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Conformally invariant field theories in two dimensions and corresponding statistical mechanical models*

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Abstract. Belavin, Zamlodochikov and Polyakov have recently proposed a class of conformally invariant field theories in two dimension with exactly determined rational critical indices. We establish a tentative identification of a subset of these theories in terms of the O(n) model and the q-state Potts model in 2-dimensions for appropriate n and q.

Keywords. Conformal invariance in field theory; statistical mechanics.

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Conformal invariance has proved to be a fruitful concept, both in relativistic field theory and in statistical mechanics. In the context of relativistic field theory, this invariance is a direct consequence of a traceless energy momentum tensor. In the context of statistical mechanics, Polyakov (1970) proposed that the correlation functions at the critical point possess conformal invariance over and above the usual scale invariance. The origin of this invariance is not entirely clear (Wolsky and Green 1974). Rather stringent consequences follow when conformal invariance is imposed on the Wilson-Kadanoff operator algebra (Wilson 1969; Kadanoff (1969). For instance, the functional form of the three-point function is severely constrained, and appears to be in conformity with known exact results (Fisher 1979).

In a recent work Belavin *et al* (1984) (BZP) have considered the consequences of conformal invariance in two dimensions (1-space, 1-time). A remarkable result of their investigation is that an infinite class of field theories exist for which the scale (or anomalous) dimensions of all the operators are known exactly! This implies that all the critical indices possible for these models are known. However, the identification of this abstract class of field theories with known models remains an open problem. We address ourselves to this very problem in the present work. BZP have remarked that one of the field theories considered by them has precisely the set of scaling indices of the 2-d Ising model. In this work, we consider two classes of well-studied models in statistical mechanics, the O(n) model and q-state Potts model, for which the predominant critical indices have been calculated recently (Nienhuis 1982, 1983; den Nijs 1979; Nienhuis *et al* 1980; Pearson 1980; den Nijs 1983; Black and Emery 1981; Nienhuis 1982; Wu 1982)

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in the form of explicit functions of n and q. We find that it is possible to identify, at least tentatively, an infinite subset of the field theories proposed by BZP with appropriate O(n) and q-state Potts models. At a deeper level, the aim of a microscopic theory should be to take the suggested models in statistical mechanics, to identify the appropriate quantum field theory in 1-space dimension*, and to explicitly check whether the algebraic structure proposed by BZP (highlighted below) follows. At the present stage, we believe that it is important to identify phenomenologically, the possible models, by actually comparing the critical indices of BZP with the conjectured values for the O(n) and q-state Potts models.

Let us first consider the field theories proposed by BZP. These authors suggest that the complete set of operators for a Wilson-Kadanoff operator expansion can be regarded as

$$\{A_i\} = \underset{n=1,\infty}{U} [\phi_n], \tag{1}$$

where $[\phi_n]$ is a (infinite) set called a conformal class. Each conformal class consists of an ancestor field $\phi_n(z,\bar{z})$ where $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$, and the descendants thereof. The ancestor field is characterized by a simple transformation law under the conformal transformation $z \to y \equiv y(z)$, y being analytic in z, such that

$$\phi_n(z,\bar{z}) \to \left[\frac{\mathrm{d}y(z)}{\mathrm{d}z}\right]^{\Delta_n} \left[\frac{\mathrm{d}\bar{y}(\bar{z})}{\mathrm{d}\bar{z}}\right]^{\bar{\Delta}_n} \phi_n(y,\bar{y}). \tag{2}$$

Here Δ_n and $\bar{\Delta}_n$ are the partial scale dimensions such that the net scale dimension of ϕ_n is $\Delta_n + \bar{\Delta}_n$. In order to obtain the descendants, consider the energy-momentum tensor $T^{\mu\nu}(z)$ with the properties $\partial_\mu T^{\mu\nu}(x) = 0$ and $T^\mu_\mu = 0$. Define

$$T(x) = T_{00}(x) - T_{11}(x) + 2i\, T_{01}(x),$$
$$\bar{T}(x) = T_{00}(x) - T_{11}(x) - 2i\, T_{01}(x). \tag{3}$$

