}$$

acts on the vector as unit operator in the linear space of vectors.

Given a $n\times n$ matrix the map suggested can also be extended. The matrix can be treated as n^2-dimensional vector and, vice versa, the vector of dimension n^2 can be mapped by this procedure onto the $n\times n$ matrix.

Let us consider a linear operator acting on the vector $\vec{\mathcal{M}}$ and related to a linear transform of the matrix M. First, we study the correspondence of the linear transform of the form

$$M \to gM = M^l_g \tag{4}$$

to the transform of the vector

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}^l_g = \mathcal{L}^l_g \vec{\mathcal{M}}. \tag{5}$$

One can show that the $n^2\times n^2$ matrix $\mathcal{L}^l_g$ is determined by the tensor product of the $n\times n$ matrix g and $n\times n$ unit matrix, i.e.,

$$\mathcal{L}^l_g = g \otimes 1. \tag{6}$$

Analogously, the linear transform of the matrix M of the form

$$M \to Mg = M^r_g \tag{7}$$

induces the linear transform of the vector $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}^r_g = \hat{t}_{\vec{\mathcal{M}}\mathcal{M}} M^r_g = \mathcal{L}^r_g \vec{\mathcal{M}}, \tag{8}$$

where the $n^2\times n^2$ matrix $\mathcal{L}^r_g$ reads

$$\mathcal{L}^r_g = 1 \otimes g^{\mathrm{tr}}. \tag{9}$$

Similarity transformation of the matrix M of the form

$$M \to gMg^{-1} \tag{10}$$

induces the corresponding linear transform of the vector $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}} \to \vec{\mathcal{M}}_s = \mathcal{L}^s_g \vec{\mathcal{M}}, \tag{11}$$

where the $n^2\times n^2$ matrix $\mathcal{L}^s_g$ reads

$$\mathcal{L}^s_g = g \otimes (g^{-1})^{\mathrm{tr}}. \tag{12}$$

One can ask how to determine the inverse map of vector $\vec{\mathcal{M}}$ onto matrix M, i.e., how to define the operator $\hat{p}_{\vec{\mathcal{M}}M}$. In fact, the reconstruction can be defined by means of star-product of vectors $\vec{\mathcal{M}}$ in a linear space. One can define the associative product of two N-vectors $\vec{\mathcal{M}}_1$ and $\vec{\mathcal{M}}_2$ using the rule

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}_1 \star \vec{\mathcal{M}}_2, \tag{13}$$

where

$$\vec{\mathcal{M}}_k = \sum_{l,s=1}^{N} K_{ls}^{k} (\vec{\mathcal{M}}_1)_l (\vec{\mathcal{M}}_2)_s. \tag{14}$$

If one applies a linear transform to the vectors $\vec{\mathcal{M}}_1$, $\vec{\mathcal{M}}_2$, $\vec{\mathcal{M}}$ of the form

$$\vec{\mathcal{M}}_1 \to \vec{\mathcal{M}}_1' = \mathcal{L}\vec{\mathcal{M}}_1, \qquad \vec{\mathcal{M}}_2 \to \vec{\mathcal{M}}_2' = \mathcal{L}\vec{\mathcal{M}}_2, \qquad \vec{\mathcal{M}} \to \vec{\mathcal{M}}' = \mathcal{L}\vec{\mathcal{M}},$$

the invariance of the star-product kernel yields

$$\vec{\mathcal{M}}_1' \star \vec{\mathcal{M}}_2' = \vec{\mathcal{M}}', \qquad \text{if} \qquad \mathcal{L} = G \otimes G^{-1\mathrm{tr}}, \quad G \in GL(n).$$

The kernel K_{ls}^{k} (structure constants) which determines the associative star-product satisfies the quadratic equation. Thus if one wants to make the correspondence of the vector star-product to the standard matrix product (row by column), the matrix M must be constructed appropriately. For example, if the vector star-product is commutative, the matrix M corresponding to the N-vector $\vec{\mathcal{M}}$ can be chosen as diagonal $N\times N$ matrix. This consideration shows that the map of matrices on the vectors provides star-product of the vectors (defines the structure constants or the kernel of star-product) and, conversely, if one has the vectors, the map of the vectors onto the matrices with the standard multiplication rule is determined by the structure constants (or by the kernel of the vector star-product).

The constructed map of matrices on the vectors gives a possibility to enlarge the dimensionality of the group acting in the linear space of matrices in comparison with the standard one. Thus, given a $n\times n$ matrix M the left action, the right action, and similarity transformation of the matrix are related to the complex group $GL(n)$. On the other hand, the linear transformations in the linear space of n^2-vectors $\vec{\mathcal{M}}$ obtained by using the introduced map are determined by the matrices belonging to the group $GL(n^2)$. There are transformations on the vectors which cannot be simply represented on matrices. If $M \to \Phi(M)$ is a linear homogeneous function of the matrix M, we may represent it by

$$\Phi_{ab} = B_{aa',\,bb'} M_{a'b'}.$$

Under rather clear conditions, $B_{aa',bb'}$ can be expressed in terms of its non-normalized left and right eigenvectors:

$$B_{aa',bb'} = \sum_{\nu} x_{aa'}(\nu) y^{\dagger}_{bb'}(\nu),$$

which corresponds to

$$\Phi(M) = xMy^{\dagger} = \sum_{\nu=1}^{n^2} x(\nu) M y^{\dagger}(\nu).$$

There are possible linear transforms on the matrices and corresponding linear transforms on the induced vector space which belong not to a group but to an algebra of matrices. One can describe the map of $n \times n$ matrices M (source space) onto vectors $\vec{\mathcal{M}}$ (target space) using specific basis in the space of the matrices. The basis is given by the matrices E_{jk} $(j,k = 1,2,\ldots,n)$ with all matrix elements equal to zero except the element in jth row and kth column which is equal to unity. One has the obvious property

$$M_{jk} = \mathrm{Tr}\left(ME_{jk}\right). \tag{15}$$

In our procedure, the basis matrix E_{jk} is mapped onto the basis column-vector $\vec{\mathcal{E}}_{jk}$, which has all components equal to zero except the unity component related to the position in the matrix determined by the numbers j and k. Then one has

$$\vec{\mathcal{M}} = \sum_{j,k=1}^{n} \mathrm{Tr}\left(ME_{jk}\right) \vec{\mathcal{E}}_{jk}. \tag{16}$$

For example, for similarity transformation of the finite matrix M, one has

$$\vec{\mathcal{M}}^{s}_{g} = \sum_{j,k=1}^{N} \mathrm{Tr}\left(gMg^{-1}E_{jk}\right) \vec{\mathcal{E}}_{jk}. \tag{17}$$

Now we will define the notion of 'composite' vector which corresponds to dividing a quantum system into subsystems.

We will use the following terminology.

In general, the given linear space of dimensionality $N = mn$ has a structure of bipartite system, if the space is equipped with the operator $\hat{p}_{\vec{\mathcal{M}}M}$ and the matrix M (obtained by means of the map) has matrix elements in factorizable form

$$M_{id} \to x_i y_d. \tag{18}$$

This $M = x \otimes y$ corresponds to the special case of nonentangled states. Otherwise, one needs

$$M = \sum_{\nu} x(\nu) \otimes y(\nu).$$

In fact, to consider in detail the entanglement phenomenon, in the bipartite system of spin, one has to introduce a hierarchy of three linear spaces. The first space of pure spin states is two-dimensional linear space of complex vectors

$$\mid \vec{x}\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{19}$$

In this space, the scalar product is defined as follows:

$$\langle \vec{x} \mid \vec{y}\rangle = x_1^* y_1 + x_2^* y_2. \tag{20}$$

So it is two-dimensional Hilbert space. We do not equip this space with a vector star-product structure. In the primary linear space, one introduces linear operators $\hat{M}$ which are described by 2×2 matrices M. Due to the map discussed in the previous section, the matrices are represented by 4-vectors $\vec{\mathcal{M}}$ belonging to the second complex 4-dimensional space. Star-product of the vectors $\vec{\mathcal{M}}$ determined by the kernel $\mathcal{K}_{ls}^{k}$ is defined in such a manner in order to correspond to the standard rule of multiplication of the matrices.

In addition to the star-product structure, we introduce the scalar product of the vectors $\vec{\mathcal{M}}_1$ and $\vec{\mathcal{M}}_2$, in view of the definition

$$\langle \vec{\mathcal{M}}_1 \mid \vec{\mathcal{M}}_2\rangle = \mathrm{Tr}\,(M_1^{\dagger} M_2), \tag{21}$$

which is the trace formula for scalar product of matrices.

This means introducing the metric $g^{\alpha\beta}$ in the standard notation for scalar product

$$\langle \vec{\mathcal{M}}_1 \mid \vec{\mathcal{M}}_2\rangle = \sum_{\alpha,\beta=1}^{4} (M_1)_{\alpha}^{*} g^{\alpha\beta} (M_2)_{\beta}, \tag{22}$$

where the matrix $g^{\alpha\beta}$ is of the form

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{\alpha j} g^{j\beta} = \delta^{\alpha\beta}. \tag{23}$$

The scalar product is invariant under action of the group of nonsingular 4×4 matrices ℓ, which satisfy the condition

$$\ell^{-1} = g\ell^{\dagger}g. \tag{24}$$

The product of matrices ℓ satisfies the same condition since $g^2 = 1$.

Thus, the space of operators $\hat{M}$ in primary two-dimensional space of spin states is mapped onto linear space which is equipped with a scalar product (metric Hilbert space structure) and associative star-product (kernel satisfying quadratic associativity equation). In the linear space of the 4-vectors $\vec{\mathcal{M}}$, we introduce linear operators (superoperators), which can be associated with algebra of 4×4 complex matrices.

3 Density Operators and Positive Maps

In this section, we focus on density matrices. This means that our matrix M is considered as a density matrix ρ which describes a quantum state. We consider here the action of the unitary transformation $U(n)$ of the density matrices and corresponding transformations on the vector space. If one has structure of bipartite system, we also consider the action of local gauge transformation both in the 'source space' of density matrices and in the 'target space' of the corresponding vectors.

The $n\times n$ density matrix ρ has matrix elements

$$\rho_{ik} = \rho^{\dagger}_{ki}, \qquad \mathrm{Tr}\,\rho = 1, \qquad \langle\psi|\rho|\psi\rangle \geq 0. \tag{25}$$

Since the density matrix is hermitian, it can be always identified as an element of the convex subset of the linear space associated with the Lie algebra of $U(n)$ group, on which the group $U(n)$ acts with the adjoint representation

$$\rho \to \rho_U = U\rho U^{\dagger}. \tag{26}$$

The system is said to be bipartite, if the space of representation is equipped with an additional structure. It means that for

$$n^2 = n_1 \cdot n_2, \qquad n_1 = n_2 = n$$

one can make first the map of $n\times n$ matrix ρ onto n^2-dimensional vector $\vec{\rho}$ according to the previous procedure, i.e., one equips the space by an operator

$\hat{t}_{\vec{\rho}\rho}$. Given this vector one makes a relabeling of the vector $\vec{\rho}$ components according to the rule

$$\vec{\rho} \to \rho_{id,ke}, \quad i,k = 1,2,\ldots,n_1, \quad d,e = 1,2,\ldots,n_2, \tag{27}$$

i.e., obtaining the quadratic matrix

$$\rho_q = \hat{p}_{\rho_q\vec{\rho}}\vec{\rho}. \tag{28}$$

The unitary transform (26) of the density matrix induces the linear transform of the vector $\vec{\rho}$ of the form

$$\vec{\rho} \to \vec{\rho}_U = (U \otimes U^*)\vec{\rho}. \tag{29}$$

There exist linear transforms (called positive maps) of the density matrix, which preserve its trace, hermicity, and positivity. It is the transform introduced in [18]

$$\rho_0 \to \rho_s = \sum_k p_k U_k \rho_0 U_k^\dagger, \quad \sum_k p_k = 1, \tag{30}$$

where U_k are unitary matrices and p_k are positive numbers.

