THE FUNDAMENTAL THEOREM ON THE RELATION BETWEEN SPIN AND STATISTICS

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Abstract

The fundamental theorem on the connection of spin and statistics is deduced from the basic symmetry between emission and absorption processes in quantum field theory. The new theorem which prescribes the physical connection of Bose fields with integral spin and Fermi fields with half-integral spin, is valid for all field theories; and it contains Pauli's theorem as a special case.

I. Introduction

It is one of the most remarkable features of elementary particle physics that all known particles obey either Bose or Fermi statistics and that there is a fundamental connection between the spins of particles and the type of statistics that they obey, the integral spin particles like the photon obeying Bose statistics and the half-integral spin particles obeying Fermi statistics. Our knowledge about the multitude of new particles like the mesons and baryons that have been discovered during the recent past strengthens this spin-statistics relation. It would indeed be satisfying to be able to deduce this relation from the basic assumptions of quantum theory. In the present paper we state and prove a fundamental theorem within a very general quantum-theory formulation, to the effect that all integral spin particles must satisfy Bose statistics and half-integral spin particles should obey Fermi statistics.

There have been previous works on this topic by several writers, notably W. Pauli (1940, 1955). Pauli's work presupposes relativistic field theory: it is not applicable to a non-relativistic theory like the second quantized non-relativistic field which is so useful in the study of many-body systems. It would be good to be able to obtain the spin-statistics connection without recourse to the imposition of relativistic invariance, since the spin-statistics connection is of paramount importance in many cases where relativity seems to play no discernible role, like in many-body physics. Even within the framework of relativistic local field theories Pauli had to make the purely technical assumption that the fields belong to finite-dimensional representations of the Lorentz group. While the electron, the neutrino, and the photon are well described by such fields it is by no means clear that this is necessarily true of the subnuclear particles. Pauli's work has been refined and extended to general field theory in recent years (*see*, for example, Schweber, 1961), but these extensions also suffer from the limitations mentioned above.

In the present investigation we show that the principle of symmetry between emission and absorption processes characteristic of quantum theories already leads to the spin-statistics relation in *all* cases. In particular it implies that the second-quantized Schrodinger field describes bosons of integral spin or fermions of half-integral spin.

II. Preliminary Motions

(i) *Symmetry between emission and absorption processes: The S-principle:* —Our experience with the interactions of the photon and the electron tell us that a quantum-mechanical process which involves the emission of a particle must of necessity imply the existence of a related process of absorption of the antiparticle. Thus the interaction responsible for the emission of a photon implies an interaction of the same strength responsible for the absorption of the photon. The nuclear beta decay interaction, in which a positron is emitted entails an interaction with the same coupling constant, the nuclear K capture, in which an electron is absorbed. We shall refer to this property of symmetry between particle emission and antiparticle absorption processes as the S-principle of quantum mechanics.

It is to be noted that relativistic finite-component fields contain both positive and negative frequency parts so that particle absorption and antiparticle emission are described by the same field operator. The use of a local (finite-component) relativistic field to describe the system automatically imposes the S-principle.

The S-principle is by no means restricted to relativistic fields: even in non-relativistic theories we expect it to hold good. The imposition of the S-principle is equivalent to the validity of the Substitution Law and of Crossing Symmetry.

We shall insist that the S-principle remain valid in our quantum theory but we shall otherwise leave the nature of the field sufficiently general. We show below that any quantized field satisfying the S-principle must have the correct spin-statistics relation. For the purpose of this paper we shall confine our attention to Lagrangian field theories.

(ii) *Lagrangian field theories.*—Without loss of generality we can construct a Lagrangian appropriate to first order equations of motion for the quantized field $\psi_r(x)$. Then the Lagrangian density may be written in the form

$$\mathrm{L}(x) = \frac{i}{2}\Gamma_{rs}\left\{\psi_r^\dagger(x)\frac{\partial\psi_s(x)}{\partial t} - \frac{\partial\psi_r^\dagger(x)}{\partial t}\psi_s(x)\right\} + \mathrm{H}(x) \tag{1}$$

where H is a function of ψ, $\psi^\dagger$ and grad ψ, grad $\psi^\dagger$ only. The quantity Γ_{rs} is a numerical matrix. In a relativistic theory Γ will be one of a set of four vector matrices Γ^μ, but no such restrictions are imposed in the general case. However, rotational invariance demands that $\psi_r^\dagger\Gamma_{rs}\psi_s$ transform as a scalar under three-dimensional rotations.