It is well known that $T(z)$ may be regarded as the generators of the conformal group (Ferrara *et al* 1973). The ladder operators L_n are defined as

$$L_n(z) = \oint T(y)\,(y-z)^{n+1}\,\mathrm{d}y,$$
$$\bar{L}_n(\bar{z}) = \oint \bar{T}(\bar{y})\,(\bar{y}-\bar{z})^{n+1}\,\mathrm{d}\bar{y}, \tag{4}$$

for integer n, where the contours encircle z and $\bar{z}$ respectively. The descendants of ϕ_n are defined as

$$\phi_n^{[-m_1,\ldots,-m_N][-\bar{m}_1,\ldots,-\bar{m}_M]}(z,\bar{z}) = L_{-m_1} L_{-m_2} L_{-m_N} \bar{L}_{-\bar{m}_1} \ldots \bar{L}_{-\bar{m}_M} \phi_n(z,\bar{z}) \tag{5}$$

The scale dimension of the descendant is $\Delta_n + \bar{\Delta}_n + \Sigma m_i + \Sigma \bar{m}_i$. The ladder operators with $m > 0$ annihilate ϕ_n and we have an eigenvalue like equation

$$L_0 \phi_n = \Delta_n \phi_n. \tag{6}$$

The algebra of L_n's can be derived from that of $T^{\mu\nu}$'s and would appear to be model-dependent. However a remarkably universal algebra is found

$$[L_n(z), L_m(z)] = (n-m)L_{n+m}(z) + \frac{C}{12}(n^3 - n)\delta_{n+m,0}, \tag{7a}$$

*The standard route for this is through the commutation relation between the transfer matrix and a quantum Hamiltonian.

$$[L_n(z), \bar{L}_n(\bar{z})] = 0, \tag{7b}$$

with the "central charge" C as the only model-dependent constant. The algebra of $\bar{L}_n$ is essentially the same as in (7a). In model field theories, C has its origins in the Schwinger term in current commutators. The algebra in (7) is known in the context of dual models as the Virasoro algebra (Virasoro 1973; for reviews of dual models, see Jacob 1974).

The special models remarked upon by BZP arise when the constant C takes on particular values such that the conformal classes $[\phi_n]$ contain less fields than usual. The situation arises if the descendants of ϕ_n in a certain generation themselves possess the conformal transformation property of the ancestors equation (2), albeit with different scale indices. All the possible scale indices have been computed exactly for these cases. The resulting expressions due to V Kac are summarized below. With p integer, define

$$C = 1 - \frac{6}{p(p+1)}. \tag{8}$$

Let

$$\Delta(n,m) = [(pm-(p+1)n)^2 - 1]/4p(p+1) = \Delta(p-n, p+1-m) \tag{9}$$

with $1 \leqslant n \leqslant p-1$, $1 \leqslant m \leqslant p$. The scale dimensions of the ancestor fields $\phi_{n,m}$ are found to be

$$d_{n,m} = \Delta(n,m) + \bar{\Delta}(n,m), \tag{10}$$

and the "Lorentz spin" of the field is $\Delta(n,m) - \bar{\Delta}(n,m)$. $\bar{\Delta}$ is defined as in (9) but with different values n and m in general. Clearly operators with dimensions $d_{n,m}+r$ exist for arbitrary integer r.

We consider next the problem of identification of the models in terms of the $O(n)$ and q-Potts models. For the latter models there exist recent calculations for the thermal and magnetic field "eigenvalues" y_T and y_H which match the exact solutions, wherever known, and also the series data. In terms of y_T and y_H, all the usual thermodynamic exponents follow on using scaling and hyperscaling. For completeness we note the results (d = spatial dimension)

$$\begin{aligned} &\nu = 1/y_T;\ \alpha = 2 - d/y_T;\ \beta = (d-y_H)/y_T;\ \gamma = (2y_H - d)/y_T \\ &\delta = y_H/(d-y_H);\ \eta = d+2-2y_H. \end{aligned} \tag{11}$$

The scale dimensions of the spin (or order parameter) and energy operators are related to y_T and y_H through

$$\begin{aligned} d_S &\equiv d\,[\text{Spin}] = d - y_H, \\ d_E &\equiv d\,[\text{Energy}] = d - y_T. \end{aligned} \tag{12}$$

For the particular case $O(1)$ (or 2-Potts), the problem reduces to the Ising case where the scale dimensions of other operators are known (Kadanoff 1966) in addition to d_S and d_E, but in the general case there is no other information available, as far as we are aware.