If the initial density matrix is diagonal, i.e., it belongs to Cartan subalgebra of Lie algebra of the unitary group, the diagonal elements of the obtained matrix give smoother probability distribution than the initial one. There exists the generic transform (see [18, 19])

$$\rho_0 \to \rho = \sum_k V_k \rho_0 V_k^\dagger, \quad \sum_k V_k^\dagger V_k = 1. \tag{31}$$

For large number of terms in the sum, the above map gives the most stochastic density matrix

$$\rho_0 \to \rho_s = (n)^{-1}1.$$

The transform (30) is the partial case of the transform (31). We discuss the transforms separately since they are used in the literature in the presented form.

One can see that the constructed map of density matrices onto vectors provides the corresponding transforms of the vectors, i.e.,

$$\vec{\rho}_0 \to \vec{\rho}_s = \sum_k p_k (U_k \otimes U_k^*)\vec{\rho}_0 \tag{32}$$

and

$$\vec{\rho}_0 \to \vec{\rho} = \sum_k (V_k \otimes V_k^*)\vec{\rho}_0. \tag{33}$$

It is obvious that the linear transforms of the vectors, which preserve their properties to correspond to the density matrix, are essentially larger than the standard unitary transform of the density matrix.

Formulas (32) and (33) mean that the positive map superoperators acting on the density matrix in the vector representation are described by $n^2 \times n^2$ matrices

$$\mathcal{L}_s = \sum_k p_k (U_k \otimes U_k^*) \tag{34}$$

and

$$\mathcal{L} = \sum_k V_k \otimes V_k^*, \tag{35}$$

respectively.

Positive map is called 'noncompletely positive' if

$$\mathcal{L} = \sum_k V_k \otimes V_k^* - \sum_s v_s \otimes v_s^*, \qquad \sum_k V_k^\dagger V_k - \sum_s v_s^\dagger v_s = 1.$$

This map is related to nonphysical evolution of a subsystem.

4 Positive Map and Random Matrices

Formula (34) can be considered in the context of random matrix representation. In fact, the matrix $\mathcal{L}_s$ can be interpreted as the weighted mean value of the random matrix $U_k \otimes U_k^*$. The dependence of matrix elements and positive numbers p_k on index k means that we have a probability distribution function p_k and averaging of the random matrix $U_k \otimes U_k^*$ by means of the distribution function. So the matrix $\mathcal{L}_s$ reads

$$\mathcal{L}_s = \langle U \otimes U^* \rangle. \tag{36}$$

Let us consider example of 2×2 unitary matrix. We can consider the matrix of $SU(2)$ group of the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{37}$$

The 4×4 matrix $\mathcal{L}_s$ takes the form

$$\mathcal{L}_s = \begin{pmatrix} \ell & m & m^* & 1-\ell \\ -n & s & -q & n \\ -n^* & -q^* & s^* & n^* \\ 1-\ell & -m & -m^* & \ell \end{pmatrix}. \tag{38}$$

The matrix elements of the matrix $\mathcal{L}_s$ are the means

$$\begin{aligned} m &= \langle \alpha\beta^* \rangle, \\ \ell &= \langle \alpha\alpha^* \rangle, \\ n &= \langle \alpha\beta \rangle, \\ s &= \langle \alpha^2 \rangle, \\ q &= \langle \beta^2 \rangle. \end{aligned} \tag{39}$$

Moduli of these matrix elements are smaller than unity.

Determinant of the matrix $\mathcal{L}_s$ reads

$$\det \mathcal{L}_s = (1-2\ell)\Big(|q|^2 - |s|^2\Big) + 4\,\mathrm{Re}\Big[q^* m^* n + mns^*\Big]. \tag{40}$$

If one represents the matrix $\mathcal{L}_s$ in block form

$$\mathcal{L}_s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{41}$$

then

$$A = \begin{pmatrix} \ell & m \\ -n & s \end{pmatrix}, \qquad B = \begin{pmatrix} m^* & 1-\ell \\ -q & n \end{pmatrix}, \tag{42}$$

and

$$D = \sigma_2 A^* \sigma_2, \qquad C = -\sigma_2 B^* \sigma_2, \tag{43}$$

where σ_2 is Pauli matrix.

One can check that the product of two different matrices $\mathcal{L}_s$ can be cast in the same form. This means that the matrices $\mathcal{L}_s$ form the 9-parameter compact semigroup. For example, in the case $\ell = 1/2$ and $m = 0$, one has the matrices

$$A = \begin{pmatrix} 1/2 & 0 \\ -n & s \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1/2 \\ -q & n \end{pmatrix}. \tag{44}$$

Determinant of the matrix $\mathcal{L}_s$ in this case is equal to zero. All the matrices $\mathcal{L}_s$ have the eigenvector

$$\vec{\rho}_0 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}, \tag{45}$$

i.e.,

$$\mathcal{L}_s \vec{\rho}_0 = \vec{\rho}_0. \tag{46}$$

This eigenvector corresponds to the density matrix

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \tag{47}$$

which is obviously invariant of the positive map.

For random matrix, one has correlations of the random matrix elements, e.g., $\langle \alpha \alpha^* \rangle \neq \langle \alpha \rangle \langle \alpha^* \rangle$.

The matrix $\mathcal{L}_p$

$$\mathcal{L}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

maps the vector

$$\rho_{\rm in} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} \tag{49}$$

onto the vector

$$\vec{\rho}_{\rm t} = \begin{pmatrix} \rho_{11} \\ \rho_{21} \\ \rho_{12} \\ \rho_{22} \end{pmatrix}. \tag{50}$$

This means that the positive map (48) connects the positive density matrix with its transposed. This map can be presented as the connection of the matrix ρ with its transposed of the form

$$\rho \to \rho^T = \rho^* = \frac{1}{2}\Big(\rho + \sigma_1 \rho \sigma_1 - \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3\Big).$$

There is no unitary transform connecting these matrices.

This noncompletely positive map in N-dimensional case is given by generalized formula

$$\rho \to \rho_s = -\varepsilon\rho + \frac{1+\varepsilon}{N}\,1_N, \qquad \varepsilon > 0.$$

The standard unitary transform can be interpreted as average random transform with probability distribution

$$p_k = \delta(k), \tag{51}$$

where $\delta(k)$ is either Kronecker symbol δ_{k0} for discrete index k or Dirac delta-function for continuous index k.

For standard unitary transform, one cannot find the matrix U satisfying the equation

$$U \otimes U^* = \mathcal{L}_p. \tag{52}$$

But if one makes averaging with generic distribution function [not with probability distribution (51)], the equation

$$\langle U \otimes U^* \rangle = \mathcal{L}_p \tag{53}$$

has the solution.

The standard unitary transform of density matrix is 3-parameter subset of this 9-parameter semigroup.

Thus we constructed matrix representation of positive map of density operators of spin-1/2 system. To construct this representation, one needs to use the map of matrices on the vectors discussed in the previous section. Formulas (31) and (35) can be interpreted also in the context of random matrix representation, but we use the uniform distribution for averaging in this case. So one has equality (35) in the form

$$\mathcal{L} = \langle V \otimes V^* \rangle \tag{54}$$

and the equality

$$\langle V^\dagger V \rangle = 1, \tag{55}$$

which provides constrains for used random matrices V.

Using random matrix formalism, the positive (but not completely positive) maps can be presented in the form

$$\mathcal{L} = \langle V \otimes V^* \rangle - \langle v \otimes v^* \rangle, \qquad \langle V^\dagger V \rangle - \langle v^\dagger v \rangle = 1.$$

In [18] the positive maps (30) and (31) were used to describe non-Hamiltonian evolution of quantum states for open systems. If the map corresponds to an extended Hamiltonian evolution, the leading terms are of order t^2 (Zeno effect) and consequently, in the $\lambda^2 t$-approximation of Prigogine and van Hove, one can derive rate equations from macroscopic equations of motion.

We have to point out that, in general, such evolution is not described by first-order-in-time partial differential equation. Like in the previous case, if there are added structures of the matrix in the form

$$\rho_{id,ke} \to x_i y_d z_k t_e, \tag{56}$$

which means association with the initial linear space two extra linear spaces in which x_i, z_k are considered as vector components in the n_1-dimensional linear space and y_d, t_e are vector components in n_2-dimensional vector space, we will tell that one has bipartite structure of the initial space of state [bipartite structure of the space of adjoint representation of the group $U(n)$]. Usually the adjoint representation of any group is defined per se without any reference to possible substructures. Here we introduce the space with extra structure. In addition to be space of adjoint representation of the group $U(n)$, it has structure of bipartite system. The generalization to multipartite (N-partite) structure is straightforward. One needs only the representation of positive integer n^2 in the form

$$n^2 = \prod_{k=1}^{N} n_k^2. \tag{57}$$

If one considers more general map given by superoperator (35) rewritten in the form

$$\mathcal{L} = \langle V \otimes V^* \rangle, \qquad \langle V^\dagger V \rangle = 1,$$

the number of parameters determining the matrix $\mathcal{L}$ can be easily evaluated. For example, if

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad V^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix},$$

where matrix elements are the complex numbers, the normalization condition provides 4 constrains for the real and imaginary parts of matrix elements of

the following matrix:

$$\mathcal{L} = \begin{pmatrix} \langle |a|^2 \rangle & \langle ab^* \rangle & \langle ba^* \rangle & \langle bb^* \rangle \\ \langle ac^* \rangle & \langle aa^* \rangle & \langle bc^* \rangle & \langle bd^* \rangle \\ \langle ca^* \rangle & \langle cb^* \rangle & \langle da^* \rangle & \langle db^* \rangle \\ \langle cc^* \rangle & \langle cd^* \rangle & \langle dc^* \rangle & \langle dd^* \rangle \end{pmatrix},$$

namely,

$$\langle |a|^2 \rangle + \langle |b|^2 \rangle = 1, \qquad \langle |c|^2 \rangle + \langle |d|^2 \rangle = 1, \qquad \langle a^* c \rangle + \langle a^* d \rangle = 0.$$

Due to structure of the matrix $\mathcal{L}$, there are 6 complex parameters

$$\langle ab^* \rangle, \quad \langle ac^* \rangle, \quad \langle ad^* \rangle, \quad \langle bc^* \rangle, \quad \langle bd^* \rangle, \quad \langle cd^* \rangle$$

or 12 real parameters.

Geometrical picture of positive map can be clarified if one considers transform of the positive density matrix onto another density matrix as transform of ellipsoid into another ellipsoid. The generic positive transform means a generic transform of the ellipsoid, which changes its orientation, values of semiaxis, and position in the space. But the transform is not making from the ellipsoid the surface like hyperboloid or paraboloid. For pure states, the positive density matrix defines the quadratic form which is maximally degenerated. In this sense, we say "ellipsoid" also including all its degenerate forms corresponding to density matrix of ranks less than n (in n-dimensional case). The number of parameters defining the map $\langle V \otimes V^* \rangle$ in n-dimensional case is equal to $n^2(n^2-1)$.

