

# ON THE RELATION BETWEEN SCHRÖDINGER AND VON NEUMANN EQUATIONS

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## Abstract

Relation of the density matrix obeying von Neumann equation and the wave function obeying Schrödinger equation is discussed in connection with the superposition principle of quantum states. The definition of the ray-addition law is given and its relation to the addition law of vectors in Hilbert space of states and the role of a constant phase factor of the wave function is elucidated. The superposition law of density matrices, Wigner functions, and tomographic probabilities describing quantum states in the probability representation of quantum mechanics is studied. Examples of spin-1/2 and Schrödinger-cat states of the harmonic oscillator are discussed. The connection of the addition law with the entanglement problem is considered.

## 1 Introduction

Pure state of a quantum system is described by a wave function obeying the linear Schrödinger evolution equation [1] or by a vector in Hilbert space of states [2]. For the description of mixed states, the notion of density matrix was introduced by von Neumann [3] (see also [4]). This density matrix for a pure state is equal to the product of the wave function and its complex conjugate (at different argument). The replacement of the wave function by the density matrix can be considered as nonlinear (quadratic) change of variables and due to such nonlinearity a constant phase factor of the wave function disappears in the density-matrix representation of pure quantum state. The density matrix obeys to the linear von Neumann evolution equation (quantum Liouville equation). It is a common statement that the constant phase factor of a

wave function is irrelevant for physical applications. This statement is reformulated by saying that the quantum state is described by a ray and not by a vector in Hilbert space. A set of rays has no obvious linear structure in contradistinction to Hilbert space.

The wish of making the description of quantum states more similar to the classical-mechanics description has led to introducing the notion of quasidistributions in the phase space by means of Wigner functions [5]. Also the Husimi–Kano function [6, 7] was introduced and this function takes only nonnegative values. In connection with quantum optics, a quasidistribution function with singularities was introduced in [8], which is also called P-distribution function [9].

The quasidistributions discussed are not the joint-probability distributions of position and momentum. Due to the uncertainty relation, the existence of such joint-probability distribution is forbidden in quantum mechanics since such dynamical variables cannot be measured simultaneously.

Recently, in connection with the tomographic procedure of measuring quantum states [10–12] the probability representation of quantum mechanics was suggested [13], in which the quantum state is described by a positive probability distribution of the position, which depends, in addition, on parameters of the phase-space reference frame, in which the position is considered. Since the probability-distribution function depends only on the position and is independent of the conjugated variable (momentum), the uncertainty relation is consistent with the existence of such probability density. Both the quasidistributions and tomographic probabilities have the common property of being determined by the density matrix and therefore they do not depend on a constant phase factor of the vector in Hilbert space of pure quantum state, i.e., they depend on the ray.

On the other hand, the essence of quantum mechanics is the superposition principle of pure quantum states [2]. The constant phase-factors of the Hilbert-space vectors play an essential role when one makes the addition of two or more vectors. For example, the relative phase of the two vectors is important in superposing them giving different kinds of the Schrödinger cats, either of the type of even and odd coherent states [14] or of the type of cats created in the Kerr medium [15].

The problem, we want to discuss in this paper, is how the superposition of quantum states is described in the formalism of the density operator or of the Wigner functions or other quasidistributions.

In particular, this problem has to be discussed in connection with the tomographic-probability representation of the quantum state. The tomographic probability is nothing else as the linear integral transform of the density matrix. But the superposition of two pure quantum states means a rule of combining two tomographic probabilities, i.e., the specific addition law of two probabilities in the set of all probabilities.

The aim of our paper is to give the definition of adding the density operators for the superposition of both pure and impure states. We also elucidate the superposition of two tomographic probabilities using their addition rule corresponding to the interference of quantum states. From this viewpoint, we want to study the relation between two linear equations, namely, the Schrödinger equation for the wave function and von Neumann equation for the density operator. As our analysis shows, the Schrödinger and von Neumann descriptions of pure states are nonequivalent. The density matrices contain the same information as the wave functions only if one formulates the addition law of the pure density matrices describing the superposition of quantum states. In this sense, the constant phase factor of the wave function is essential. The both descriptions become equivalent if the addition law of the pure density matrices (rays) is given. The usual addition law of solutions of the linear Schrödinger equation for wave functions yields the wave function corresponding to pure superposition state. The usual addition law of pure solutions of the linear von Neumann equation for a density matrix yields the density matrix corresponding to impure quantum states. We would like to demonstrate that there exists another one-parametric addition law of pure solutions of the linear von Neumann equation which yields the density matrix of pure state. In this paper, we consider the addition law in different representations of the density matrix including the Wigner function and the tomographic probability.

The paper is organized as follows.

In Sec. 2, we formulate the superposition principle in terms of vectors in Hilbert space of states. In Sec. 3, we discuss the properties of rays. An example of the ray-description of spin state is given in Sec. 4. In Sec. 5, we review the properties of density matrices. In Sec. 6, we consider the Wigner function of the quantum state. A review of the state description in terms of tomographic probabilities for continuous variable (position) is done in Sec. 7 and for discrete variable (spin), in Sec. 8. The superposition rule for density matrices (called “ $\varphi$ -additional law”) in the coordinate representation is formulated in Sec. 9 and the superposition of pure states in terms of the Wigner function is considered in Sec. 10. In Sec. 11, the  $\varphi$ -addition law of tomographic probabilities is given. In Sec. 12, we study the relation between the linear Schrödinger and von Neumann equations in connection with the superposition principle for pure quantum states. In Sec. 13, the formula for the  $\varphi$ -addition of two nonorthogonal projector operators is derived and the superposition principle in terms of rays is discussed. The example of even and odd Schrödinger cat states is studied in Sec. 14 and of 1/2-spin states, in Sec. 15. The main results of our study are summarized in Sec. 15.