The Lagrangian density (1) would lead to satisfactory equations of motion whether ψ and $\psi^\dagger$ satisfy commutation or anti-commutation relations. In fact, both the commutation relations and the equations of motion can be derived from the Weiss-Schwinger Action Principle:

$$i\,\delta\psi_r(x) = [\psi_r(x), \delta\mathrm{A}] \tag{2}$$

$$\mathrm{A} = \int d^4x\, \mathrm{L}(x). \tag{3}$$

For Bose fields we take the field variations $\delta\psi$ to commute with the field operators and we obtain

$$\delta\psi_r(x) = \int [\psi_r(x), \psi_s^\dagger(y)]\,\Gamma_{st}\,\delta\psi_t(y)\,\delta(x^0 - y^0)d^4y \tag{4}$$

implying

$$\delta(x^0 - y^0)\,[\psi_r(x), \psi_s^\dagger(y)] = (\Gamma^{-1})_{rs}\,\delta(x - y) \tag{5}$$

provided Γ is non-singular. Similarly for Fermi fields we take the field variations $\delta\psi$ to anticommute with the field operators and we obtain:

$$\delta\psi_r(x) = \int \{\psi_r(x), \psi_s^\dagger(y)\}\,\Gamma_{st}\,\delta\psi_t(y)\,\delta(x^0 - y^0)\,d^4y \tag{6}$$

implying

$$\delta(x^0 - y^0)\,\{\psi_r(x), \psi_s^\dagger(y)\} = (\Gamma^{-1})_{rs}\,\delta(x - y). \tag{7}$$

By considering variations which vanish at end points we can get equations of motion from (2),

III. The Connection between Spin and Statistics

Let us now turn to the discussion of the S-principle and the spin-statistics connection for quantum field theory in various cases:

(i) *Relativistic finite component fields.*—Let $\psi(x)$ be a field which transforms as a finite-dimensional representation of the homogeneous Lorentz group and which satisfies a covariant wave equation. It then follows that $\psi(x)$ should contain both positive and negative frequency solutions, to be associated with annihilation and creation operators for particles and anti-particles respectively. The existence of an equal number of the two kinds of solutions can be deduced from the fact that covariant wave equations retain their form not only under the usual (real) homogeneous Lorentz transformations but also under Complex Lorentz transformations. For finite dimensional representations of the Lorentz group these complex Lorentz transformations are implemented by finite matrices which can in particular be defined for the 'strong reflection' transformation:

$$x \longrightarrow -x.$$

The strong reflection transformation can be obtained as a real element of the complex Lorentz group which corresponds to rotation through π in the 1–2 plane and Lorentz rotation through $i\pi$ in the 3–0 plane. But under such a transformation a "plane wave" solution labelled by the momentum k will be taken into another plane wave solution labelled by the momentum $-k$. This establishes our contention that for a covariant equation, (not necessarily of the first degree in either the derivatives or the fields but consisting entirely of fields which furnish finite dimensional representations of the Lorentz group), for every solution we can find a partner which has its four-momentum reversed.

It follows that the S-principle is automatically satisfied by any quantized relativistic field which furnishes a finite dimensional representation of the Lorentz group. We now wish to demonstrate that all these fields satisfy the correct spin-statistics relation. This is the result of Pauli, but we shall derive it in a somewhat simpler fashion.

For a covariant relativistic field Γ is the time-component of a matrix "four-vector" Γ^μ. For the Dirac spinor the matrices Γ^μ are the same as the α^μ: where $\alpha^\circ = 1$; $\alpha = \beta\gamma$ and there exists a representation (the "Majorana" representation) in which all the α^μ are real and symmetric, while the "scalar" matrix β is antisymmetric. In this representation the spinor fields may be chosen to be real, let us do so. It follows that for any multispinor which transforms as the product of an odd number of Dirac spinors, the matrix four-vector Γ^{μ} would be symmetric and the matrix scalar be antisymmetric; and for any multispinor which transforms as the product of an even number of Dirac spinors, the matrix four-vector Γ^{μ} would be antisymmetric and the matrix scalar would be symmetric (Johnson and Sudarshan, 1961). This is consistent with the known symmetry properties of the matrices in the wave equations for spin 0, $\frac{1}{2}$, 1.

For real fields equations (5) and (7) require that Γ be, respectively, antisymmetric and symmetric for Bose and Fermi fields. But Γ is antisymmetric for integral spin fields and symmetric for half-integral spin fields. This establishes the familiar relation between spin and statistics.

(ii) *Relativistic infinite component fields.*—If $\phi(x)$ is an infinite component field which satisfied a covariant wave equation, it does not follow that it contains both positive and negative frequencies. The proof of the existence of both signs of the frequencies does not hold in the case of infinite dimensional representations. It is possible that if k is an allowed four-momentum $-k$ is not an allowed four-momentum; the time-like solutions of the Majorana equations provide one such example.