The result for the $O(n)$ model (Nienhuis 1982, 1983) may be stated as follows. Let $1 \leqslant t \leqslant 2$ and $n \equiv -2\cos(2\pi/t)$, then

$$\begin{aligned} y_T &= 4 - 2t, \\ y_H &= 1 + \frac{3}{4t} + \frac{t}{4}. \end{aligned} \tag{13}$$

For the q-Potts model (den Nijs 1979; Nienhuis *et al* 1980; Pearson 1980; Black and Emery 1981; Nienhuis 1982; Wu 1982; den Nijs 1983) let $0 \leqslant u \leqslant 1$ and $u \equiv \frac{2}{\pi} \cos^{-1} \frac{\sqrt{q}}{2}$, then

$$y_T = 3(1-u)/(2-u),$$

$$y_H = \frac{1}{4}(3-u)(5-u)/(2-u). \tag{14}$$

The branch $-1 < u < 0$ gives rise to the same value of q and is believed to be relevant in the context of the tricritical point for the q-state Potts lattice gas (den Nijs 1979; Nienhuis *et al* 1980; Pearson 1980; Black and Emery 1981; Nienhuis 1982; Wu 1982; den Nijs 1983). In both cases we may eliminate the parameter (t or u) and find two parametric relations between d_S and d_E (on using (12)). Thus for the $O(n)$ case

$$8(2+d_E)d_S - d_E(4-d_E) = 0 \tag{15}$$

and for the q-Potts case

$$4(2-d_S)(1+d_E) - (d_E+2)(d_E+4) = 0. \tag{16}$$

We now consider the indices equation (10). It is not known a priori, as to which values of n and m correspond to the energy operator or the spin operator. We therefore scan all values of n and m for a fixed p (subject to $1 \leqslant n \leqslant p-1, 1 \leqslant m \leqslant p$) and try to satisfy the two parametric relations (15) and (16) separately. We set $\bar{\Delta}(n,m) = \Delta(n,m)$ in the following, which implies that all the operators possess Lorentz spin zero. (We have also examined the case $\bar{\Delta} \neq \Delta$ and comment on this later). Firstly let us discard the values $\Delta(1,1) = 0$ leading to $d_S = 0 = d_E$ (the gaussian model) which always satisfies (15) and (16).

O(n) model

The two cases p even or odd are distinct

(a) p-even: Equation (15) is satisfied provided

$$d_E = 2\Delta(3,1) = 2(p+2)/p,$$

$$d_S = 2\Delta(p/2, p/2) = (p^2-4)/(8p(p+1)). \tag{17}$$

This implies $t = 2(p+1)/p$ (> 2) and $n = 2\cos(\pi/p+1)$. The resultant t is outside the range $1 \leqslant t \leqslant 2$ and also $d_E > 2$ so that $\nu < 0$. These values of d therefore do not correspond to any sensible model and we discard this case.

(b) p-odd: Equation (15) is satisfied provided

$$d_E = 2\Delta(1,3) = 2(p-1)/(p+1),$$

$$d_S = 2\Delta(p/2-1/2, p/2+1/2) = \tfrac{1}{8}(p-1)(p+3)/p(p+1). \tag{18}$$

This implies $t = 2p/(p+1)$ and $n = 2\cos(\pi/p)$. The case $p = 3$ corresponds to the Ising model and has already been remarked upon by BZP.

q-Potts model

(a) p-even: Equation (16) is satisfied provided

$$d_E = 2\Delta(1,2) = (p-2)/[2(p+1)],$$

$$d_S = 2\Delta(p/2, p/2) = \tfrac{1}{8}(p^2-4)/p(p+1). \tag{19}$$

This implies that $u = -2/p$ and $q = 4\cos^2(\pi/p)$. As noted above $u < 0$ corresponds to the q-state Potts model with dilution.

(b) *p-odd*: Equation (16) is satisfied provided

$$\begin{aligned} d_E &= 2\Delta(2,1) = (p+3)/2p \\ d_S &= 2\Delta(p/2-1/2,\, p/2+1/2) = \tfrac{1}{8}(p-1)(p+3)/p(p+1). \end{aligned} \tag{20}$$

This corresponds to $u = 2/(p+1)$ and $q = 4\cos^2(\pi/p+1)$. The case $p = 5$ corresponds to $q = 3$ and has been noted already by Dotsenko according to BZP.