## 2 State Vectors

In the conventional quantum mechanics, pure state is described by a vector  $|\psi\rangle$  in Hilbert space. The eigenvector  $|q\rangle$  of the position operator  $\hat{q}$

$$\hat{q}|\psi\rangle = q|\psi\rangle \quad (1)$$

is used to determine the wave function  $\psi(q)$  of the state  $|\psi\rangle$  in the coordinate representation

$$\psi(q) = \langle q|\psi\rangle. \quad (2)$$

The superposition principle states that, for given quantum states  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , the state

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \quad (3)$$

is also realized. In (3), the complex numbers  $c_1$  and  $c_2$  are considered as given,

$$c_1 = |c_1|e^{i\theta_1}, \quad c_2 = |c_2|e^{i\theta_2}. \quad (4)$$

If the states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are normalized and orthogonal, the state  $|\psi\rangle$  is also normalized if

$$|c_1|^2 + |c_2|^2 = 1. \quad (5)$$

For the generic situation, the normalized superposition state  $|\psi\rangle$  is given by the formula

$$|\psi\rangle = \frac{c_1}{N_{12}}|\psi_1\rangle + \frac{c_2}{N_{12}}|\psi_2\rangle, \quad (6)$$

where the normalization factor is

$$N_{12}^2 = |c_1|^2 + |c_2|^2 + c_1 c_2^* \langle \psi_2 | \psi_1 \rangle + c_1^* c_2 \langle \psi_1 | \psi_2 \rangle. \quad (7)$$

For the wave function  $\psi(q)$  of the generic superposition state, one has the addition law

$$\psi(q) = \frac{c_1}{N_{12}}\psi_1(q) + \frac{c_2}{N_{12}}\psi_2(q). \quad (8)$$

## 3 Rays

The expectation values of physical observables are bilinear in state vectors. In view of this, the expectation values are insensitive to a change of the phase factor of the pure-state vector

$$|\psi\rangle \longrightarrow |\psi'\rangle = e^{i\phi}|\psi\rangle. \quad (9)$$

It is a common statement that a state in quantum mechanics is described by a ray in Hilbert space. The ray can be associated with a representative vector in Hilbert space, which we denote

with  $|\psi\rangle_r$ . The ray-representative vector  $|\psi\rangle_r$  is common for all the vectors given by Eq. (9). Let us choose the ray-representative vector  $|\psi\rangle_r$  using the convention that the number

$$\langle 0 | \psi \rangle_r = \psi_r(0) \quad (10)$$

is a real nonnegative one.

It implies that, for given wave function  $\psi(q)$  of the state  $|\psi\rangle$ , the ray-representative vector is described by the wave function  $\psi_r(q)$  given by the formula

$$\psi_r(q) = \psi(q) \frac{|\langle 0 | \psi \rangle|}{\langle 0 | \psi \rangle}. \quad (11)$$

The choice of the ray-representative vector  $|\psi\rangle_r$  due to Eq. (10) is not translationally invariant. One could use other choices, for example, considering the state's wave function in the momentum representation.

## 4 Two-Dimensional Hilbert Space

As another example, we consider the rays for spin state of spin  $j = 1/2$ .

The generic normalized spin state is described by the vector  $|\psi\rangle$  in two-dimensional Hilbert space

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a = \left\langle \frac{1}{2}, \frac{1}{2} \middle| \psi \right\rangle, \quad b = \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \psi \right\rangle. \quad (12)$$

We used base vectors  $|j, m\rangle$  with spin projection  $m = \pm 1/2$ . In (12), the complex numbers

$$a = |a| e^{i\varphi_a}, \quad b = |b| e^{i\varphi_b}$$

satisfy the normalization condition

$$|a|^2 + |b|^2 = 1. \quad (13)$$

We determine the ray-representative vector

$$|\psi\rangle_r = \begin{pmatrix} |a| \\ b e^{-i\varphi_a} \end{pmatrix}, \quad \langle \psi |_r = (|a| \quad b^* e^{i\varphi_a}). \quad (14)$$

So one has the relation of the vector (12) with the ray-representative vector

$$|\psi\rangle = e^{i\varphi_a} |\psi\rangle_r. \quad (15)$$

## 5 Density Matrix

For pure state  $|\psi\rangle$ , the density operator reads

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi| \quad (16)$$

and the density matrix in the coordinate representation is

$$\rho_\psi(q, q') = \langle q | \hat{\rho}_\psi | q' \rangle = \psi(q)\psi^*(q') . \quad (17)$$

For given density operator  $\hat{\rho}_\psi$ , the state vector can be obtained up to arbitrary phase factor. It implies that the ray-representative vector  $|\psi\rangle_r$ , which is insensitive to the phase factor, can be determined by the density operator. In fact, using (11) one has the wave function of the ray-representative vector

$$\psi_r(q) = \frac{\rho_\psi(q, 0)}{\sqrt{\rho_\psi(0, 0)}} . \quad (18)$$

Since the set of states  $|q\rangle$  is complete, i.e.,

$$\int |q\rangle\langle q| dq = \hat{1} , \quad (19)$$

one has the invariant expression for the ray-representative vector in terms of the density operator of pure state

$$|\psi\rangle_r = \int dq |q\rangle \frac{\langle q | \hat{\rho}_\psi | 0 \rangle}{\sqrt{\langle 0 | \hat{\rho}_\psi | 0 \rangle}} . \quad (20)$$

The positive density operator  $\hat{\rho}_\psi$  of pure state satisfies the relations determining projector operator

$$\hat{\rho}_\psi^\dagger = \hat{\rho}_\psi , \quad \hat{\rho}_\psi^2 = \hat{\rho}_\psi , \quad \text{Tr } \hat{\rho}_\psi = 1 . \quad (21)$$

The density matrix of pure state has rank equal unity and the unique nonzero eigenvalue equals to unity.

The density operator  $\hat{\rho}$  of impure state is characterized by the impurity parameter

$$\mu_0 = \text{Tr } \hat{\rho}^2 < 1 \quad (22)$$

and it has rank larger than unity. If  $\mu_0 = 1$  the state is pure.

One can introduce other parameters

$$\mu_k = \text{Tr } \hat{\rho}^{k+2} , \quad k = 0, 1, 2, 3, \dots \quad (23)$$

For impure state, one has

$$\mu_0 < \mu_1 < \mu_2 < \dots < 1 .$$

For discrete spin observable, relation (14) determining the ray-representative vector can be rewritten in the form

$$\langle j, m | \psi \rangle_r = \frac{\langle j, m | \hat{\rho}_\psi | j, j \rangle}{\sqrt{\langle j, j | \hat{\rho}_\psi | j, j \rangle}} , \quad (24)$$

where

$$j = \frac{1}{2} ; \quad m = \pm \frac{1}{2} .$$

In invariant form, one has the ray-representative vector  $|\psi\rangle_r$  in terms of the density operator of pure state  $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$ , namely,

$$|\psi\rangle_r = \frac{1}{2j+1} \sum_{m=-j}^j |j, m\rangle \frac{\langle j, m | \hat{\rho}_\psi | j, j \rangle}{\sqrt{\langle j, j | \hat{\rho}_\psi | j, j \rangle}}. \quad (25)$$

Formula (25) is valid also in the case of arbitrary spin  $j$ . This formula for spin states is analogous to formula (20) for the position-observable case.