5 Local and Nonlocal Transforms

In this section, we discuss the transforms of the density operators of bipartite system. We concentrate on the case of two spin-1/2 systems. Let the first system be in the state with 2×2 density matrix

$$\rho_A = \begin{pmatrix} \rho_{11}^A & \rho_{12}^A \\ \rho_{21}^A & \rho_{22}^A \end{pmatrix} \tag{58}$$

and the second system is in the state with 2×2 density matrix

$$\rho_B = \begin{pmatrix} \rho_{11}^B & \rho_{12}^B \\ \rho_{21}^B & \rho_{22}^B \end{pmatrix}. \tag{59}$$

According to the suggested map, we associate with the state density matrices (58) and (59) the 4-vectors with components

$$\vec{\rho}_A = \begin{pmatrix} \rho^A_{11} \\ \rho^A_{12} \\ \rho^A_{21} \\ \rho^A_{22} \end{pmatrix}, \qquad \vec{\rho}_B = \begin{pmatrix} \rho^B_{11} \\ \rho^B_{12} \\ \rho^B_{21} \\ \rho^B_{22} \end{pmatrix}. \tag{60}$$

The density matrices (58) and (59) [i.e., vectors (60)] belong to linear spaces of adjoint representations of groups $U_A(2)$ and $U_B(2)$, respectively.

For the product state (simply separable state) of composite system with 4×4 density matrix

$$\rho_{AB} = \rho_A \otimes \rho_B, \tag{61}$$

the corresponding 16-vector associated to 4×4 density matrix

$$\rho_{AB} = \begin{pmatrix} \rho^A_{11}\rho^B_{11} & \rho^A_{11}\rho^B_{12} & \rho^A_{12}\rho^B_{11} & \rho^A_{12}\rho^B_{12} \\ \rho^A_{11}\rho^B_{21} & \rho^A_{11}\rho^B_{22} & \rho^A_{12}\rho^B_{21} & \rho^A_{12}\rho^B_{22} \\ \rho^A_{21}\rho^B_{11} & \rho^A_{21}\rho^B_{12} & \rho^A_{22}\rho^B_{11} & \rho^A_{22}\rho^B_{12} \\ \rho^A_{21}\rho^B_{21} & \rho^A_{21}\rho^B_{22} & \rho^A_{22}\rho^B_{21} & \rho^A_{22}\rho^B_{22} \end{pmatrix} \tag{62}$$

has the form

$$\vec{\rho}_{AB} = C\Big(\vec{\rho}_A \otimes \vec{\rho}_B\Big), \qquad \vec{\rho}_A \otimes \vec{\rho}_B = C\vec{\rho}_{AB}. \tag{63}$$

Here the 16×16 matrix C acting on the 16-component vector $\vec{\rho}_A \otimes \vec{\rho}_B$, which is standard tensor-product of two vectors, has the form

$$C = \begin{pmatrix} 1_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_2 \end{pmatrix}. \tag{64}$$

The matrix C consists of 2×2-block zero and unity matrices.

The linear space of Hermitian matrices is equipped also by commutator structure defining Lie algebra of the group $U(n)$. The kernel, which defines

this structure (Lie product structure) is determined by the kernel, which determines star-product.

In the space of 16-vectors, one defines the scalar product as follows:

$$\vec{\rho}_1 \cdot \vec{\rho}_2 = \sum_{\alpha,\beta=1}^{16} \rho^*_{1\alpha} C_{\alpha\beta} \rho_{2\beta}. \tag{65}$$

The product is invariant, in view of linear transform (group transform)

$$\vec{\rho}_1 \to \vec{\rho}'_1 = L\vec{\rho}_1 \qquad \vec{\rho}_2 \to \vec{\rho}'_2 = L\vec{\rho}_2, \tag{66}$$

if the 16×16 matrix L satisfies the condition

$$L^{-1} = CL^{\dagger}C. \tag{67}$$

Vector (63) belongs to linear space of adjoint representation of the unitary group $U(4)$ and star-product of the vectors is identified with the associative algebra generated by the Lie algebra. The local gauge transformations are defined as tensor product of independent unitary transforms of the density matrices ρ_A (58) and ρ_B (59). These transformations are described by the group $U(2)\times U(2)$. In the space of vectors $\vec{\rho}_A$ and $\vec{\rho}_B$, the local transform superoperators have the matrix form

$$\hat{U}_A(2)\vec{\rho}_A = \Big(U_A(2) \otimes U^*_A(2)\Big)\vec{\rho}_A \tag{68}$$

and

$$\hat{U}_B(2)\vec{\rho}_B = \Big(U_B(2) \otimes U^*_B(2)\Big)\vec{\rho}_B. \tag{69}$$

In the 16-dimensional space of vectors $\vec{\rho}_{AB}$, the local transforms are described by the superoperator

$$\vec{\rho}_{AB} \to \vec{\rho}^{\,\mathrm{loc}}_{AB} = \mathcal{L}^{\mathrm{loc}}\vec{\rho}_{AB}, \tag{70}$$

with the matrix

$$\mathcal{L}^{\mathrm{loc}} = C\Big(U_A(2) \otimes U^*_A(2)\Big) \otimes \Big(U_B(2) \otimes U^*_B(2)\Big)C. \tag{71}$$

The local positive map transforms induce in the space of adjoint representation of the group $U(4)$ the transform of the vectors $\vec{\rho}_{AB}$ associated to the matrices

$$\mathcal{L}_{pU} = C\Big(\sum_k p_k U^A_k \otimes U^{A*}_k\Big) \otimes \Big(\sum_{k'} \omega_{k'} U^B_{k'} \otimes U^{B*}_{k'}\Big)C \tag{72}$$

and

$$\mathcal{L}_{pV} = C\Big(\sum_k V_k^A \otimes V_k^{A*}\Big) \otimes \Big(\sum_{k'} V_{k'}^B \otimes V_{k'}^{B*}\Big)C, \tag{73}$$

respectively.

The matrix (72) can be expressed in terms of semigroup matrices $\mathcal{L}_A$ and $\mathcal{L}_B$ as follows:

$$\mathcal{L}_{pU} = C\Big(\mathcal{L}_A \otimes \mathcal{L}_B\Big)C, \tag{74}$$

where

$$\mathcal{L}_A = \langle U^A \otimes U^{A^*}\rangle, \qquad \mathcal{L}_B = \langle U^B \otimes U^{B^*}\rangle. \tag{75}$$

Analogously

$$\mathcal{L}_{pV} = C\Big(\langle V^A \otimes V^{A*}\rangle \otimes \langle V^B \otimes V^{B*}\rangle\Big)C. \tag{76}$$

The matrices $\mathcal{L}_{pV}$ form 18-parameter semigroup.

6 Distributions as Vectors

The notion of entanglement can be better clarified using the concept of distance between the quantum states. In this section, we consider the notion of distance between the quantum states in terms of vectors. First, let us discuss the notion of distance between conventional probability distributions. This notion is well known in the classical probability theory.

Given probability distribution $P(k)$, $k = 1, 2, \ldots N$, one can introduce vector $\vec{P}$ in the form of column with components $P_1 = P(1)$, $P_2 = P(2), \ldots,$ $P_N = P(N)$. The vector satisfies the condition

$$\sum_{k=1}^{N} P_k = 1. \tag{77}$$

The set of the vectors does not form a linear space but only a convex subset. Nevertheless, in this set one can introduce distance between two distributions using the vector intuition

$$D^2 = \Big(\vec{P}_1 - \vec{P}_2\Big)^2 = \sum_k P_{1k}P_{1k} + \sum_k P_{2k}P_{2k} - 2\sum_k P_{1k}P_{2k}. \tag{78}$$

One can use another identification of distribution with vectors.

Since all $P(k) \geq 0$, one can introduce $\mathcal{P}_k = \sqrt{P(k)}$ as components of vector $\vec{\mathcal{P}}$. The $\vec{\mathcal{P}}$ can be thought as column with nonnegative components. Then the distance between the two distributions takes the form

$$\mathcal{D}^2 = \left(\vec{\mathcal{P}}_1 - \vec{\mathcal{P}}_2\right)^2 = 2 - 2\sum_k \sqrt{P_1(k)P_2(k)}. \tag{79}$$

Two different definitions (77) and (78) can be used for the notion of distance between the distributions.

Let us discuss now the notion of distance between the quantum states determined by density matrices. In the density-matrix space (in the set of linear space of adjoint $U(n)$ representation), one can introduce distances analogously. The first case is

$$\mathrm{Tr}\,(\rho_1 - \rho_2)^2 = D^2 \tag{80}$$

and the second case is

$$\mathrm{Tr}\,(\sqrt{\rho_1} - \sqrt{\rho_2})^2 = \mathcal{D}^2. \tag{81}$$

In fact, the distances introduced can be written naturally as norms of vectors associated to density matrices

$$D^2 = |\vec{\rho}_1 - \vec{\rho}_2|^2 \tag{82}$$

and

$$\mathcal{D}^2 = \left((\vec{\sqrt{\rho}}_1) - (\vec{\sqrt{\rho}}_2)\right)^2, \tag{83}$$

respectively.

In the above expressions, we use scalar product of vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ as well as scalar products of vectors $(\vec{\sqrt{\rho}}_1)$ and $(\vec{\sqrt{\rho}}_2)$, respectively.

Both definitions immediately follow by identification of matrices either ρ_1 and ρ_2 with vectors according to the map of the previous sections or matrices $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ with vectors. Since the density matrices ρ_1 and ρ_2 have nonnegative eigenvalues, the matrices $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ are defined without ambiguity. This means that the vectors $(\vec{\sqrt{\rho}}_1)$ and $(\vec{\sqrt{\rho}}_2)$ are also defined without ambiguity.

One can easily see that the distance given as the norm of real vector is invariant of the orthogonal group including the improper transform, which are discrete transforms like permutations of the vector components and changing sign of the components. The norm of the vector is invariant with respect to all these transforms. Thus, the invariance group of the norm contains the standard local rotations of the orthogonal group and the discrete transforms.

7 Invariance of Intrinsic Measure

Given the density matrix ρ_{AB} in the linear space equipped with bipartite structure. Then the partial traces exist

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB}, \qquad \rho_B = \mathrm{Tr}_A\, \rho_{AB}. \tag{84}$$

The Hilbert–Schmidt distance between ρ_{AB} and $\rho_A \otimes \rho_B$, i.e.,

$$e = \mathrm{Tr}\, (\rho_{AB} - \rho_A \otimes \rho_B)^2 \tag{85}$$

was considered to define the parameter e as intrinsic measure of the state entanglement. It was not proved that this measure is invariant under local group $U(n_1)\times U(n_2)$ transformations. The proof is straightforward. One uses the structure of the tensor product in the form

$$U_{n_1 n_2} = U(n_1) \otimes U(n_2) \to \begin{pmatrix} U_{11}V & U_{12}V & \dots & U_{1n}V \\ U_{21}V & U_{22}V & \dots & U_{2n}V \\ \dots & \dots & \dots & \dots \\ U_{n1}V & U_{n2}V & \dots & U_{nn}V \end{pmatrix}, \tag{86}$$

where U_{ik} are matrix elements of the $U(n_1)$ group and V is the matrix of $U(n_2)$ group.