In the case of infinite component relativistic wave equations the S-principle is not automatically satisfied. But we now require that it be satisfied by demanding that free field equations satisfied by the field $\phi(x)$ contain a solution $u'(x)$ with the momentum $-k$ if it contains a solution $u(x)$ with the momentum k. We further demand that the interchange of the components $u'(x)$ and $u(x)$ should leave the Lagrangian unaltered except for a sign change. The interchange of $u'(x)$ and $u(x)$ with opposite frequencies should be associated with a change in sign of the free field terms in the Lagrangian with first-order time derivatives. Such a requirement is automatically satisfied for the case of the finite component relativistic wave equations.

As the precise expression of the S-principle we demand: The field must be decomposable into two parts

$$\phi(x) = \chi(x) \oplus \chi'(x) \tag{8}$$

such that for every solution with momentum k in χ there exists a solution with momentum $-k$ in χ'; and the action must remain unchanged when $\chi(x)$ and $\chi'(-x)$ are interchanged.

This S-principle is automatically valid for finite component relativistic fields. For example for the Dirac field (in the Dirac representation of the gamma matrices) we have

$$L(x) = \frac{1}{2}\left(\psi^{\dagger}\frac{\partial\psi}{\partial t} - \frac{\partial\psi^{\dagger}}{\partial t}\psi\right) + H$$

so that with the choice

$$\chi(x) = \psi(x); \quad \chi'(x) = i\gamma_5\psi^{\dagger T}(x)$$

the conditions for the validity of the S-principle are fulfilled. Similarly for the scalar field in the Duffin-Kemmer-Petiau form we have

$$L(x) = \frac{1}{2}\left(\phi^{\circ}\frac{\partial\phi}{\partial t} - \frac{\partial\phi^{0}}{\partial t}\phi\right) + H(x).$$

With the choice

$$\chi = \frac{1}{\sqrt{2}}(\omega^{\frac{1}{2}}\phi + i\omega^{-\frac{1}{2}}\phi^{0})$$

$$\chi' = \frac{1}{\sqrt{2}}(\omega^{\frac{1}{2}}\phi - i\omega^{-\frac{1}{2}}\phi^{0})$$

where

$$(\omega^{\pm\frac{1}{2}}\phi)(x) = (2\pi)^{-3}\int d^3k\,(m^2+k^2)^{\pm\frac{1}{2}}\int e^{ik(x-y)}\phi(y)\,\delta(x^0-y^0)\,d^4y.$$

We could satisfy the S-principle.

Returning to infinite components fields, since the system is rotationally invariant χ will furnish a representation, reducible in general, of the rotation group. So will χ'. The Lagrangian (1) can be written in terms of χ and χ' in the form

$$L(x) = \tfrac{1}{2}\{\chi'^{T}(x)\,\eta\dot{\chi}(x) - \dot{\chi}'^{T}(x)\,\eta\chi(x)\} + H(x). \qquad (9)$$

The fundamental matrix η that appears in (9) in such that $L(x)$ is an invariant with respect to rotations in space. By a suitable choice of representation η can therefore be reduced to a block-diagonal form, each block corresponding to an irreducible representation of the rotation group. Within each block η would be a multiple of the identity for tensor fields and an antisymmetric matrix for spinor fields. This is an elementary property of the invariant bilinear form for the representations of the rotation group. Hence we conclude:

$$\eta^{T} = \begin{cases} +\eta & : \text{ tensor fields} \\ -\eta & : \text{ spinor fields} \end{cases} \qquad (10)$$

as a consequence of rotation invariance. The imposition of the S-principle now implies that the Lagrangian (9) can be written in the form:

$$L(x) = \tfrac{1}{4}\{\chi'^{T}(x)\,\eta\dot{\chi}(x) - \dot{\chi}'^{T}(x)\,\eta\,\chi(x) - \chi^{T}(x)\,\eta\chi'(x) + \dot{\chi}^{T}(x)\,\eta\chi'(x)\} + H(x). \tag{11}$$

This leads to the commutation relations

$$\tfrac{1}{2}\left(\eta_{sn}\,[\chi'_r(x),\ \chi_n(y)] - \eta_{ns}\,[\chi_n(y),\ \chi'_s(x)]\right)\delta(x^0 - y^0) = \delta_{rs}\delta(x - y) \tag{12}$$

provided we consider Bose fields. But this relation is inconsistent if η is antisymmetric: consistency demands that for commutation relations we must have, according to (10), tensor fields.

In a similar fashion for Fermi fields we deduce the anticommutation relations

$$\tfrac{1}{2}\left(\eta_{sn}\,\{\chi'_r(x),\ \chi_n(y)\} - \eta_{ns}\,\{\chi_n(y),\ \chi'_r(x)\}\right)\delta(x^0 - y^0) = \delta_{rs}\delta(x - y) \tag{13}$$

which demands, for consistency, that η be antisymmetric. According to (10) we must then have quantization by anticommutators only for spinor fields.