## 6 Wigner Function

The Wigner quasidistribution function is introduced in terms of the density matrix as

$$W(q, p) = \int \rho \left( q + \frac{u}{2}, q - \frac{u}{2} \right) e^{-ipu} du, \quad (26)$$

where the Planck constant is taken as  $\hbar = 1$ .

For pure state, the Wigner function  $W_\psi(q, p)$  is expressed in terms of the wave function

$$W_\psi(q, p) = \int \psi \left( q + \frac{u}{2} \right) \psi^* \left( q - \frac{u}{2} \right) e^{-ipu} du. \quad (27)$$

The density matrix of pure state is determined by the inverse of Eq. (26) as

$$\rho_\psi(q, q') = \frac{1}{2\pi} \int W_\psi \left( \frac{q+q'}{2}, p \right) e^{ip(q-q')} dp. \quad (28)$$

But the wave function  $\psi(q)$  is determined by the Wigner function  $W_\psi(q, p)$  up to a phase factor. The wave function  $\psi_r(q)$  corresponding to the ray-representative vector  $|\psi\rangle_r$  can be obtained explicitly in terms of the Wigner function  $W_\psi(q, p)$  in the form

$$\psi_r(q) = \frac{1}{\sqrt{2\pi}} \frac{\int W_\psi \left( \frac{q}{2}, p \right) e^{ipq} dp}{\sqrt{\int W_\psi(0, p) dp}}. \quad (29)$$

For impure state, the parameter  $\mu_k$  (23) is expressed in terms of the Wigner function

$$\begin{aligned} \mu_k &= \text{Tr} \hat{\rho}^{k+2} \\ &= 2^{-(k+3)} \pi^{-(k+2)} \int W(q_1, p_1) W(q_2, p_2) \cdots W(q_{k+2}, p_{k+2}) \\ &\quad \times \exp \left[ 2i \sum_{n,m=1}^{k+2} p_m c_{mn} q_n \right] dq_1 \cdots dq_{k+2} dp_1 \cdots dp_{k+2}. \end{aligned} \quad (30)$$

where the  $(k+2) \times (k+2)$ -matrix  $c_{mn}$  reads

$$c_{mn} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \end{pmatrix}.$$

## 7 Tomographic Probability for Pure State

Pure state  $|\psi\rangle$  can be described by the tomographic-probability distribution  $w_\psi(X, \mu, \nu)$  of the position  $X$  measured in an ensemble of the different reference frames in the phase space labeled by the real parameters  $\mu$  and  $\nu$  due to the relation

$$X = \mu q + \nu p. \quad (31)$$

The parameters of the linear canonical transform (31)  $\mu$  and  $\nu$  can be considered as a combination of rotation and scaling of the initial reference frame in the phase space

$$\mu = e^\lambda \cos \theta, \quad \nu = e^{-\lambda} \sin \theta. \quad (32)$$

The tomographic probability  $w_\psi(X, \mu, \nu)$  is related to the wave function  $\psi(q)$  by the relation [16, 17]

$$w_\psi(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(q) \exp \left( \frac{i\mu}{2\nu} q^2 - \frac{iXq}{\nu} \right) dq \right|^2. \quad (33)$$

For the generic impure state with the Wigner function  $W(q, p)$ , the tomographic probability  $w(X, \mu, \nu)$  is given by the relation [12]

$$w(X, \mu, \nu) = \frac{1}{2\pi} \int W(q, p) \delta(X - \mu q - \nu p) dq dp. \quad (34)$$

The inverse of this relation is [12]

$$W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp [i(X - \mu q - \nu p)] dX d\mu d\nu. \quad (35)$$

For normalized pure and impure states,

$$\int W(q, p) \frac{dq dp}{2\pi} = 1, \quad (36)$$

and the tomographic probability is normalized for any  $\mu$  and  $\nu$

$$\int w(X, \mu, \nu) dX = 1. \quad (37)$$

The function  $w(X, \mu, \nu)$  is nonnegative

$$w(X, \mu, \nu) \geq 0 \quad (38)$$

for any reference frame in the phase space with parameters  $\mu$  and  $\nu$ .

This function is homogeneous function [18]

$$w(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|} w(X, \mu, \nu). \quad (39)$$

Since the probability distribution  $w(X, \mu, \nu)$  determines the Wigner function  $W(q, p)$ , it can be used for a complete description of quantum states.

The density matrix is also determined in terms of the tomographic probability

$$\rho(q, q') = \frac{1}{2\pi} \int w(Y, \mu, q - q') \exp \left[ i \left( Y - \mu \frac{q + q'}{2} \right) \right] d\mu dY. \quad (40)$$

Due to this formula, the tomographic probability  $w_\psi(X, \mu, \nu)$  can be used to obtain the ray-representative vector for pure state  $|\psi\rangle_r$ . In view of Eqs. (18), (40), one has

$$\psi_r(q) = \frac{1}{\sqrt{2\pi}} \frac{\int w_\psi(Y, \mu, q) \exp \left[ i \left( Y - \frac{\mu q}{2} \right) \right] d\mu dY}{\sqrt{\int w_\psi(Y, \mu, 0) e^{iY} d\mu dY}}. \quad (41)$$