In addition to the indices listed above, we have also found other cases numerically. We checked all the nontrivial solutions for $p \leqslant 30$ for both the $O(n)$ and the q-Potts models on a computer. The other cases do not fall into any obvious systematic class and hence we cannot say very much about these. We allowed for the possibility that $\Delta \neq \bar{\Delta}$ in (10) and considered all possible pairs $\Delta(n',m')$ such that their difference (i.e. the Lorentz spin) is either an integer or a half integer. For the $O(n)$ model we find the following solutions (1) $p = 8$; $d_S = 2\Delta(4,4)$, $d_E = \Delta(5,4) + \Delta(3,4)$ $(n = 0)$ (2) $p = 27$; $d_S = 2\Delta(13,14)$, $d_E = 2\Delta(13,15)$ $(n = 2\cos(2\pi/7))$ (3) $p = 30$; $d_S = 2\Delta(15,15)$, $d_E = 2\Delta(11,10)$ $(n = 2\cos(17\pi/45))$. For the q-Potts model we find (1) $p = 3$; $d_S = 2\Delta(1,2)$, $d_E = \Delta(1,3) + \Delta(1,1)$ $(q = 4)$ (2) $p = 4$; $d_S = 2\Delta(2,2)$, $d_E = \Delta(3,1) + \Delta(1,1)$ $(q = 4\cos^2[2\pi/5])$ (3) $p = 14$; $d_S = 2\Delta(1,1)$, $d_E = \Delta(1,4) + \Delta(1,1)$ $(q = 0)$ (4) $p = 20$; $d_S = 2\Delta(8,8)$, $d_E = 2\Delta(7,8)$ $(q = 2)$ (5) $p = 21$; $d_S = 2\Delta(10,11)$, $d_E = 2\Delta(8,7)$ $(q = 4\cos^2(9\pi/42))$. Let us remark that the case (1) for $O(n)$ model and the case (3) for the Potts model correspond to the polymer and resistor network models, which are not covered by (18) and (19)*.

In conclusion we would like to make the following remarks (i) For an infinite set of values of C corresponding to odd p, we have found that amongst the set of scale dimensions possible for the field theory, there are two pairs which lie on the parametric families for the $O(n)$ and q-Potts models. (ii) The dimension for the order parameter is the *same* for the $O(n)$ model and the q-Potts model which correspond to a given p. If we accept that the BZP list of critical indices is complete, then it follows that the $O(n)$ model with $n = 2\cos(\pi/p)$ and the q-Potts model with $q = 4\cos^2(\pi/p+1)$ (for p odd) possesses a common set of critical indices. This tempts us to conjecture that these models are isomorphic, with a common operator representing the order parameter and an appropriate identification of other operators. (iii) There are presumably other operators apart from energy and spin in the theory corresponding to the indices found by BZP. (iv) For the case of even p, the q-Potts lattice gas family appears to give a realization to the field theory. (v) The values of q for which we find a correspondence are $q = 4\cos^2(\pi/2r)$ for integer r. This sets of q's is a subset of Beraha numbers (Beraha *et al* 1975; Tuette 1974) $q = 4\cos^2(\pi/r)$. The Beraha numbers are believed to be the limit points of the real roots of the chromatic polynomial in the n-colouring problem. We find this coincidence intriguing. We are studying the field theoretic formulation for the $O(n)$ and q-Potts models with the inverse problem in mind, viz to compute the central charge given to n and q, and hope to present the results soon.

After the completion of our work we became aware of two recent references relating to our work. Friedan *et al* (1984) have found results similar to those of BZP by

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*For $p = 15$, the critical indices $d_E = \frac{3}{5}$ and $d_S = 21/160$ can also be attained by taking $d_E = \Delta(7,7) + \Delta(7,9)$, $d_S = 2\Delta(7,8)$ instead of $d_E = 2\Delta(2,1)$ and $d_S = 2\Delta(7,8)$ as given by equation (20).

considering the constraints imposed by unitarity. They have remarked on the correspondence between the exponents of the tricritical Potts model and the Kac formula for $p = 4$ and 6. Dotsenko and Fateev (1984)* have considered the identification between the exponents of the $O(n)$ and q-Potts models for nonintegral n and q, with the Kac formula, also using the conjectures in Nienhuis (1982); den Nijs 1979 and Nienhuis *et al* (1980).

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*This preprint also contains the identification of the exponents of the $O(n)$ and q-state Potts models belonging to the regular series found by us. Our computer search, locates, in addition, the non-regular (or sporadic) classes.