Due to unitarity

$$U^{-1}_{n_1 n_2} = U^{\dagger}_{n_1 n_2} \tag{87}$$

and using the form of this matrix given by Eq. (86), one has

$$\mathrm{Tr}_B \left(U_{n_1 n_2} \rho_{AB} U^{\dagger}_{n_1 n_2} \right) = \tilde{\rho}_A = U(n_1) \rho_A U^{\dagger}(n_1), \tag{88}$$

$$\mathrm{Tr}_A \left(U_{n_1 n_2} \rho_{AB} U^{\dagger}_{n_1 n_2} \right) = \tilde{\rho}_B = V \rho_B V^{\dagger}. \tag{89}$$

This means that

$$\mathrm{Tr}\, (\rho_{AB} - \rho_A \otimes \rho_B)^2 = \mathrm{Tr}\, (\tilde{\rho}_{AB} - \tilde{\rho}_A \otimes \tilde{\rho}_B)^2 . \tag{90}$$

Thus the entanglement e is invariant under local transformations. One can introduce another measure of entanglement, which is also invariant under local transformations

$$\widetilde{e} = \left| \rho^{1/2}_{AB} - \mathrm{tr}_A\, \rho^{1/2}_{AB}\, \mathrm{tr}_B\, \rho^{1/2}_{AB} \right|^2 .$$

Since we have shown that norm of a vector in adjoint representation of the unitary group is invariant under action of the group, which is larger than the local group, the measures of entanglement introduced are invariant under action of local transformations.

8 Separable Systems and Separability Criterion

According to known definition, the system density matrix is called separable (for composite system) if one has decomposition of the form

$$\rho_{AB} = \sum_k p_k \Big(\rho_A^{(k)} \otimes \rho_B^{(k)} \Big), \qquad \sum_k p_k = 1, \qquad 1 \geq p_k \geq 0. \tag{91}$$

The formula does not demand orthogonality of the density operators $\rho_A^{(k)}$ and $\rho_B^{(k)}$ for different k. Since every density matrix is a convex set of pure density matrices, one could demand that $\rho_A^{(k)}$ and $\rho_B^{(k)}$ be pure. This formula can be interpreted in the context of random matrix representation. In fact, one has

$$\rho_{AB} = \langle \rho_A \otimes \rho_B \rangle, \tag{92}$$

where ρ_A and ρ_B are considered as random density matrices of the subsystems A and B, respectively.

There are several criteria for the system to be separable. We suggest in the next sections a new approach to the problem of separability and entanglement based on the tomographic probability description of quantum states. The states which cannot be represented in the form (91) by definition are called entangled states [17]. Thus the states are entangled if in formula (91) at least one coefficient (or more) p_i is negative which means that the positive ones can take values more than unity.

Let us discuss the condition for the system state to be separable. According to Peres criterion [20], the system is separable if partial transpose of the matrix ρ_{AB} (91) gives the positive density matrix. This condition is necessary but not sufficient. Let us discuss this condition within the framework of positive-map matrix representation. On the example of spin-1/2 bipatite system, we have shown that the map of density matrix onto its transpose can be included in the matrix semigroup of matrices $\mathcal{L}_s$. One should point out that this map cannot be obtained by means of the averaging with all positive probability distributions p_k. On the other hand, it is obvious that generic criterion, which contains the Peres one as a partial case, can be formulated as follows.

Let us map the density matrix ρ_{AB} of a bipartite system onto vector $\vec{\rho}_{AB}$. Let us act on the vector $\vec{\rho}_{AB}$ by an arbitrary matrix, which represents the

positive maps in subsystems A and B. Thus we get a new vector

$$\vec{\rho}_{AB}^{(p)} = \Big(\mathcal{L}_A \otimes \mathcal{L}_B\Big)\vec{\rho}_{AB}. \tag{93}$$

Let us construct the density matrix $\rho_{AB}^{(p)}$ using inverse map of vectors onto matrices. If the initial density matrix is separable, the new density matrix $\rho_{AB}^{(p)}$ must be positive (and separable).

In the case of bipartite spin-1/2 system, by choosing $\mathcal{L}_A = 1$ and $\mathcal{L}_B$ being matrix coinciding with the matrix $g^{\alpha\beta}$, we obtain the Peres criterion as a partial case of the criterion of separability formulated above. Thus, our criterion means that separable matrix keeps positivity under action of tensor product of two semigroups. In the case of bipartite spin-1/2 system, the 16×16 matrix of the semigroup tensor product is determined by 18 parameters.

Let us discuss the positive map (54) which is determined by the semigroup for n-dimensional system. It can be realized as follows.

The $n\times n$ Hermitian generic matrix ρ is mapped onto complex n^2-vector $\vec{\rho}$ by the map described above. The complex vector $\vec{\rho}$ is mapped by means of multiplying by the unitary matrix S onto real vector $\vec{\rho}_{\rm r}$, i.e.,

$$\vec{\rho}_{\rm r} = S\vec{\rho}, \qquad \vec{\rho} = S^{-1}\vec{\rho}_{\rm r}. \tag{94}$$

The matrix S is composed from n unity blocks and the blocks

$$S_b^{(jk)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{95}$$

where j corresponds to column and k corresponds to row in the matrix ρ.

For example, in the case $n = 2$, one has the vector $\vec{\rho}_{\rm r}$ of the form

$$\vec{\rho}_{\rm r} = \begin{pmatrix} \rho_{11} \\ \sqrt{2}\,{\rm Re}\,\rho_{12} \\ \sqrt{2}\,{\rm Im}\,\rho_{12} \\ \rho_{22} \end{pmatrix}. \tag{96}$$

One has the equalities

$$\vec{\rho}_{\rm r}^{\,2} = \vec{\rho}^{\,2} = {\rm Tr}\,\rho^2. \tag{97}$$

The semigroup preserves the trace of the density matrix. Also the discrete transforms, which are described by the matrix with diagonal matrix blocks

of the form

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{98}$$

preserve positivity of the density matrix.

For the spin case, the semigroup contains 12 parameters.

Thus, the direct product of the semigroup (54) and the discrete group of the transform D is the positive map preserving positivity of the density operator.

9 Symbols, Star-Product and Entanglement

In this section, we describe how entangled states and separable states can be considered using properties of symbols and density operators of different kinds, e.g., from the viewpoint of Wigner function or tomogram. The general scheme of constructing the operator symbols is as follows [15].

Given a Hilbert space H and an operator $\hat{A}$ acting on this space, let us suppose that we have a set of operators $\hat{U}(\mathbf{x})$ acting on H, a n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ labels the particular operator in the set. We construct the c-number function $f_{\hat{A}}(\mathbf{x})$ (we call it the symbol of operator $\hat{A}$) using the definition

$$f_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right]. \tag{99}$$

Let us suppose that relation (99) has an inverse, i.e., there exists a set of operators $\hat{D}(\mathbf{x})$ acting on the Hilbert space such that

$$\hat{A} = \int f_{\hat{A}}(\mathbf{x})\hat{D}(\mathbf{x})\ d\mathbf{x}, \qquad \mathrm{Tr}\,\hat{A} = \int f_{\hat{A}}(\mathbf{x})\,\mathrm{Tr}\,\hat{D}(\mathbf{x})\ d\mathbf{x}. \tag{100}$$

Then, we will consider relations (99) and (100) as relations determining the invertible map from the operator $\hat{A}$ onto function $f_{\hat{A}}(\mathbf{x})$. Multiplying both sides of Eq. (2) by the operator $\hat{U}(\mathbf{x}')$ and taking trace, one has the consistency condition satisfied for the operators $\hat{U}(\mathbf{x}')$ and $\hat{D}(\mathbf{x})$

$$\mathrm{Tr}\left[\hat{U}(\mathbf{x}')\hat{D}(\mathbf{x})\right] = \delta\left(\mathbf{x}' - \mathbf{x}\right). \tag{101}$$

The consistency condition (101) follows from the relation

$$f_{\hat{A}}(\mathbf{x}) = \int K(\mathbf{x}, \mathbf{x}')f_{\hat{A}}(\mathbf{x}')\,d\mathbf{x}'. \tag{102}$$

The kernel in (102) is equal to the standard Dirac delta-function, if the set of functions $f_{\hat{A}}(\mathbf{x})$ is a complete set.

In fact, we could consider relations of the form

$$\hat{A} \to f_{\hat{A}}(\mathbf{x}) \tag{103}$$

and

$$f_{\hat{A}}(\mathbf{x}) \to \hat{A}. \tag{104}$$

The most important property of the map is the existence of associative product (star-product) of functions.

We introduce the product (star-product) of two functions $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$ corresponding to two operators $\hat{A}$ and $\hat{B}$ by the relationships

$$f_{\hat{A}\hat{B}}(\mathbf{x}) = f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) := \mathrm{Tr}\left[\hat{A}\hat{B}\hat{U}(\mathbf{x})\right]. \tag{105}$$

Since the standard product of operators on a Hilbert space is an associative product, i.e., $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, it is obvious that formula (105) defines an associative product for the functions $f_{\hat{A}}(\mathbf{x})$, i.e.,

$$f_{\hat{A}}(\mathbf{x}) * \Big(f_{\hat{B}}(\mathbf{x}) * f_{\hat{C}}(\mathbf{x})\Big) = \Big(f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x})\Big) * f_{\hat{C}}(\mathbf{x}). \tag{106}$$

Using formulas (99) and (100), one can write down a composition rule for two symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$, which determines star-product of these symbols. The composition rule is described by the formula

$$f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) = \int f_{\hat{A}}(\mathbf{x}'')f_{\hat{B}}(\mathbf{x}')K(\mathbf{x}'', \mathbf{x}', \mathbf{x})\, d\mathbf{x}'\, d\mathbf{x}''. \tag{107}$$

The kernel in the integral of (107) is determined by the trace of product of the basic operators, which we use to construct the map

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \mathrm{Tr}\left[\hat{D}(\mathbf{x}'')\hat{D}(\mathbf{x}')\hat{U}(\mathbf{x})\right]. \tag{108}$$

Formula (108) can be extended to the case of star-product of N symbols of operators $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_N$. Thus one has

$$\begin{aligned} W_{\hat{A}_1}(\mathbf{x}) * W_{\hat{A}_2}(\mathbf{x}) * \cdots * W_{\hat{A}_N}(\mathbf{x}) \;&=\; \int W_{\hat{A}_1}(\mathbf{x}_1)W_{\hat{A}_2}(\mathbf{x}_2)\cdots W_{\hat{A}_N}(\mathbf{x}_N) \\ &\quad \times K\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{x}\right)\, d\mathbf{x}_1\, d\mathbf{x}_2 \cdots d\mathbf{x}_N \end{aligned} \tag{109}$$

where the kernel has the form

$$K\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{x}\right) = \mathrm{Tr}\left[\hat{D}(\mathbf{x}_1)\hat{D}(\mathbf{x}_2)\cdots\hat{D}(\mathbf{x}_N)\hat{U}(\mathbf{x})\right]. \tag{110}$$

Since this kernel determines the associative star-product of N symbols, it can be expressed in terms of the kernel of star-product of two symbols. The trace of an operator $\hat{A}^N$ is determined by the kernel as follows:

$$\begin{aligned}\mathrm{Tr}\,\hat{A}^N &= \int W_{\hat{A}}(\mathbf{x}_1)W_{\hat{A}}(\mathbf{x}_2)\cdots W_{\hat{A}}(\mathbf{x}_N) \\ &\quad \times \mathrm{Tr}\left[\hat{D}(\mathbf{x}_1)\hat{D}(\mathbf{x}_2)\cdots\hat{D}(\mathbf{x}_N)\right] d\mathbf{x}_1\, d\mathbf{x}_2 \cdots d\mathbf{x}_N. \end{aligned}\tag{111}$$

When the operator $\hat{A}$ is a density operator of a quantum state, formula (111) determines the generalized purity parameter of the state. When the operator $\hat{A}$ is equal to the product of two density operators and $N = 1$, formula (111) determines the fidelity.