For pure state, one has the equality

$$\frac{1}{2\pi} \int w_\psi(X, \mu, \nu) w_\psi(Y, -\mu, -\nu) e^{i(X+Y)} d\mu d\nu dX dY = 1, \quad (42)$$

which follows from the corresponding property of the Wigner function  $W_\psi(q, p)$  of pure state

$$\int W_\psi^2(q, p) \frac{dq dp}{2\pi} = 1. \quad (43)$$

## 8 Tomographic Probabilities for Spin State

The tomographic probability was introduced also for spin states [19, 20]. For given density matrix of spin  $j$ -state

$$\rho_{mm'}^{(j)} = \langle j, m | \hat{\rho}^{(j)} | j, m' \rangle, \quad m, m' = -j, -j + 1, \dots, j - 1, j, \quad (44)$$

the tomographic probability  $w(m, \alpha, \beta)$  is given by the formula

$$w(m, \alpha, \beta) = \sum_{m_1=-j}^j \sum_{m_2=-j}^j \mathcal{D}_{mm_1}^{(j)} \rho_{m_1 m_2}^{(j)} \mathcal{D}_{mm_2}^{(j)*}, \quad (45)$$

where we use the matrix elements of the rotation-group representation

$$\mathcal{D}_{m_1 m_2}^{(j)} = e^{im_1 \gamma} d_{m_1 m_2}^{(j)}(\beta) e^{im_2 \alpha}, \quad (46)$$

with

$$d_{m_1 m_2}^{(j)}(\beta) = \left[ \frac{(j+m_1)!(j-m_1)!}{(j+m_2)!(j-m_2)!} \right]^{1/2} \left( \cos \frac{\beta}{2} \right)^{m_1+m_2} \left( \sin \frac{\beta}{2} \right)^{m_1-m_2} P_{j-m_1}^{(m_1-m_2, m_1+m_2)}(\cos \beta). \quad (47)$$

In (45) and (46),  $\alpha$ ,  $\beta$ , and  $\gamma$  are the Euler angles and in (47)  $P_n^{(a,b)}$  is the Jacobi polynomial. Due to the structure of (45), the tomographic probability  $w(m, \alpha, \beta)$  does not depend on the angle  $\gamma$ . In fact, this probability is the function of the parameters  $\alpha$  and  $\beta$  on the sphere  $S^2$ , being also the function of a random spin projection  $m$ . The inverse relation of the density matrix in terms of the tomographic probability has the form [20]

$$\begin{aligned} \rho_{m'_1 m'_2}^{(j)} &= (-1)^{m'_2} \sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} (2j_3+1)^2 \sum_{m_1=-j}^j \int (-1)^{m_1} w(m_1, \alpha, \beta) \mathcal{D}_{0m_3}^{(j_3)}(\alpha, \beta, \gamma) \\ &\quad \times \begin{pmatrix} j & j & j_3 \\ m_1 & -m_1 & 0 \end{pmatrix} \begin{pmatrix} j & j & j_3 \\ m'_1 & -m'_2 & m_3 \end{pmatrix} \frac{d\omega}{8\pi^2}, \end{aligned} \quad (48)$$

where

$$\int d\omega = \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta \, d\beta \int_0^{2\pi} d\gamma, \quad (49)$$

and we used  $3j$ -symbols according to [21]. Using (24) one can express the ray-representative vector  $|\psi\rangle_r$  of a pure spin state  $|\psi\rangle$  in terms of the tomographic probability.

The example of the density matrix of pure spin state is the Hermitian matrix with two angle parameters

$$0 \leq \theta < \pi, \quad 0 \leq \gamma < 2\pi,$$

which determine the point on the sphere  $S^2$

$$\rho_\psi = \begin{pmatrix} \cos^2 \theta & e^{i\gamma} \cos \theta \sin \theta \\ e^{-i\gamma} \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}. \quad (50)$$

The ray-representative vector corresponding to this density matrix for  $0 \leq \theta < \pi/2$  reads

$$|\psi\rangle_r = \begin{pmatrix} \cos \theta \\ e^{-i\gamma} \sin \theta \end{pmatrix} \quad (51)$$

and the tomographic probability  $w(m, \alpha, \beta)$  for this density matrix is

$$w\left(\frac{1}{2}, \alpha, \beta\right) = \cos^2 \frac{\beta}{2} \cos^2 \theta + \sin^2 \frac{\beta}{2} \sin^2 \theta + \cos \frac{\beta}{2} \sin \frac{\beta}{2} \cos(\alpha + \gamma) \sin 2\theta \quad (52)$$

and

$$w\left(-\frac{1}{2}, \alpha, \beta\right) = 1 - w\left(\frac{1}{2}, \alpha, \beta\right). \quad (53)$$

## 9 Superposition Principle in Terms of Density Matrix

Given two orthonormal pure states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  with density matrices  $\rho_{\psi_1}(q, q')$  and  $\rho_{\psi_2}(q, q')$ , let us consider their superposition (3). We address the question: how the addition law for the density matrices can be formulated in order to take into account the superposition principle, which is easily expressed in terms of the usual addition law of state vectors in Hilbert space?

To get the answer, we calculate the density matrix of the superposed state given by Eq. (3). One has

$$\rho_\psi(q, q') = |c_1|^2 \rho_{\psi_1}(q, q') + |c_2|^2 \rho_{\psi_2}(q, q') + c_1 c_2^* \psi_1(q) \psi_2^*(q') + c_1^* c_2 \psi_1^*(q) \psi_2(q'). \quad (54)$$

In order to express the density matrix in terms of only density matrices  $\rho_{\psi_1}$  and  $\rho_{\psi_2}$ , we express the wave functions  $\psi_1(q)$  and  $\psi_2(q)$  in terms of the wave functions of the ray-representative vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$ . Using Eq. (11) one obtains

$$\rho_\psi(q, q') = |c_1|^2 \rho_{\psi_1}(q, q') + |c_2|^2 \rho_{\psi_2}(q, q') + [c_1 c_2^* e^{i\varphi} \psi_{1r}(q) \psi_{2r}^*(q') + \text{c.c.}] , \quad (55)$$

where the phase  $\varphi$  is the relative phase of the vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$ ,

$$e^{i\varphi} = \frac{\langle 0 | \psi_1 \rangle}{|\langle 0 | \psi_1 \rangle|} \frac{|\langle 0 | \psi_2 \rangle|}{\langle 0 | \psi_1 \rangle} . \quad (56)$$

On the other hand, the functions  $\psi_{1r}(q)$  and  $\psi_{2r}(q')$  are expressed in terms of density matrices by (18). Thus, we have

$$\rho_\psi(q, q') = |c_1|^2 \rho_{\psi_1}(q, q') + |c_2|^2 \rho_{\psi_2}(q, q') + \left[ c_1 c_2^* e^{i\varphi} \frac{\rho_{\psi_1}(q, 0) \rho_{\psi_2}^*(q', 0)}{\sqrt{\rho_{\psi_1}(0, 0) \rho_{\psi_2}(0, 0)}} + \text{c.c.} \right] . \quad (57)$$

The phase  $\varphi$  is not contained in the density matrices  $\rho_{\psi_1}(q, q')$  and  $\rho_{\psi_2}(q, q')$ .

