THE ELASTIC BEHAVIOUR OF ISOTROPIC SOLIDS

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1. INTRODUCTION

IT is a classic result in the phenomenological theory of elasticity that *two* independent constants suffice to describe the stress-strain relationships for an isotropic solid. If, for example, the bulk modulus and the shear modulus of the material are known, Young's modulus and Poisson's ratio may be calculated therefrom. In the present paper, the ideas regarding the nature of the strains and stresses in solids on which the derivation of this result is based are critically examined and it is shown that they are untenable. A re-formulation of the phenomenological theory of elasticity not open to the same objections is then presented and its consequences are developed. It emerges that *three* independent constants are needed to describe the stress-strain relationships of an isotropic body; in particular, it is shown that the bulk modulus of the material cannot be evaluated from the experimental data for the velocities of propagation of longitudinal and transverse waves respectively in the solid and its density.

2. SOME GENERAL CONSIDERATIONS

As is well known, the elastic constants of solids can be determined independently by static and dynamic methods. The latter are based on measurements of the velocity of propagation of waves of different types in the material. In all studies of this nature we are clearly concerned with *heterogeneous* strains, in other words with strains which are not of the same magnitude throughout the solid at any given instant; clearly, there could be no wave-propagation if the strains were the same everywhere. On the other hand, in the static methods of measuring elastic constants the strains may be homogeneous or heterogeneous according to the nature of the experiment. The change in volume of a solid under hydrostatic pressure is a case of the first kind, while the twisting of a rod by couples applied at its two ends is clearly a case of heterogeneous strain. The examples cited are sufficient to show that any theory of elastic behaviour has necessarily to concern itself with *heterogeneous* strains; a theory which restricts itself to the consideration of homogeneous strains would be fundamentally incomplete.

The twisting of a rod by couples applied at its two ends also serves to illustrate certain fundamental aspects of the theory of elasticity. As just mentioned, it is an example of heterogeneous strain, and indicates that the movements of the parts of the solid in such strains may be angular movements or rotations, the magnitude of which varies through the volume of the solid. Thus, we are forced to recognize that the strains in a solid cannot, in general, be described solely as *elongations* but may also include *twists*. Further, in the case referred to, the external stresses applied to the body are *couples*. It follows that the internal stresses may also be of the same nature. In other words, the stresses in an elastic solid cannot be assumed to be exclusively in the nature of *tractive forces* but may also include *torques*.

The arguments in the classical theory of elasticity by which the familiar result quoted in the opening sentence of the paper are derived may be summed up briefly as follows: that it is sufficient to consider the case of homogeneous strains; that any homogeneous strain may be analysed into a "pure strain" and a rotation and that the latter should be ignored in formulating the stress-strain relationships; and finally that the tractive forces assumed to act on elements of area in the solid are so related that no torques tending to rotate the volume-elements of the solid are present. Everyone of these statements is at variance with the considerations set forth above. It follows that the argument with all its consequences is unacceptable.

3. FORMULATION OF THE THEORY

If now we denote by u_x, u_y, u_z the three components of the displacements of a point (x, y, z) of the material and by $u_x + u_x'$, $u_y + u_y'$, $u_z + u_z'$ the corresponding displacements of a neighbouring point situated at $(x + x', y + y', z + z')$, then it is a well-known result that the strains in the neighbourhood of the point (x, y, z) can be represented by the scheme of equations

$$
\begin{aligned}
u_x' &= u_{xx}x' + u_{xy}y' + u_{xz}z' \\
u_y' &= u_{yx}x' + u_{yy}y' + u_{yz}z' \\
u_z' &= u_{zx}x' + u_{zy}y' + u_{zz}z'
\end{aligned}
\tag{1}
$$

where u_{xy} stands, for brevity, for the differential coefficient $\frac{\partial u_x}{\partial y}$.

In view of what has been said in the previous section, all the nine components of strain figuring in the equations (1) are required for a complete specification of the deformations in which rotations are not ignored. Then the changes in the state of a volume element contemplated in (1) can be analysed into (i) changes of volume, (ii) changes in shape not involving rotations or alterations of volume, and (iii) rotations.