10 Weyl Symbol

In this section, we will consider a known example of the Heisenberg–Weyl-group representation. As operator $\hat{U}(\mathbf{x})$, we take the Fourier transform of displacement operator $\hat{D}(\xi)$

$$\hat{U}(\mathbf{x}) = \int \exp\left(\frac{x_1 + ix_2}{\sqrt{2}}\boldsymbol{\xi}^* - \frac{x_1 - ix_2}{\sqrt{2}}\boldsymbol{\xi}\right)\hat{D}(\boldsymbol{\xi})\pi^{-1}\, d^2\boldsymbol{\xi}, \tag{112}$$

where $\boldsymbol{\xi}$ is a complex number, $\boldsymbol{\xi} = \xi_1 + i\xi_2$, and the vector $\mathbf{x} = (x_1, x_2)$ can be considered as $\mathbf{x} = (q, p)$, with q and p being the position and momentum. One can see that

$$\mathrm{Tr}\,\hat{U}(\mathbf{x}) = 1.$$

The displacement operator may be expressed through the creation and annihilation operators in the form

$$\hat{D}(\boldsymbol{\xi}) = \exp(\boldsymbol{\xi}\hat{a}^\dagger - \boldsymbol{\xi}^*\hat{a}). \tag{113}$$

The displacement operator is used to create coherent states from the vacuum state. For the creation and annihilation operators, one has

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \qquad \hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}, \tag{114}$$

where $\hat{q}$ and $\hat{p}$ may be thought as the coordinate and momentum operators for the carrier space of an harmonic oscillator. The operator $\hat{a}$ and its Hermitian conjugate $\hat{a}^\dagger$ satisfy the boson commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}.$$

Let us introduce the Weyl symbol for an arbitrary operator $\hat{A}$ using the definition given by Eq. (99)

$$W_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right]. \tag{115}$$

The form of operator $\hat{U}(\mathbf{x})$ is given by Eq. (112). One can check that Weyl symbols of the identity operator $\hat{\mathbf{1}}$, position operator $\hat{q}$ and momentum operator $\hat{p}$ have the form

$$W_{\hat{\mathbf{1}}}(q,p) = 1, \qquad W_{\hat{q}}(q,p) = q, \qquad W_{\hat{p}}(q,p) = p. \tag{116}$$

The inverse transform, which expresses the operator $\hat{A}$ through its Weyl symbol, is of the form

$$\hat{A} = \int W_{\hat{A}}(\mathbf{x})\hat{U}(\mathbf{x})\,\frac{d\mathbf{x}}{2\pi}\,. \tag{117}$$

One can check that for $W_{\hat{\mathbf{1}}}(\mathbf{x}) = 1$, formula (117) reproduces the identity operator, i.e.,

$$\int \hat{U}(\mathbf{x})\,\frac{d\mathbf{x}}{2\pi} = \hat{\mathbf{1}}. \tag{118}$$

Comparing (117) with (100), one can see that the operator $\hat{D}(\mathbf{x})$ is connected with $\hat{U}(\mathbf{x})$ by the relationship

$$\hat{D}(\mathbf{x}) = \frac{\hat{U}(\mathbf{x})}{2\pi}\,. \tag{119}$$

Let us consider now star-product of two Weyl symbols (it is usually called Moyal star-product). If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through Weyl symbols by formulas

$$\hat{A}_1 = \int W_{\hat{A}_1}(\mathbf{x}')\hat{U}(\mathbf{x}')\,\frac{d\mathbf{x}'}{2\pi}\,, \qquad \hat{A}_2 = \int W_{\hat{A}_2}(\mathbf{x}'')\hat{U}(\mathbf{x}'')\,\frac{d\mathbf{x}''}{2\pi}\,, \tag{120}$$

with vectors

$$\mathbf{x}' = (x_1', x_2') \qquad \text{and} \qquad \mathbf{x}'' = (x_1'', x_2''),$$

the operator $\hat{A}$ (product of operators $\hat{A}_1$ and $\hat{A}_2$) has Weyl symbol given by

$$W_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right] = \frac{1}{4\pi^5}\int d\mathbf{x}'\, d\mathbf{x}''\, d^2\xi\, d^2\xi'\, d^2\xi''\, W_{\hat{A}_1}(\mathbf{x}')W_{\hat{A}_2}(\mathbf{x}'')$$
$$\times \exp\left\{2^{-1/2}\Big[(\xi_1' - i\xi_2')(x_1' + ix_2') - (\xi_1' + i\xi_2')(x_1' - ix_2')\right.$$
$$+(\xi_1'' - i\xi_2'')(x_1'' + ix_2'') - (\xi_1'' + i\xi_2'')(x_1'' - ix_2'') + (\xi_1 - i\xi_2)(x_1 + ix_2)$$
$$\left.-(\xi_1 + i\xi_2)(x_1 - ix_2)\Big]\right\}\mathrm{Tr}\left[\hat{D}(\boldsymbol{\xi}')\hat{D}(\boldsymbol{\xi}'')\hat{D}(\boldsymbol{\xi})\right], \quad (121)$$

where $\boldsymbol{\xi} = \xi_1 + i\xi_2$, with $\boldsymbol{\xi}' = \xi_1' + i\xi_2'$ and $\boldsymbol{\xi}'' = \xi_1'' + i\xi_2''$.

Using properties of displacement operators

$$\hat{D}(\boldsymbol{\xi}')\hat{D}(\boldsymbol{\xi}'') = \hat{D}(\boldsymbol{\xi}' + \boldsymbol{\xi}'')\exp\left(i\ \mathrm{Im}\,(\boldsymbol{\xi}'\boldsymbol{\xi}''^*)\right), \qquad \mathrm{Tr}\left[\hat{D}(\boldsymbol{\xi})\right] = \pi\delta^2(\boldsymbol{\xi}), \quad (122)$$

one can get known explicit form of the kernel, which determines star-product of Weyl symbols.Thus we described the construction of Weyl symbols, including Wigner function, by means of the star-product formalism.

11 Tomographic Representation

In this section, we will consider an example of the probability representation of quantum mechanics [21]. In the probability representation of quantum mechanics, the state is described by a family of probabilities [22–24]. According to the general scheme, one can introduce for the operator $\hat{A}$ the function $f_{\hat{A}}(\mathbf{x})$, where

$$\mathbf{x} = (x_1, x_2, x_3) \equiv (X, \mu, \nu),$$

which we denote here as $w_{\hat{A}}(X, \mu, \nu)$ depending on the position X and the parameters μ and ν of the reference frame

$$w_{\hat{A}}(X, \mu, \nu) = \mathrm{Tr}\left[\hat{A}\hat{U}(\mathbf{x})\right]. \quad (123)$$

We call the function $w_{\hat{A}}(X, \mu, \nu)$ the tomographic symbol of the operator $\hat{A}$. The operator $\hat{U}(x)$ is given by

$$\begin{aligned}\hat{U}(\mathbf{x}) \equiv \hat{U}(X, \mu, \nu) &= \exp\left(\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right)\exp\left(\frac{i\theta}{2}\left(\hat{q}^2 + \hat{p}^2\right)\right) \mid X\rangle\langle X \mid \\ &\times \exp\left(-\frac{i\theta}{2}\left(\hat{q}^2 + \hat{p}^2\right)\right)\exp\left(-\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right) \\ &= \hat{U}_{\mu\nu} \mid X\rangle\langle X \mid \hat{U}^{\dagger}_{\mu\nu}. \quad (124)\end{aligned}$$

The angle θ and parameter λ in terms of the reference frame parameters are given by

$$\mu = e^{\lambda} \cos\theta, \qquad \nu = e^{-\lambda} \sin\theta.$$

Moreover, $\hat{q}$ and $\hat{p}$ are position and momentum operators

$$\hat{q} \mid X\rangle = X \mid X\rangle \tag{125}$$

and $\mid X\rangle\langle X \mid$ is the projection density. One has the canonical transform of quadratures

$$\hat{X} = \hat{U}_{\mu\nu}\, \hat{q}\, \hat{U}_{\mu\nu}^{\dagger} = \mu\hat{q} + \nu\hat{p},$$

$$\hat{P} = \hat{U}_{\mu\nu}\, \hat{p}\, \hat{U}_{\mu\nu}^{\dagger} = \frac{1+\sqrt{1-4\mu^2\nu^2}}{2\mu}\, \hat{p} - \frac{1-\sqrt{1-4\mu^2\nu^2}}{2\nu}\, \hat{q}.$$

Using the approach of [25] one can obtain the relationship

$$\hat{U}(X,\mu,\nu) = \delta(X - \mu\hat{q} - \nu\hat{p}).$$

In the case we are considering, the inverse transform determining the operator in terms of tomogram [see Eq. (100)] will be of the form

$$\hat{A} = \int w_{\hat{A}}(X,\mu,\nu)\hat{D}(X,\mu,\nu)\, dX\, d\mu\, d\nu, \tag{126}$$

where

$$\hat{D}(\mathbf{x}) \equiv \hat{D}(X,\mu,\nu) = \frac{1}{2\pi} \exp\left(iX - i\nu\hat{p} - i\mu\hat{q}\right), \tag{127}$$

i.e.,

$$\hat{D}(X,\mu,\nu) = \frac{1}{2\pi} \exp(iX)\hat{D}\Big(\boldsymbol{\xi}(\mu,\nu)\Big). \tag{128}$$

The unitary displacement operator in (128) reads now

$$\hat{D}\Big(\boldsymbol{\xi}(\mu,\nu)\Big) = \exp\Big(\boldsymbol{\xi}(\mu,\nu)\hat{a}^{+} - \boldsymbol{\xi}^{*}(\mu,\nu)\hat{a}\Big),$$

where $\boldsymbol{\xi}(\mu,\nu) = \xi_1 + i\xi_2$, with $\xi_1 = \mathrm{Re}\,(\boldsymbol{\xi}) = \nu/\sqrt{2}$ and $\xi_2 = \mathrm{Im}\,(\boldsymbol{\xi}) = -\mu/\sqrt{2}$.

Trace of the above operator which provides the kernel determining the trace of an arbitrary operator in the tomographic representation reads

$$\mathrm{Tr}\, \hat{D}(\mathbf{x}) = e^{iX}\delta(\mu)\delta(\nu).$$

The function $w_{\hat{A}}(X,\mu,\nu)$ satisfies the relation

$$w_{\hat{A}}\left(\lambda X,\lambda\mu,\lambda\nu\right)=\frac{1}{|\lambda|}\,w_{\hat{A}}(X,\mu,\nu). \tag{129}$$

This means that the tomographic symbols of operators are homogeneous functions of three variables.