Thus, we formulate the  $\varphi$ -addition law of two pure density matrices given by Eqs. (55), which corresponds to the superposition of quantum states using additional parameter. We derive the one-parametric  $\varphi$ -addition law, which provides the density matrix of pure state as result of “summing” the density matrices of two pure states.

One should point out that the density matrix

$$\rho'(q, q') = |c_1|^2 \rho_{\psi_1}(q, q') + |c_2|^2 \rho_{\psi_2}(q, q') \quad (58)$$

is the density matrix of impure state.

The  $\varphi$ -addition law can be called purification of density matrix (58), which is obtained by the standard mixture of two pure density matrices.

## 10 Purification of Wigner Function

We discuss the purification of the density matrix obtained by the superposition of two pure states in terms of the Wigner function. Given two Wigner functions  $W_{\psi_1}(q, p)$  and  $W_{\psi_2}(q, p)$  satisfying the condition

$$\int W_{\psi_{1,2}}^2(q, p) \frac{dq dp}{2\pi} = 1. \quad (59)$$

Let us have the Wigner function

$$W'(q, p) = |c_1|^2 W_{\psi_1}(q, p) + |c_2|^2 W_{\psi_2}(q, p), \quad (60)$$

where the complex numbers  $c_1$  and  $c_2$  satisfy the normalization condition (5). One can check that for the generic case the purity  $\mu'$  of the state (60)

$$\int W'^2(q, p) \frac{dq dp}{2\pi} = \mu' < 1. \quad (61)$$

The quasidistribution (60) corresponds to the mixture of two pure states that provides impure state. Employing the result of the previous section, we perform the purification of the Wigner function applying the following  $\varphi$ -addition law. We construct the Wigner function using (29)

$$W_\varphi(q, p) = |c_1|^2 W_{\psi_1}(q, p) + |c_2|^2 W_{\psi_2}(q, p) + 2 \operatorname{Re} \left\{ c_1 c_2^* e^{i\varphi} W_{12}(q, p) \right\}, \quad (62)$$

where

$$\begin{aligned} W_{12}(q, p) &= \frac{1}{2\pi} \left[ \int W_{\psi_1}(0, p_1) W_{\psi_2}(0, p_2) dp_1 dp_2 \right]^{-1/2} \\ &\quad \times \int du dp_1 dp_2 W_{\psi_1} \left( \frac{q+u/2}{2}, p_1 \right) W_{\psi_2} \left( \frac{q-u/2}{2}, p_2 \right) \\ &\quad \times \exp \left\{ i \left[ p_1 \left( q + \frac{u}{2} \right) - p_2 \left( q - \frac{u}{2} \right) \right] \right\}. \end{aligned} \quad (63)$$

Formulas (62) and (63) give the  $\varphi$ -addition law of Wigner functions that corresponds to the purification of the impure Wigner function (60).

## 11 Purification of Tomographic Probabilities

In this section, we address the problem of the purification of tomographic probabilities.

If one has two pure states with tomographic probabilities  $w_{\psi_1}(X, \mu, \nu)$  and  $w_{\psi_2}(X, \mu, \nu)$ , correspondingly, the tomographic probability

$$w'(X, \mu, \nu) = |c_1|^2 w_{\psi_1}(X, \mu, \nu) + |c_2|^2 w_{\psi_2}(X, \mu, \nu) \quad (64)$$

can be constructed. One can check that the tomographic probability  $w'(X, \mu, \nu)$  gives the impurity parameter

$$\int w'(X, \mu, \nu) w'(Y, -\mu, -\nu) e^{i(X+Y)} dX dY d\mu d\nu = \mu' < 1. \quad (65)$$

This means that the sum rule for the probabilities given by (64) gives the mixed-states' tomographic probability.

We address the question of the state purification in terms of tomographic probabilities which means that we formulate the superposition of quantum states by the special law of adding the probabilities. This  $\varphi$ -addition law of probabilities can be formulated using Eq. (41). One has

$$w_\varphi(X, \mu, \nu) = |c_1|^2 w_{\psi_1}(X, \mu, \nu) + |c_2|^2 w_{\psi_2}(X, \mu, \nu) + 2 \operatorname{Re} \left\{ c_1 c_2^* e^{i\varphi} w_{12}(X, \mu, \nu) \right\}, \quad (66)$$

where

$$\begin{aligned} w_{12}(X, \mu, \nu) = & \frac{1}{4\pi^2|\nu|} \left[ \int w_{\psi_1}(Y_1, \mu_1, 0) w_{\psi_2}(Y_2, \mu_2, 0) e^{i(Y_1-Y_2)} d\mu_1 d\mu_2 dY_1 dY_2 \right]^{-1/2} \\ & \times \int \exp \left\{ i \left[ Y_1 - \frac{\mu_1 q_1}{2} - Y_2 + \frac{\mu_2 q_2}{2} + \frac{\mu}{2\nu} (q_1^2 - q_2^2) - \frac{X}{\nu} (q_1 - q_2) \right] \right\} \\ & \times w_{\psi_1}(Y_1, \mu_1, q_1) w_{\psi_2}(Y_2, \mu_2, q_2) d\mu_1 d\mu_2 dY_1 dY_2 dq_1 dq_2. \end{aligned} \quad (67)$$

The  $\varphi$ -addition law (66), (67) corresponds to the superposition of pure quantum states described in terms of tomographic probabilities.

## 12 Linear Quantum Evolution Equations and Superposition Principle

Basic equations of quantum mechanics in arbitrary representation are the Schrödinger evolution equation for the wave function  $\psi$

$$i\dot{\psi} = H\psi; \quad \hbar = 1 \quad (68)$$

and the von Neumann evolution equation for the density matrix  $\rho$

$$\dot{\rho} + i[H, \rho] = 0, \quad (69)$$

where  $H$  is the Hamiltonian of the system.

Both equations are linear. Due to linearity of Eq. (68), the superposition of two solutions with complex coefficients

$$\psi = c_1 \psi_1 + c_2 \psi_2 \quad (70)$$

is again a solution of Eq. (68).

This mathematical property corresponds to the interference picture known for the superposition of two pure quantum states.