Likewise, the stresses in the interior of the solid require nine components for their full specification. Denoting by $T_{x\nu}$, $T_{y\nu}$ and $T_{z\nu}$ the components of the tractive forces parallel to the three axes of co-ordinates on any elementary area whose normal has a specified direction ν, these tractions are related to the stresses acting on the three co-ordinate planes by means of the relations

$$\begin{aligned} T_{x\nu} &= T_{xx}\cos(x,\nu) + T_{xy}\cos(y,\nu) + T_{xz}\cos(z,\nu) \\ T_{y\nu} &= T_{yx}\cos(x,\nu) + T_{yy}\cos(y,\nu) + T_{yz}\cos(z,\nu) \\ T_{z\nu} &= T_{zx}\cos(x,\nu) + T_{zy}\cos(y,\nu) + T_{zz}\cos(z,\nu) \end{aligned} \quad (2)$$

As mentioned earlier, the three components of the angular momenta of any volume element will not vanish in dynamic experiments or for heterogeneous strains involving rotations and which accordingly involve torques. We therefore retain all the nine stress components in our formulation and do not make the usual reduction in their number from nine to six.

At this stage, we introduce a slight change in notation which enables us to pass on from symbols with double subscripts to symbols involving a single suffix only. We use for the stress components

$T_{xx}\ T_{yy}\ T_{zz}\ T_{yz}\ T_{zy}\ T_{zx}\ T_{xz}\ T_{xy}\ T_{yx}$ the symbols

$T_1\ T_2\ T_3\ T_4\ T_5\ T_6\ T_7\ T_8\ T_9$ respectively and similarly write the strain variables

$u_{xx}\ u_{yy}\ u_{zz}\ u_{yz}\ u_{zy}\ u_{zx}\ u_{xz}\ u_{xy}\ u_{yx}$ as

$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9$ respectively.

With this notation the stress-strain relations take a neat form. Since the stresses in the solid are dependent on the strain produced in the solid, the stress components can be expanded as a power series in the strain variables. If we measure the stresses from an initial state corresponding to the undeformed condition of the solid and consider infinitesimal strains only, so that squares and higher powers of the strain variables can be neglected in comparison with first order terms, the stresses at any point of the solid are linear functions of the strain components at that point. The stress-strain relations can then be expressed as

$$T_m = \sum_{n=1}^{9} d_{mn}\, u_n \quad (m = 1, 2 \ldots 9) \quad (3)$$

and these involve 81 constants. Here the constant d_{mn} relates the stress T_m to the strain u_n and is the ratio of the two for a deformation in which all strain components other than u_n vanish.

The 81 constants figuring in (3) are not all independent, but reduce in the first instance to forty-five for all solids in view of the relations

$$d_{mn} = d_{nm} \ (m, n = 1, 2. \ldots 9) \tag{4}$$

These relations follow from the well-known theorem of reciprocity relating forces and the corresponding displacements in dynamical systems.[4] The reciprocity relations further enable us to write down the expression for the deformation energy per unit volume in the neighbourhood of any point and this is given by

$$U = \tfrac{1}{2} \sum_{m=1}^{9} T_m u_m$$

or

$$2U = \sum_m \sum_n d_{mn} u_m u_n \tag{5}$$

4. THE THREE ELASTIC CONSTANTS OF ISOTROPIC SOLIDS

The isotropic nature of a body results in a great reduction of the number of independent constants occurring in the stress-strain relationships. Most of these constants in fact are zero and the others become equal to each other in sets for isotropic materials. Some of these relations can be deduced easily from simple symmetry considerations, without going into the full details of the analytic apparatus needed to derive them. For example, the cubic symmetry possessed by the material endows it with the same property for all the three directions of the axes of co-ordinates and therefore the stress-relationships should remain invariant under any permutation of the symbols x, y, z in both the strain variables (u_{xy}) as well as in the stress components T_{xy}. We thus get

$$\begin{aligned} d_{11} &= d_{22} = d_{33}; \\ d_{12} &= d_{23} = d_{31}; \\ d_{45} &= d_{67} = d_{89}; \\ d_{44} &= d_{55} = d_{66} = d_{77} = d_{88} = d_{99} \end{aligned} \tag{6}$$