If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through the corresponding functions by the formulas

$$\begin{aligned}\hat{A}_1 &= \int w_{\hat{A}_1}(X',\mu',\nu')\hat{D}(X',\mu',\nu')\,dX'\,d\mu'\,d\nu',\\ \hat{A}_2 &= \int w_{\hat{A}_2}(X'',\mu'',\nu'')\hat{D}(X'',\mu'',\nu'')dX''\,d\mu''\,d\nu'',\end{aligned} \tag{130}$$

and $\hat{A}$ denotes the product of $\hat{A}_1$ and $\hat{A}_2$, then the function $w_{\hat{A}}(X,\mu,\nu)$, which corresponds to $\hat{A}$, is star-product of functions $w_{\hat{A}_1}(X,\mu,\nu)$ and $w_{\hat{A}_2}(X,\mu,\nu)$. Thus this product

$$w_{\hat{A}}(X,\mu,\nu)=w_{\hat{A}_1}(X,\mu,\nu)*w_{\hat{A}_2}(X,\mu,\nu)$$

reads

$$w_{\hat{A}}(X,\mu,\nu)=\int w_{\hat{A}_1}(\mathbf{x}'')w_{\hat{A}_2}(\mathbf{x}')K(\mathbf{x}'',\mathbf{x}',\mathbf{x})\,d\mathbf{x}''\,d\mathbf{x}', \tag{131}$$

with kernel given by

$$K(\mathbf{x}'',\mathbf{x}',\mathbf{x})=\mathrm{Tr}\left[\hat{D}(X'',\mu'',\nu'')\hat{D}(X',\mu',\nu')\hat{U}(X,\mu,\nu)\right]. \tag{132}$$

The explicit form of the kernel reads

$$\begin{aligned}&K(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X\mu,\nu)\\ &=\frac{\delta\Big(\mu(\nu_1+\nu_2)-\nu(\mu_1+\mu_2)\Big)}{4\pi^2}\,\exp\left(\frac{i}{2}\Big\{(\nu_1\mu_2-\nu_2\mu_1)+2X_1+2X_2\right.\\ &\left.-\left[\frac{1-\sqrt{1-4\mu^2\nu^2}}{\nu}\,(\nu_1+\nu_2)+\frac{1+\sqrt{1-4\nu^2\mu^2}}{\mu}\,(\mu_1+\mu_2)\right]X\Big\}\right).\end{aligned} \tag{133}$$

The kernel for star-product of N operators is

$$\begin{aligned}&K\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,\ldots,X_N,\mu_N,\nu_N,X,\mu,\nu\right)\\ &=\frac{\delta\left(\mu\sum_{j=1}^{N}\nu_j-\nu\sum_{j=1}^{N}\mu_j\right)}{(2\pi)^N}\,\exp\left(\frac{i}{2}\,\left\{\sum_{k<j=1}^{N}(\nu_k\mu_j-\nu_j\mu_k)+2\sum_{j=1}^{N}X_j\right.\right.\\ &\left.\left.-\left[\frac{1-\sqrt{1-4\mu^2\nu^2}}{\nu}\left(\sum_{j=1}^{N}\nu_j\right)+\frac{1+\sqrt{1-4\mu^2\nu^2}}{\mu}\left(\sum_{j=1}^{N}\mu_j\right)\right]X\right\}\right).\end{aligned} \tag{134}$$

The above kernel can be expressed in terms of the kernel determining star-product of two operators.

12 Multipartite Systems

Let us assume that for multimode (N-mode) system one has

$$\hat{U}(\vec{y}) = \prod_{k=1}^{N} \hat{U}\left(\vec{x}^{(k)}\right), \tag{135}$$

$$\hat{D}(\vec{y}) = \prod_{k=1}^{N} \hat{D}\left(\vec{x}^{(k)}\right), \tag{136}$$

where

$$\vec{y} = \left(x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_m^{(N)}\right). \tag{137}$$

This means that symbol of density operator of the composite system reads

$$f_\rho(\vec{y}) = \mathrm{Tr}\left[\hat{\rho} \prod_{k=1}^{N} \hat{U}(\vec{x}^{(k)})\right]. \tag{138}$$

The inverse transform reads

$$\hat{\rho} = \int d\vec{y}\, f_\rho(\vec{y}) \prod_{k=1}^{N} \hat{D}(\vec{x}^{(k)}), \qquad d\vec{y} = \prod_{k=1}^{N} \prod_{s=1}^{m} dx_s^{(k)}. \tag{139}$$

If the symbol corresponds to a system Wigner function $W(\vec{q}, \vec{p})$, the operator $\hat{U}(\vec{x}^{(k)})$, where $\vec{x}^{(k)} = (q_k, p_k)$, was discussed above. It has the form

$$\hat{U}(\vec{x}^{(k)}) = \mathcal{D}(\alpha_k)(-1)^{a_k^\dagger a_k}\mathcal{D}(-\alpha_k), \tag{140}$$

where

$$\alpha_k = \frac{1}{\sqrt{2}}\,(q_k + ip_k), \qquad \mathcal{D}(\alpha_k) = e^{\alpha_k a_k^\dagger - \alpha_k^* a_k}, \qquad [a_k, a_m] = 0, \qquad [a_k, a_m^\dagger] = \delta_{km}. \tag{141}$$

The operator

$$\hat{D}(\vec{x}^k) = \frac{1}{\pi}\,\hat{U}(\vec{x}^{(k)}). \tag{142}$$

The tomographic symbols are also defined by analogous formulas with specific $\vec{x}$ and operators $\hat{U}(\vec{x}_k)$, $\hat{D}(\vec{x}_k)$.

Now we formulate properties of symbols in the case of entangled and separable states, respectively.

Given a composite m-partite system with density operator $\hat{\rho}$.

If the nonnegative operator can be presented in the form of 'probabilistic sum'

$$\hat{\rho} = \sum_{\vec{z}} \mathcal{P}(\vec{z}) \hat{\rho}_{\vec{z}}^{(a_1)} \otimes \hat{\rho}_{\vec{z}}^{(a_2)} \otimes \cdots \otimes \hat{\rho}_{\vec{z}}^{(a_m)}, \tag{143}$$

with positive probability distribution function $\mathcal{P}(\vec{z})$, where components of $\vec{z}$ can be either discrete or continuous, we call the state 'separable state'. This means that the symbol of the state can be presented in the form

$$f_\rho(\vec{y}) = \sum_{\vec{z}} \mathcal{P}(\vec{z}) \prod_{k=1}^{m} f_\rho^{(a_k)}(\vec{x}_k, \vec{z}). \tag{144}$$

For example, the Wigner function of separable state of bipartite system has the form

$$W(q_1, q_2, p_1, p_2) = \sum_{\vec{z}} \mathcal{P}(\vec{z}) W^{(1)}(q_1, p_1, \vec{z}) W^{(2)}(q_2, p_2, \vec{z}). \tag{145}$$

Analogous formula can be written for the tomogram of separable state.

13 Spin Tomography

Below we concentrate on bipartite spin systems.

The tomographic probability (spin tomogram) completely determines the density matrix of a spin state. It has been introduced in [10, 11, 15].

The tomographic probability for spin-j state is defined via the density matrix by the formula

$$\langle jm \mid D^\dagger(g) \rho D(g) \mid jm \rangle = W^{(j)}(m, \vec{0}), \qquad m = -j, -j+1, \ldots, j, \tag{146}$$

where $D(g)$ is the matrix of $SU(2)$-group representation depending on the group element g determined by three Euler angles. The set of the tomogram values for each $\vec{0}$ is an overcomplete set. We need only finite number of independent locations which will give information on the density matrix of the spin state. Due to structure of the formula, there are only two Euler angles involved. They are combined into the unit vector

$$\vec{0} = (\cos\phi \sin\vartheta, \sin\phi \sin\vartheta, \cos\vartheta). \tag{147}$$

This is the map from S^3 to S^2.

The physical meaning of the probability $W(m,\vec{0})$ is the following.

It is the probability to find, in the state with the density matrix ρ, the spin projection on direction $\vec{0}$ equal to m. For bipartite system, the tomogram is defined as follows:

$$W(m_1m_2\vec{0}_1\vec{0}_2) = \langle j_1m_1j_2m_2 \mid D^\dagger(g_1)D^\dagger(g_2)\rho D(g_1)D(g_2) \mid j_1m_1j_2m_2\rangle. \tag{148}$$

It completely determines the density matrix ρ. It has the meaning of joint probability distribution for spin j_1 and j_2 projections m_1 and m_2 on directions $\vec{0}_1$ and $\vec{0}_2$. Since the map $\rho \rightleftharpoons W$ is linear and invertible, the definition of separable system can be rewritten in the following form of decomposition of the joint probability into sum of products (of factorized probabilities):

$$W(m_1m_2\vec{0}_1\vec{0}_2) = \sum_k p_k W^{(k)}(m_1\vec{0}_1)\tilde{W}^{(k)}(m_2\vec{0}_2). \tag{149}$$

This form can be considered to formulate the criterion of separability of the two spin state.

The state is separable iff the tomogram can be written in the form (149) with $\sum_k p_k = 1$, $p_k \geq 0$. It seems that we simply use the definition but, in fact, we cast the problem of separability into the form of property of the positive joint probability distribution of two random variables. This is area of probability theory and one can use results and theorems on the joint probability distributions. If one does not use any theorem, one has to study solvability of relation (149) considered as the equation for unknown probability distribution p_k and unknown probability functions $W^{(k)}(m_1\vec{0}_1)$ and $W^{(k)}(m_2\vec{0}_2)$.

14 Example of Spin-1/2 Bipartite System

For spin-1/2 state, the generic density matrix can be presented in the form

$$\rho = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\vec{n}\right), \qquad \vec{n} = (n_1, n_2, n_3), \tag{150}$$

where $\vec{\sigma}$ are Pauli matrices and $\vec{n}^2 \leq 1$, with vector $\vec{n}$ for a pure state being unit vector. This decomposition means that we use as basis in 4-dimensional

vector space the vectors corresponding to Pauli matrices and unit matrix, i.e.,

$$\vec{\sigma}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{\sigma}_2 = \begin{pmatrix} 0 \\ -i \\ i \\ 0 \end{pmatrix}, \qquad \vec{\sigma}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{151}$$

The density matrix vector

$$\vec{\rho} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} \tag{152}$$

is decomposed in terms of the basis vectors

$$\vec{\rho} = \frac{1}{2}\left(\vec{1} + n_1\vec{\sigma}_1 + n_2\vec{\sigma}_2 + n_3\vec{\sigma}_3\right). \tag{153}$$

It means that tomogram of spin-1/2 state can be given in the form

$$W\left(\frac{1}{2}, \vec{0}\right) = \left(\frac{1}{2} + \frac{\vec{n}\cdot\vec{0}}{2}\right), \qquad W\left(-\frac{1}{2}, \vec{0}\right) = \left(\frac{1}{2} - \frac{\vec{n}\cdot\vec{0}}{2}\right). \tag{154}$$

Inserting these probability values into relation (149) for each value of k we get the relationships:

$$W\left(\frac{1}{2}, \frac{1}{2}, \vec{0}_1, \vec{0}_2\right) = \frac{1}{4} + \frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1 + \frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 + \sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^8\cdot\vec{0}_2\right), \tag{155}$$

$$W\left(\frac{1}{2}, -\frac{1}{2}, \vec{0}_1, \vec{0}_2\right) = \frac{1}{4} + \frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1 - \frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 - \sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^*\cdot\vec{0}_2\right), \tag{156}$$

$$W\left(-\frac{1}{2}, \frac{1}{2}, \vec{0}_1, \vec{0}_2\right) = \frac{1}{4} - \frac{1}{2}\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1 + \frac{1}{2}\left(\sum_k p_k\vec{n}_k^*\right)\cdot\vec{0}_2 - \sum_k p_k\left(\vec{n}_k\cdot\vec{0}_1\right)\left(\vec{n}_k^*\cdot\vec{0}_2\right). \tag{157}$$

One has the normalization property

$$\sum_{m_1,\, m_2=-1/2}^{1/2} W(m_1 m_2 \vec{0}_1\vec{0}_2) = 1. \tag{158}$$

One easily gets

$$W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right)=\frac{1}{2}+\left(\sum_k p_k\vec{n}_k\right)\cdot\vec{0}_1. \tag{159}$$