On the other hand, due to linearity of Eq. (69), the linear combination of two density matrices  $\rho_1$  and  $\rho_2$ , which are solutions of the equation, is also a solution of Eq. (69)

$$\rho = p_1 \rho_1 + p_2 \rho_2; \quad p_1 + p_2 = 1. \quad (71)$$

The important point is that the density matrix and the wave function are related each other. Due to this, the superposition (70) of wave functions implies a combination of the density matrices of pure states  $\rho_{\psi_1}$  and  $\rho_{\psi_2}$ , which must be a solution of Eq. (69). This combination for orthogonal pure states is given by the one-parameter addition rule

$$\rho_{12\varphi} = |c_1|^2 \rho_{\psi_1} + |c_2|^2 \rho_{\psi_2} + e^{i\varphi} c_1 c_2^* \rho_{12} + e^{-i\varphi} c_1^* c_2 \rho_{21}, \quad (72)$$

where the matrices  $\rho_{21}$  and  $\rho_{12}$  are given by the ray-representative vectors

$$\rho_{21} = \psi_{2r} \psi_{1r}^*, \quad \rho_{12} = \psi_{1r} \psi_{2r}^*. \quad (73)$$

The term  $|c_1|^2 \rho_{\psi_1} + |c_2|^2 \rho_{\psi_2}$  satisfies the evolution equation (69). The phase-dependent interference term, which itself is not density matrix, also satisfies (69). The sum of the two terms (71) provides the combination of solutions of the evolution equation (69), which does not coincide with the rule (72), (73). We call the linear combination (71), which is a solution of (69), the mixture of the density-matrix solutions.

We call linear combination (72) the superposition of two density matrices. Thus, we observe on the example of the quantum linear evolution equation an interesting mathematical phenomenon. There exist two different addition rules for the solutions of the same equation, which provide a new solution.

One addition is a standard addition of two solutions, which provides a mixture of two solutions. Another addition is a one-parameter  $\varphi$ -addition, which involves extra interference term dependent on the external relative phase of vectors in Hilbert space. The Hilbert-space structure is intrinsically present in the density-matrix formalism and in the evolution equation (69). In fact, one can consider eigenvectors of the density operator as a basis of Hilbert space. The superposition of these vectors provides the  $\varphi$ -addition rule for the density operators, i.e., obtaining the new solutions by means of the  $\varphi$ -addition of two initial solutions. Nevertheless, it is not clear for a given linear mathematical equation, which is not associated with the density-matrix formalism of quantum mechanics, whether this linear equation possesses only standard linear combination of solutions (mixture) or nontrivial parameter-dependent superposition similar to the interference term of quantum density matrix in Eq. (72).

## 13 Addition Law for Rays

In this section, we formulate the addition law for non-orthogonal rays. Given two rays  $|\psi_1\rangle_r$  and  $|\psi_2\rangle_r$ , which are represented by normalized wave functions  $\psi_{1r}(q)$  and  $\psi_{2r}(q)$  in the coordinate representation. One has for ray-representative vectors in Hilbert space of states

$$|\psi_1\rangle_r = \int dq \ |q\rangle \psi_{1r}(q), \quad |\psi_2\rangle_r = \int dq \ |q\rangle \psi_{2r}(q), \quad (74)$$

and for non-orthogonal rays a complex number

$$\langle\psi_2|\psi_1\rangle_r = \int \psi_{2r}^*(q) \psi_{1r}(q) dq \quad (75)$$

is given. We want to formulate an analog of the superposition law of two vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$  in Hilbert space in terms of rays. This means that for given complex numbers  $c_1$  and  $c_2$  with the condition

$$|c_1|^2 + |c_2|^2 = 1,$$

we have to obtain the expression for the wave function  $\psi_r(q)$  corresponding to the ray of the normalized superposition vector  $|\psi\rangle$

$$|\psi\rangle = [c_1|\psi_1\rangle + c_2|\psi_2\rangle] N_{12}^{-1}.$$

The normalization constant  $N_{12}$  depends on the relative phase of the vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$ .

The  $\varphi$ -additional law of the two rays  $\psi_{1r}(q)$  and  $\psi_{2r}(q)$  yields the wave function

$$\begin{aligned} \psi_r(q) &= \left[ \frac{|c_1|^2 \psi_{1r}^2(0) + |c_2|^2 \psi_{2r}^2(0) + 2 \operatorname{Re} c_1 c_2^* e^{i\varphi} \psi_{1r}(0) \psi_{2r}(0)}{1 + 2 \operatorname{Re} c_1 c_2^* e^{i\varphi} \langle\psi_2|\psi_1\rangle_r} \right]^{-1/2} \\ &\times \left[ \frac{c_1 e^{i\varphi} \psi_{1r}(q) + c_2 \psi_{2r}(q)}{c_1 e^{i\varphi} \psi_{1r}(0) + c_2 \psi_{2r}(0)} \right]. \end{aligned} \quad (76)$$

For  $q = 0$ , the functions  $\psi_{1r}(q)$  and  $\psi_{2r}(q)$  take nonnegative real values:

$$\psi_{1r}(0) = \langle 0|\psi_1\rangle_r, \quad \psi_{2r}(0) = \langle 0|\psi_2\rangle_r. \quad (77)$$

In invariant form, the addition law for two ray-representative vectors reads

$$\begin{aligned} |\psi\rangle_r &= \left[ \frac{|c_1|^2 \langle 0|\psi_1\rangle_r^2 + |c_2|^2 \langle 0|\psi_2\rangle_r^2 + 2 \operatorname{Re} c_1 c_2^* e^{i\varphi} \langle 0|\psi_1\rangle_r \langle 0|\psi_2\rangle_r}{1 + 2 \operatorname{Re} c_1 c_2^* e^{i\varphi} \langle\psi_2|\psi_1\rangle_r} \right]^{1/2} \\ &\times \left[ \frac{c_1 e^{i\varphi} |\psi_1\rangle_r + c_2 |\psi_2\rangle_r}{c_1 e^{i\varphi} \langle 0|\psi_1\rangle_r + c_2 \langle 0|\psi_2\rangle_r} \right]. \end{aligned} \quad (78)$$

One can see that the normalization constant  $N_{12}$  [denominator in the first factor in (78)] for the  $\varphi$ -additional law of two non-orthogonal rays depends on the angle parameter  $\varphi$ . Since the projector operators are expressed in terms of rays

$$\hat{P}_1 = |\psi_1\rangle_r \langle\psi_1|_r; \quad \hat{P}_2 = |\psi_2\rangle_r \langle\psi_2|_r; \quad \hat{P}_\varphi = |\psi\rangle_r \langle\psi|_r, \quad (79)$$

the  $\varphi$ -addition law (78) is equivalent to the  $\varphi$ -addition law of two projectors  $\hat{P}_1$  and  $\hat{P}_2$ . The projectors coincide with the density operators of the pure states. In view of this property, the  $\varphi$ -addition law of projectors for orthogonal states  $(\hat{P}_1\hat{P}_2 = 0)$  can be presented in the form of Eq. (57).