We have thus deduced for infinite component fields also the normal spin-statistics relation. Integral spin (tensor) fields obey Bose statistics and half integral spin (spinor) fields obey Fermi statistics. It is interesting that essentially only rotational invariance and the antisymmetry of the kinetic terms in the Lagrangian have been used in deducing the result. To have a consistent Lagrangian density we must have the normal connection between spin and statistics: integral spin for Bose fields and half-integral spin for Fermi fields.

In the recent literature statements have appeared to the effect that the normal spin-statistics connection may not or cannot be realized. Scrutiny of the "quantization" in these cases shows that the symmetry between emission and absorption processes embodied in the S-principle is not realized in these pathological treatments.

(iii) *Non-relativistic quantized Schrodinger fields.*—We now show that proper quantization of the Schrodinger equation to get a non-relativistic second-quantized field also leads to the physical spin-statistics connection. This is a very important result because it has been believed that this connection could be obtained only in a relativistic theory. The argument in this case is very similar to the case of the relativistic infinite component fields (where again Pauli's method fails but where we were able to establish the result, on the basis of the S-principle).

For the second quantized Schrodinger field with spin, the S-principle is not automatically satisfied. We demand the symmetry between emission and absorption, so that the field contains positive and negative frequency solutions in pairs by requiring that the field $\psi(x)$ be decomposable in the form

$$\psi(x) = \chi(x) \oplus \chi'(x). \tag{14}$$

Such that for every solution with momentum k in $\chi(x)$ there exists a solution with momentum $-k$ in $\chi'(x)$: and that the action remain unchanged under the interchange of $\chi(x)$ and $\chi'(-x)$. The Lagrangian density should then be of the form

$$\begin{aligned} L(x) = \tfrac{1}{4}\{\chi'^{T}(x)\,\zeta\dot{\chi}(x) - \dot{\chi}'^{T}(x)\,\zeta\chi(x) - \chi^{T}(x)\,\zeta\dot{\chi}'(x) \\ + \dot{\chi}^{T}(x)\,\zeta\chi'(x)\} + H(x). \end{aligned} \tag{15}$$

The fundamental matrix ζ is such that $\chi'^{T}\zeta\chi$ is an invariant with respect to rotations. Again, ζ can be reduced to block-diagonal form, each block corresponding to irreducible representations of the rotation group. Within each such block we would have

$$\begin{aligned} \zeta^{T} = +\zeta \quad &: \text{ tensor fields} \\ -\zeta \quad &: \text{ spinor fields.} \end{aligned} \tag{16}$$

From this point onwards the derivation is identical with that of the previous section. We conclude that integral spin (tensor) fields must satisfy Bose statistics and half-integral spin (spinor) fields must satisfy Fermi statistics.

(iv) *The fundamental theorem.*—We are now able to assert the following theorem: "In all rotationally invariant quantum field theories which maintain the symmetry between emission and absorption processes, Bose fields must have integral spin and Fermi fields must have half-integral spin."

IV. Discussion

In the preceding discussion we have confined attention to the study of more conventional quantum field theories and obtained the *fundamental theorem on the connection between spin and statistics* from the general principles of quantum theory including the symmetry between emission and absorption (the S-principle). This theorem goes beyond the work of Pauli and its refinements in that it *applies equally well to all quantum field theories in which "spin" is defined*: namely all rotationally invariant quantum field theories. No special mention is made of fields with only time-like or light-like states, since such a qualification was unnecessary. It can be shown that the theorem holds for theories with space-like states also; in particular for quantum theory of tachyon (faster-than-light) fields. In a paper in preparation I have formulated the quantum theory of the infinite-component Majorana fields where the field is quantized uniformly according to Bose statistics for the integral spin case and according to Fermi statistics for the half-integral spin case.

It is very satisfying to have the technical restrictions implicit in the statement of Pauli's theorem eliminated. Since the spin-statistics connection was true even for non-relativistic fields for which Pauli's theorem did not apply, it was only to be anticipated that when the theorem was properly deduced these artificial restrictions would be removed. In this paper this important task has been accomplished.

It is remarkable that there is a deep connection between the symmetry between emission and absorption which is characteristic of quantum field theory and the "statistics" obeyed by the field. We have stated this connection in terms of the S-principle within the Lagrangian formulations of quantum field theory, but it is possible to transcribe it into a requirement on the transition amplitude (the "S-matrix") under "crossing". This transcription may make our fundamental theorem appeal to a wider audience, but in my opinion such a formulation obscures the underlying physical principles.

In passing we may also note that while the four-momentum of a particle-state of a quantized field depends in an essential manner on its interactions, the spin (or rather, the distinction between integral spin and half-integral spin) of the particle states is independent of the interactions: it is an intrinsic property of the field. The statistics depends only on this intrinsic property of the field and no requirement is imposed on the sign of the energy density or any such quantity which depends on the interaction structure.

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