This means that derivative in $\vec{0}_1$ on the left-hand side gives

$$\frac{\partial}{\partial\vec{0}_1}\left[W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]=\left(\sum_k p_k\vec{n}_k\right). \tag{160}$$

Analogously

$$\frac{\partial}{\partial\vec{0}_2}\left[W\left(\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(-\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]=\left(\sum_k p_k\vec{n}_k^{(\star)}\right). \tag{161}$$

Taking the sum of (156) and (157)) one sees that

$$\frac{1}{2}\frac{\partial}{\partial\vec{0}_i}\frac{\partial}{\partial\vec{0}_j}\left[W\left(\frac{1}{2},-\frac{1}{2},\vec{0}_1,\vec{0}_2\right)+W\left(-\frac{1}{2},\frac{1}{2},\vec{0}_1,\vec{0}_2\right)\right]=-\sum_k p_k(n_k)_i(n_k^{(\star)})_j. \tag{162}$$

Since we look for solution where $p_k\geq 0$, we can introduce

$$\vec{N}_k=\sqrt{p_k}\vec{n}_k,\quad \vec{N}_k^{(\star)}=\sqrt{p_k}\vec{n}_k^{(\star)}. \tag{163}$$

This means that the derivative in (162) can be presented as tensor

$$-T_{ij}=\sum_k(N_k)_i(N_k^{(\star)})_j. \tag{164}$$

One has

$$\sum_k p_k\vec{n}_k=\sum_k\sqrt{p_k}\vec{N}_k. \tag{165}$$

$$\sum_k p_k\vec{n}_k^{\star}=\sum_k\sqrt{p_k}\vec{N}_k^{(\star)}. \tag{166}$$

The conditions of solvability of the obtained equations is a criterion for separability or entanglement of bipartite quantum spin state.

For Werner state [26] with the density matrix

$$\rho_{AB}=\begin{pmatrix}(1+p)/4 & 0 & 0 & p/2\\ 0 & (1-p)/4 & 0 & 0\\ 0 & 0 & (1-p)/4 & 0\\ p/2 & 0 & 0 & (1+p)/4\end{pmatrix},\qquad \rho_A=\rho_B=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{167}$$

one can reconstruct known results that for $p < 1/3$ the state is separable and for $p > 1/3$ the state is entangled, since in the decomposition of density operator in the form (149) the state

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{168}$$

has the weight $p_0 = (1 - 3p)/4$.

For $p > 1/3$, the coefficient p_o becomes negative.

There is some extension of the presented consideration.

Let us consider the state with the density matrix (nonnegative and Hermitian)

$$\rho = \begin{pmatrix} R_{11} & 0 & 0 & R_{12} \\ 0 & \rho_{11} & \rho_{12} & 0 \\ 0 & \rho_{21} & \rho_{22} & 0 \\ R_{21} & 0 & 0 & R_{22} \end{pmatrix}, \qquad \mathrm{Tr}\,\rho = 1. \tag{169}$$

Using procedure of mapping the matrix onto vector $\vec{\rho}$ and applying to the vector nonlocal linear transform corresponding to Peres partial transpose and making inverse map of the transformed vector onto the matrix, we obtain

$$\rho^m = \begin{pmatrix} R_{11} & 0 & 0 & \rho_{12} \\ 0 & \rho_{11} & R_{12} & 0 \\ 0 & R_{21} & \rho_{22} & 0 \\ \rho_{21} & 0 & 0 & R_{22} \end{pmatrix}. \tag{170}$$

In the case of separable matrix ρ, the matrix ρ^m is nonnegative matrix. Calculating eigenvalues of ρ^m and applying condition of their positivity, we get

$$R_{11}R_{22} \geq |\rho_{12}|^2, \qquad \rho_{11}\rho_{22} \geq |R_{12}|^2. \tag{171}$$

Violation of these inequalities gives a signal that ρ is entangled. For Werner state (167), Eq. (171) means

$$1 + p > 0, \qquad 1 - p > 2p, \tag{172}$$

which recovers the condition of separability $p < 1/3$ mentioned above.

The joint probability distribution (148) of separable state is positive after making the local and nonlocal (Peres-like) transforms connected with

positive map semigroup. But for entangled state, function (148) can take negative values after making this map in the function. This is a criterion of entanglement in terms of tomogram of the state of multiparticle system.

15 Dynamical Map and Purification

In this section, we consider connection of positive maps with purification procedure. In fact, formula

$$\rho \to \rho' = \sum_k p_k U_k \rho U_k^\dagger, \tag{173}$$

where U_k are unitary operators, can be considered in the form

$$\rho \to \rho' = \sum_k p_k \rho_k, \qquad p_k \geq 0, \qquad \sum_k p_k = 1. \tag{174}$$

Here the density operators ρ_k read

$$\rho_k = U_k \rho U_k^\dagger. \tag{175}$$

This form is the form of probabilistic addition. This mixture of density operators can be purified

$$\rho' \to \rho'' = N \left[\sum_{kj} \sqrt{p_k p_j} \, \frac{\rho_k P_0 \rho_j}{\sqrt{\mathrm{Tr}\, \rho_k P_0 \rho_j P_0}} \right], \tag{176}$$

where P_0 is a fiducial projector and

$$N^{-1} = \mathrm{Tr} \left(\sum_{kj} \sqrt{p_k p_j} \, \frac{\rho_k P_0 \rho_j}{\sqrt{\mathrm{Tr}\, \rho_k P_0 \rho_j P_0}} \right]. \tag{177}$$

The map (173) could be interpreted as the evolution in time of the initial matrix ρ_0 considering unitary operators $U_k(t)$ depending on time. Thus one has

$$\rho_0 \to \rho(t) = \sum_k p_k U_k(t) \rho_0 U_k^\dagger(t). \tag{178}$$

In this case, the purification procedure provides the dynamical map of a pure state

$$| \, \psi_0 \rangle \langle \psi_0 \, | \to | \, \psi(t) \rangle \langle \psi(t) \, |, \tag{179}$$

where $\mid \psi(t)\rangle$ obeys to a nonlinear equation and, in the general case, this equation is not differential equation in time variable like the Schrödinger equation.

For a specific case, the evolution (178) can be described by semigroup. The density matrix (178) obeys to first-order differential equation in time for this case [27–29].

The reason why there is no differential equation in time for generic case is due to the absence of the property

$$\rho_{ij}(t_2) = \sum_{mn} K_{ij}^{mn}(t_2, t_1)\rho_{mn}(t_1), \tag{180}$$

where the kernel of evolution operator satisfies

$$K_{ij}^{mn}(t_3, t_2)K_{mn}^{pq}(t_2, t_1) = K_{ij}^{pg}(t_3, t_1). \tag{181}$$

It means that trajectory (curve) is not determined by differential equation in time.

Thus via purification procedure and dynamical map of the density matrix we get the dynamical map of a pure state (nonlinear dynamical map). This map can be used in nonlinear models of quantum motion.

16 Density Matrix and Real Quadratic Forms

It is convenient to associate the Hermitian nonnegative $n\times n$ density matrix ρ with real quadratic form determined by the real matrix D using the relationships

$$r = \frac{\rho + \rho^*}{2}, \qquad iR = \frac{\rho - \rho^*}{2}, \qquad r^{\mathrm{t}} = r, \qquad R^{\mathrm{t}} = -R, \tag{182}$$

and

$$D = \begin{pmatrix} r & R \\ R^{\mathrm{t}} & r \end{pmatrix}. \tag{183}$$

The quadratic form is the scalar function (homogeneous polynomial of second order)

$$f\Big(\vec{x}, \vec{y}, r, R\Big) = \Big(\vec{x}, \vec{y}\Big) D \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \geq 0. \tag{184}$$

Nonnegativity of $f\Big(\vec{x}, \vec{y}, r, R\Big)$ takes place for all nonnegative Hermitian matrices ρ.

In fact, for complex vectors,

$$\vec{z} = \vec{x} + i\vec{y}, \tag{185}$$

one has

$$\vec{z}^{*}\rho\vec{z} = f\Big(\vec{x}, \vec{y}, r, R\Big). \tag{186}$$

All real transforms of the form

$$D \to D' = ADA^{\mathrm{T}} \tag{187}$$

keep the quadratic form nonnegative.

Let us consider the real $2n\times 2n$ matrix A given in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{188}$$

where a, b, c, and d are real $n\times n$ matrices.

If the matrix (187) has the form

$$D' = \begin{pmatrix} r' & R' \\ R'^{\mathrm{t}} & r' \end{pmatrix}, \tag{189}$$

one has the relationships:

$$ara^{\mathrm{t}} + bR^{\mathrm{t}}a^{\mathrm{t}} + aRb^{\mathrm{t}} + brb^{\mathrm{t}} = r', \tag{190}$$

$$crc^{\mathrm{t}} + dR^{\mathrm{t}}c^{\mathrm{t}} + cRd^{\mathrm{t}} + drd^{\mathrm{t}} = r'^{\mathrm{t}}, \tag{191}$$

$$arc^{\mathrm{t}} + bR^{\mathrm{t}}c^{\mathrm{t}} + aRd^{\mathrm{t}} + brd^{\mathrm{t}} = R', \tag{192}$$

$$cra^{\mathrm{t}} + dR^{\mathrm{t}}a^{\mathrm{t}} + cRb^{\mathrm{t}} + drb^{\mathrm{t}} = R'^{\mathrm{t}}. \tag{193}$$

Also $R'^{\mathrm{t}} = -R', \quad r'^{\mathrm{t}} = r'$.

For the case

$$b = c = 0, \tag{194}$$

one has possible solutions

$$a = d \tag{195}$$

and

$$a = -d. \tag{196}$$

Solution (196) for $a = 1$ describes a Peres-like transposition of the matrix ρ.

For the case $a = d = 0$, one has possible solutions

$$b = c \tag{197}$$

and

$$b = -c. \tag{198}$$

In the case (195), one has

$$r' = ara^{\mathrm{t}}, \qquad R' = aRa^{\mathrm{t}}, \tag{199}$$

$$r' + iR' = a(r + iR)a^{\mathrm{t}}. \tag{200}$$

In the case (196), one has

$$r' = ara^{\mathrm{t}}, \qquad R' = -aRa^{\mathrm{t}}, \tag{201}$$

$$r' + iR' = a(r - iR)a^{\mathrm{t}}. \tag{202}$$

Thus for block diagonal matrices A the possible transforms of the initial density matrix matrix ρ have the form

$$\rho \to \rho_a^{(\pm)} = a \left\{ \begin{array}{c} \rho \\ \rho^{\mathrm{t}} \end{array} \right\} a^{\mathrm{t}}. \tag{203}$$

In the vector form $\rho \leftrightarrow \vec{\rho}$, the transform (203) is described by superoperators

$$L_a^{\pm} = \left\{ \begin{array}{c} a \otimes a \\ (a \otimes a)L^{\mathrm{t}} \end{array} \right. . \tag{204}$$

Obviously one can apply the averaging procedure to get the matrix $\langle a \otimes a \rangle$. It is a partial case of the general transform $\langle v \otimes v \rangle$ for real v. Here superpoperator L^{t} makes from the vector $\vec{\rho} \leftrightarrow \rho$ the vector $\vec{\rho}_{\mathrm{t}} \leftrightarrow \rho^{\mathrm{t}}$, where ρ^{t} is transposed density matrix ρ.