Now we present the  $\varphi$ -addition law corresponding to Eq. (78) in invariant form. The projector  $\hat{P}_\varphi$  reads

$$\hat{P}_\varphi = \frac{|c_1|^2 \hat{P}_1 + |c_2|^2 \hat{P}_2 + (c_1 c_2^* e^{i\varphi} \hat{P}_1 \hat{P}_0 \hat{P}_2 + \text{h.c.}) [\text{Tr}(\hat{P}_1 \hat{P}_0) \text{Tr}(\hat{P}_2 \hat{P}_0)]^{-1/2}}{1 + [c_1 c_2^* e^{i\varphi} \text{Tr}(\hat{P}_1 \hat{P}_0 \hat{P}_2) + \text{c.c.}] [\text{Tr}(\hat{P}_1 \hat{P}_0) \text{Tr}(\hat{P}_2 \hat{P}_0)]^{-1/2}}. \quad (80)$$

In Eq. (80), the operator

$$\hat{P}_0 = |0\rangle\langle 0| \quad (81)$$

with matrix elements in the coordinate representation

$$\langle q | \hat{P}_0 | q' \rangle = \delta(q) \delta(q') \quad (82)$$

is used; the use of the operator (81) is connected with our convention to choose the ray-representative vector due to the relation

$$\langle 0 | \psi \rangle_r = [\text{Tr}(\hat{P}_0 \hat{P}_\psi)]^{1/2}. \quad (83)$$

For orthogonal states  $(\hat{P}_1 \hat{P}_2 = 0)$ , one obtains

$$\hat{P}_\varphi = |c_1|^2 \hat{P}_1 + |c_2|^2 \hat{P}_2 + (c_1 c_2^* e^{i\varphi} \hat{P}_1 \hat{P}_0 \hat{P}_2 + \text{h.c.}) [\text{Tr}(\hat{P}_1 \hat{P}_0) \text{Tr}(\hat{P}_2 \hat{P}_0)]^{-1/2}. \quad (84)$$

For  $j$ -spin states, the  $\varphi$ -additional law of projectors (80) holds by the replacement of the operator  $\hat{P}_0$  with the projector

$$\hat{P}_j = |j, j\rangle\langle j, j|. \quad (85)$$

Thus, for the  $j$ -spin case, one has

$$\hat{P}_\varphi = \frac{|c_1|^2 \hat{P}_1 + |c_2|^2 \hat{P}_2 + (c_1 c_2^* e^{i\varphi} \hat{P}_1 |j, j\rangle\langle j, j| \hat{P}_2 + \text{h.c.}) [\text{Tr}(\hat{P}_1 \hat{P}_j) \text{Tr}(\hat{P}_2 \hat{P}_j)]^{-1/2}}{1 + [c_1 c_2^* e^{i\varphi} \text{Tr}(\hat{P}_1 |j, j\rangle\langle j, j| \hat{P}_2 + c_1 c_2^* e^{-i\varphi} \text{Tr}(\hat{P}_2 |j, j\rangle\langle j, j| \hat{P}_1)] [\text{Tr}(\hat{P}_1 \hat{P}_j) \text{Tr}(\hat{P}_2 \hat{P}_j)]^{-1/2}}. \quad (86)$$

One can check that, for arbitrary projectors  $\hat{P}_1$  and  $\hat{P}_2$ , the Hermitian operator  $\hat{P}_\varphi$  given by (80) [or by (86) for the spin case] satisfies the relations

$$\hat{P}_\varphi^2 = \hat{P}_\varphi; \quad \text{Tr} \hat{P}_\varphi^2 = 1.$$

## 14 Example of Even and Odd Coherent States

In this section, we consider an example of even and odd Schrödinger cats (even and odd coherent states) as a result of the purification of a mixture of two coherent states.

The coherent state  $|\alpha\rangle$  of the harmonic oscillator ( $\hbar = m = \omega = 1$ ) is the normalized eigenfunction of the annihilation operator  $\hat{a}$  (see [22])

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad (87)$$

where  $\alpha$  is a complex number.

There exist the superpositions of the coherent states

$$|\alpha_+\rangle = \frac{1}{\sqrt{2(1+2e^{-2|\alpha|^2})}} (|\alpha\rangle + |-\alpha\rangle) \quad (88)$$

and

$$|\alpha_-\rangle = \frac{1}{\sqrt{2(1-2e^{-2|\alpha|^2})}} (|\alpha\rangle - |-\alpha\rangle). \quad (89)$$

Also one has the mixture of two coherent states with density operators

$$\hat{\rho}' = \frac{1}{2} (|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|). \quad (90)$$

Within the framework developed, the even (88) and odd (89) coherent states can be considered as the result of the purification of impure state (90). In view of the fact that coherent states  $|\alpha\rangle$  and  $|-\alpha\rangle$  are non-orthogonal, in order to obtain the density operators of pure states

$$|\alpha_{\pm}\rangle\langle\alpha_{\pm}| = \hat{\rho}_{\pm\alpha}, \quad (91)$$

one has to apply formula (80) where

$$\hat{P}_1 \longrightarrow \hat{\rho}_{\alpha} = |\alpha\rangle\langle\alpha|, \quad \hat{P}_2 \longrightarrow \hat{\rho}_{-\alpha} = |-\alpha\rangle\langle-\alpha|, \quad c_1 = c_2 = \frac{1}{\sqrt{2}},$$

with the angle  $\varphi$  to be taken equal zero for even coherent state and  $\pi$  for odd coherent state. In fact, in this case in formula (80)

$$\hat{P}_1 \hat{P}_0 \hat{P}_2 [\text{Tr}(\hat{P}_1 \hat{P}_0) \text{Tr}(\hat{P}_2 \hat{P}_0)]^{-1/2} = |\alpha\rangle\langle-\alpha|. \quad (92)$$

Taking into account that

$$\text{Tr} |\alpha\rangle\langle-\alpha| = \langle\alpha|-\alpha\rangle = e^{-2|\alpha|^2}, \quad (93)$$

one can reduce expression (80) to the form

$$\hat{\rho}_{\pm\alpha} = \frac{1}{2(1 \pm e^{-2|\alpha|^2})} [|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| \pm |\alpha\rangle\langle-\alpha| \pm |-\alpha\rangle\langle\alpha|], \quad (94)$$

which in turn coincides with the form of the density operators of the even and odd coherent states. Thus, we derive the density matrices of even and odd Schrödinger cats by the purification of the density matrix (90).

## 15 Example of Spin States

The generic pure state of 1/2-spin particle has the density matrix  $\rho_\psi$  (50). The  $\varphi$ -addition law of two non-orthogonal pure states with density matrices

$$\rho_{\psi 1,2} = \begin{pmatrix} \cos^2 \theta_{1,2} & e^{i\gamma_{1,2}} \cos \theta_{1,2} \sin \theta_{1,2} \\ e^{-i\gamma_{1,2}} \cos \theta_{1,2} \sin \theta_{1,2} & \sin^2 \theta_{1,2} \end{pmatrix} \quad (95)$$

can be obtained by using formula (80) where

$$\hat{P}_1 \longrightarrow \rho_{\psi 1}; \quad \hat{P}_2 \longrightarrow \rho_{\psi 2},$$

and the projector

$$\hat{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

One has

$$\text{Tr}(\hat{P}_1 \hat{P}_0) = \cos^2 \theta_1; \quad \text{Tr}(\hat{P}_2 \hat{P}_0) = \cos^2 \theta_2 \quad (96)$$

and

$$\frac{\text{Tr}(\hat{P}_1 \hat{P}_0 \hat{P}_2)}{\sqrt{\text{Tr}(\hat{P}_1 \hat{P}_0) \text{Tr}(\hat{P}_2 \hat{P}_0)}} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2. \quad (97)$$

Thus, for the superposition state, in the case

$$|c_1|^2 + |c_2|^2 = 1,$$

we obtain the density matrix  $\rho_\psi$  of the form (50) with parameters  $\theta$  and  $\gamma$  given by the relations

$$\cos^2 \theta = \frac{|c_1|^2 \cos^2 \theta_1 + |c_2|^2 \cos^2 \theta_2 + (c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}) \cos \theta_1 \cos \theta_2}{1 + (c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}) \cos(\theta_1 - \theta_2)} \quad (98)$$

and

$$\begin{aligned} \cos \theta \sin \theta e^{i\gamma} &= \left[ 1 + (c_1 c_2^* e^{i\varphi} + c_1^* c_2 e^{-i\varphi}) \cos(\theta_1 - \theta_2) \right]^{-1} \\ &\times \left[ |c_1|^2 \cos \theta_1 \sin \theta_1 e^{i\gamma_1} + |c_2|^2 \cos \theta_2 \sin \theta_2 e^{i\gamma_2} \right. \\ &\quad \left. + c_1 c_2^* e^{i\varphi} \cos \theta_1 \sin \theta_2 e^{i\gamma_2} + c_1^* c_2 e^{-i\varphi} \sin \theta_1 \cos \theta_2 e^{i\gamma_1} \right]. \end{aligned} \quad (99)$$

## 16 Entanglement and Purification of Product of Density Matrices

The density operator of a composite system  $AB$  with subsystems  $A$  and  $B$  may be chosen pure or impure. For a pure density operator  $\rho_{AB}$ , one can get the density operator  $\rho_A$  and  $\rho_B$  by partial trace operation

$$\rho_A = \text{Tr}_B(\rho_{AB}); \quad \rho_B = \text{Tr}_A(\rho_{AB}).$$

It is not necessary that  $A$  and  $B$  have the same dimensionality. Unless  $\rho_{AB}$  is a direct product of pure states of  $A$  and  $B$ , a pure  $\rho_{AB}$  yields impure  $\rho_A$  and  $\rho_B$ . But they will have the same rank  $n$  and the same nonnegative eigenvalues which sum up to unity. The density operator

$$\rho'_{AB} = \rho_A \otimes \rho_B \neq \rho_{AB}$$

is impure. Thus, the whole is greater than the parts: there is additional information in  $\rho_{AB}$ . These are the “entanglement” terms [1].

We purify the product  $\rho'_{AB}$  in the same way that we used before for the mixture of two density operators  $\rho_1$  and  $\rho_2$ . Here we have  $n$  such pure states mixed together and need  $(n - 1)$  phase angles  $\varphi_1 = 0, \varphi_2, \dots, \varphi_n$ . The diagonal form of  $\rho'_{AB}$  has only  $n$  nonzero diagonal elements  $\lambda_j$ ,  $j = 1, 2, \dots, n$ . We need to introduce the off-diagonal elements  $\sqrt{\lambda_j \lambda_k} e^{i(\varphi_j - \varphi_k)}$  in the  $(j, k)$  location. Note that while we have  $n(n - 1)/2$  off-diagonal terms, there are only  $(n - 1)$  phases  $\varphi_j$ .

The purification of the density matrix  $\rho'_{AB}$  we call as the  $\varphi$ -multiplication law of the density matrices  $\rho_A$  and  $\rho_B$ .

While purification is dependent on the phase angle, the form of the entanglement is constructed and the phase angles have to be obtained from other considerations.

## 17 Conclusions

The quantum mechanics can be formulated using for the notion of states' vectors in Hilbert states, density operators, and different types of quasidistributions.

The special type of representations uses tomographic probabilities. The superposition of states and the mixture of states are two different types of adding the solutions of the evolution equations for density matrices of the states. As it is shown in this work, to describe the pure-state superposition within the framework of the density-matrix formalism (in terms of tomographic probabilities), the one-parametric  $\varphi$ -addition rule of the states can be formulated. Employing this rule, one can work within the framework only of probabilities and take into account all the quantum interference effects.

The main result of our paper is the formulation of the  $\varphi$ -addition law for density operators of pure quantum states, which yields the rule of purification of the impure mixture of quantum states. This  $\varphi$ -addition law connects the Schrödinger and von Neumann descriptions of pure quantum states and provides a rule of the pure-state superposition in terms of density operators.

Another result of our paper consists in formulation of the purification rule of product of two density matrices that describes the phenomenon of entanglement.

We consider the superposition of two pure states which is described by one parameter in terms of the  $\varphi$ -addition law of two density matrices. If one considers a superposition of three (or

more) pure states, the number of independent angle parameters increases correspondingly. The purification of the mixture of two impure states can be done in view of the rule of purification of the mixture of several pure states using the corresponding number of the angle parameters.

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