The superoperator L^{t} in the case $n = 2$ is described by the matrix $g^{\alpha\beta}$. The solutions (197) and (198) provide analogous transforms (203) and (204) with replacement $a \to b$.

For $n = 2$, the choice $a = \sigma_1$ gives

$$\rho \to \sigma_1 \rho^{\mathrm{t}} \sigma_1 = \rho^{\mathrm{t}}, \tag{205}$$

which is exactly Peres transpose transform.

For $n = 3$, the choice

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{206}$$

provides two transforms of the Hermitian density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \to \rho' = \begin{pmatrix} \rho_{11} & \rho_{12} & -\rho_{13} \\ \rho_{21} & \rho_{22} & -\rho_{23} \\ -\rho_{31} & -\rho_{32} & \rho_{33} \end{pmatrix} \tag{207}$$

and

$$\rho \to \rho'' = \begin{pmatrix} \rho_{11} & \rho_{21} & -\rho_{31} \\ \rho_{12} & \rho_{22} & -\rho_{32} \\ -\rho_{13} & -\rho_{23} & \rho_{33} \end{pmatrix}. \tag{208}$$

Obviously, the unitary transform u of the form

$$\rho \to u\rho u^\dagger$$

does not change the nonnegative eigenvalues of the density operator but this transform differs from the transforms discussed above, e.g., if $|\det a| \neq 1$, the above transforms do not preserve the determinant of the density matrix.

The given construction can provide also noncompletely positive map. For example, if the transform of Hermitian nonnegative 3×3 matrix ρ_{jk} is described by the formula

$$\rho_{jk} \to \rho_{jk}\Big(\cosh^2\theta\cos(\theta_j - \theta_k) - \sinh^2\theta\delta_{jk}\Big)$$

(there is no sum over j, k), it corresponds to applying to vector $\vec{\rho}$ the matrix $\langle a \otimes a\rangle$ with real matrices a. The formula can be used also for arbitrary integer n. But averaging is done using quasidistribution (not the probability distribution). This means that in sum $\sum_k \varepsilon_k a_k \otimes a_k$ the numbers ε_k take both positive and negative values ± 1. It is another example of noncompletely positive map. The example of Peres transpose transform discussed in the previous sections for the case $n = 2$ belongs also to the case of positive but not completely positive maps.

17 Tomogram of the Group $U(n)$

In order to formulate a criterion of separability for a bipartite spin system with spin j_1 and j_2, we introduce the tomogram $w(\vec{l}, \vec{m}, g^{(n)})$ for the group $U(n)$, where

$$n = n_1 n_2, \qquad n_1 = 2j_1 + 1, \qquad n_2 = 2j_2 + 1,$$

and $g^{(n)}$ are parameters of the group element. Vectors $\vec{l}$ and $\vec{m}$ label a basis $\mid \vec{l}, \vec{m}\rangle$ of the fundamental representation of the group $U(n)$. For example, since this representation is irreducible, being reduced to representation of $U(n_1) \otimes U(n_2)$-subgroup of the group $U(n)$, the basis can be chosen as the product of basis vectors:

$$\mid j_1, m_1\rangle \mid j_2, m_2\rangle = \mid j_1, j_2, m_1, m_2\rangle. \tag{209}$$

Due to irreducibility of this representation of the group $U(n)$ and its subgroup, there exists a unitary transform $u^{\vec{l}\vec{m}}_{j_1 j_2 m_1 m_2} \mid \vec{l}, \vec{m}\rangle$ such that

$$\mid j_1, j_2, m_1, m_2\rangle = \sum_{\vec{l}\vec{m}} u^{\vec{l}\vec{m}}_{j_1 j_2 m_1 m_2} \mid \vec{l}, \vec{m}\rangle, \tag{210}$$

$$\mid \vec{l}\vec{m}\rangle = \sum_{m_1 m_2} (u^{-1})^{\vec{l}\vec{m}}_{j_1 j_2 m_1 m_2} \mid j_l, j_2, m_1, m_2\rangle. \tag{211}$$

One can define the $U(n)$-tomogram for a Hermitian nonnegative $n \times n$ density matrix ρ, which belongs to Lie algebra of the group $U(n)$, by a generic formula:

$$w(\vec{l}, \vec{m}, g^{(n)}) = \langle \vec{l}, \vec{m} \mid U^\dagger(g^{(n)}) \rho U(g^{(n)}) \mid \vec{l}, \vec{m}\rangle. \tag{212}$$

Formula (212) defines the tomogram in basis $\mid \vec{l}, \vec{m}\rangle$.

Now let us define the $U(n)$-tomogram using basis $\mid j_1, j_2, m_1, m_2\rangle$, i.e.,

$$w^{(j_1, j_2)}(m_1, m_2, g^{(n)}) = \langle j_1, j_2, m_1, m_2 \mid U^\dagger(g^{(n)}) \rho U(g^{(n)}) \mid j_1, j_2, m_1, m_2\rangle. \tag{213}$$

This tomogram is spin-tomogram [13] for $g^{(n)} \in U(2) \otimes U(2)$ subgroup of the group $U(n)$. Properties of this tomogram follow from its meaning to be joint probability distribution of two random spin projections m_1, m_2 depending on $g^{(n)}$ parameters.

One has normalization condition

$$\sum_{m_1, m_2} w^{(j_1, j_2)}(m_1, m_2, g^{(n)}) = 1. \tag{214}$$

Also all the probabilities are nonnegative, i.e.,

$$w^{(j_1,j_2)}(m_l, m_2, g^{(n)}) \geq 0. \tag{215}$$

Due to this, one has

$$\sum_{m_1,m_2} |w^{(j_1,j_2)}(m_l, m_2, g^{(n)})| = 1. \tag{216}$$

For spin-tomogram,

$$g^{(n)} \to \left(\vec{O}_1, \vec{O}_2\right) \tag{217}$$

and

$$w^{(j_1,j_2)}(m_l, m_2, g^{(n)}) \to w(m_1, m_2, \vec{O}_1, \vec{O}_2). \tag{218}$$

The separability and entanglement condition discussed in the previous section for bipartite spin-tomogram can be considered also from the viewpoint of the properties of $U(n)$-tomogram. If the two-spin $n\times n$ density matrix ρ is separable, it keeps to be separable under action of generic positive map of the subsystem density matrices. This map can be described as follows.

Let ρ to be mapped onto vector $\vec{\rho}$ with n^2 components. The components are simply ordered rows of the matrix ρ, i.e.,

$$\vec{\rho} = \Big(\rho_{11}, \rho_{12}, \ldots, \rho_{1n}, \rho_{21}, \rho_{22}, \ldots, \rho_{nn}, \Big). \tag{219}$$

The $n^2\times n^2$ matrix L is taken in the form

$$L = \sum_s p_s L_s^{(j_1)} \otimes L_s^{(j_2)}, \qquad p_s \geq 0, \quad \sum_s p_s = 1, \tag{220}$$

where $n_1\times n_1$ matrix $L_s^{(j_1)}$ and $n_2\times n_2$ matrix $L_s^{(j_2)}$ describe the positive maps of density matrices of spin-j_1 and spin-j_2 subsystems, respectively. We map vector $\vec{\rho}$ onto vector $\vec{\rho}_L$

$$\vec{\rho}_L = L\vec{\rho} \tag{221}$$

and construct the $n^2\times n^2$ matrix ρ_L, which corresponds to the vector $\vec{\rho}_L$. Then we consider $U(n)$-tomogram of the matrix ρ_L, i.e.,

$$w_L^{(j_1,j_2)}(m_l, m_2, g^{(n)}) = \langle j_1, j_2, m_1, m_2 \mid U^\dagger(g^{(n)}\rho_L U(g^{(n)}) \mid j_1, j_2, m_l, m_2\rangle. \tag{222}$$

Using this tomogram we introduce the function

$$F(g^{(n)}, L) = \sum_{m_1,m_2} \left| w_L^{(j_1,j_2)}(m_1, m_2, g^{(n)}) \right|. \tag{223}$$

For separable states, this function does not depend on the $U(n)$-group parameter $g^{(n)}$ and positive-map matrix elements of the matrix L.

For normalized density matrix ρ of the bipartite spin-system, this function reads

$$F(g^{(n)}, L) = 1. \tag{224}$$

For entangled states, this function depends on $g^{(n)}$ and L and it is not equal to unity. This property can be chosen as necessary and sufficient condition for separability of bipartite spin-states. In fact, the formulated approach can be extended to multipartite systems too. The generalization is as follows.

Given N spin-systems with spins $j_1, j_2, \ldots, j_N$. Let us consider the group $U(n)$ with

$$n = \prod_{k=1}^{N} n_k, \qquad n_k = 2j_k + 1. \tag{225}$$

Let us introduce basis

$$\mid \vec{m}\rangle = \prod_{k=1}^{N} \mid j_k m_k\rangle \tag{226}$$

in the linear space of the fundamental representation of the group $U(n)$. We define now $U(n)$-tomogram of a state with $n^2 \times n^2$ matrix ρ:

$$w_\rho(\vec{m}, g^{(n)}) = \langle \vec{m} \mid U^\dagger(g^{(n)}) \rho U(g^{(n)}) \mid \vec{m}\rangle. \tag{227}$$

For positive Hermitian matrix ρ with $\operatorname{Tr}\rho = 1$, we formulate a criterion of separability as follows.

Let the map matrix L to be of the form

$$L = \sum_s p_s \Big(\prod_{k=1}^{N} \otimes L_s^{(k)} \Big), \qquad p_s \geq 0, \quad \sum_s p_s = 1, \tag{228}$$

where $L_s^{(k)}$ is positive-map matrix of the density matrix of kth spin subsystem. We construct the matrix ρ_L as in the case of bipartite system using the matrix L. The function

$$F(g^{(n)}, L) = \sum_{\vec{m}} |w_{\rho_L}(\vec{m}, g^{(n)})| \geq 1 \tag{229}$$

is equal to unity for separable state and it depends on the matrix L and $U(n)$-parameters $g^{(n)}$ for entangled states. This criterion can be applied also in the case of continuous variables, e.g., for Gaussian states of photons. Function (229) can provide the measure of entanglement. Thus one can use maximum

value (or a mean value) of this function as a characteristic of entanglement. In fact, the separability criterion is related to the following positivity criterion of finite or infinite matrix A. The matrix A is positive iff the sum of moduli of diagonal matrix elements of the matrix $UAU^\dagger$ is equal to positive trace of the matrix A for arbitrary unitary matrix U.

18 Conclusions

To conclude, we formulated the notion of separability and entanglement as a criterion for joint tomographic probability of subsystem states to be represented in the specific form of sum of products of tomograms of the subsystems.

We have shown that the positive map of density matrix of multiparticle system expressed in terms of superoperator acting in Lie algebra (adjoint representation) of unitary group can be considered as a semigroup, which contains all local unitary transforms acting in subspaces corresponding to the subsystem states.

The set of separable states is shown to be invariant under action of this group.

The intrinsic measure of entanglement is shown to be invariant under action of the local group.

The formalism of vectors representing the matrices is convenient tool for the consideration. We introduced unitary spin tomogram and formulated necessary and sufficient condition of entanglement.

Acknowledgments

V I M and E C G S thank Dipartimento di Scienze Fisiche, Universitá "Federico II" di Napoli and Istitito Nazionale di Fisica Nucleare, Sezione di Napoli for kind hospitality. V I M is grateful to the Russian Foundation for Basic Research for partial support under Project No. 01-02-17745.

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