Again, the operations of reflection about any plane in space do not produce observable changes in the properties of isotropic bodies. In the simple case of a reflection about the xy plane, the z co-ordinate of any point changes its sign while its x and y co-ordinates are unaffected. Hence all the strain components like u_{yz} (u_4), u_{zy} (u_5), u_{zx} (u_6), u_{xz} (u_7) in which z

occurs *once* only as a suffix change their sign whereas the other strain variables are unaltered. If therefore we substitute these new values of the strain variables in the energy expression and equate it to the original one, we get

$$d_{14} = d_{15} = d_{16} = d_{17} = d_{24} = d_{25} = d_{26} = d_{27} = d_{34} = d_{35}$$
$$= d_{36} = d_{37} = d_{48} = d_{49} = d_{58} = d_{59} = d_{68} = d_{69} = d_{78} = d_{79} = 0 \tag{7}$$

Similarly by considering reflections about the planes $x = 0$, and $y = 0$, we could show that

$$d_{18} = d_{19} = d_{28} = d_{29} = d_{38} = d_{39} = d_{46} = d_{47} = d_{56} = d_{57} = 0 \tag{8}$$

Simple symmetry considerations thus reduce the number of non-zero and independent constants to four. Even these constants (*i.e.*) d_{11}, d_{12}, d_{44} and d_{45} however are not independent but are connected to each other by means of a linear relation. To obtain this, we use the special symmetry property possessed by isotropic solids alone, namely invariance in behaviour under all rotations in space. Considering a rotation about the z-axis through an angle θ, this operation changes the strain variables into a new set of quantities u_1', u_2'.....u_9' related to the original ones in accordance with the following scheme:

$$\begin{aligned}
u_1' &= u_1 \cos^2\theta + (u_8 + u_9) \sin\theta \cos\theta + u_2 \sin^2\theta;\\
u_2' &= u_1 \sin^2\theta - (u_8 + u_9) \sin\theta \cos\theta + u_2 \cos^2\theta;\\
u_3' &= u_3;\\
u_4' &= u_4 \cos\theta - u_7 \sin\theta;\\
u_5' &= u_5 \cos\theta - u_6 \sin\theta;\\
u_6' &= u_5 \sin\theta + u_6 \cos\theta;\\
u_7' &= u_4 \sin\theta + u_7 \cos\theta;\\
u_8' &= (u_2 - u_1) \sin\theta \cos\theta + (u_8 \cos^2\theta - u_9 \sin^2\theta);\\
u_9' &= (u_2 - u_1) \sin\theta \cos\theta + (u_9 \cos^2\theta - u_8 \sin^2\theta).
\end{aligned} \tag{9}$$

Hence under the operation of a rotation about the z-axis by an amount θ, the energy expression (5) changes into

$$\begin{aligned}
2U = d_{11}u_3^2 &+ d_{11}\{u_1 \cos^2\theta + (u_8 + u_9) \sin\theta \cos\theta + u_2 \sin^2\theta\}^2\\
&+ d_{11}\{u_1 \sin^2\theta - (u_8 + u_9) \sin\theta \cos\theta + u_2 \cos^2\theta\}^2\\
&+ 2d_{12}(u_1 + u_2)\, u_3
\end{aligned}$$

$$+ 2d_{12}\{u_1 \cos^2\theta + (u_8 + u_9)\sin\theta\cos\theta + u_2\sin^2\theta\}$$
$$\times\{u_1\sin^2\theta - (u_8 + u_9)\sin\theta\cos\theta + u_2\cos^2\theta\}$$
$$+ d_{44}\{(u_2 - u_1)\sin\theta\cos\theta + u_8\cos^2\theta - u_9\sin^2\theta\}^2$$
$$+ d_{44}\{(u_2 - u_1)\sin\theta\cos\theta + u_9\cos^2\theta - u_8\sin^2\theta\}^2$$
$$+ d_{44}(u_4^2 + u_5^2 + u_6^2 + u_7^2) + 2d_{45}(u_4u_5 + u_6u_7)$$
$$+ 2d_{45}\{(u_2 - u_1)\sin\theta\cos\theta + u_8\cos^2\theta - u_9\sin^2\theta\}$$
$$\times\{(u_2 - u_1)\sin\theta\cos\theta + u_9\cos^2\theta - u_8\sin^2\theta\} \quad (10)$$

Comparing this with the expression

$$2U = d_{11}(u_1^2 + u_2^2 + u_3^2) + 2d_{12}(u_2u_3 + u_3u_1 + u_1u_2)$$
$$+ d_{44}(u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2 + u_9^2)$$
$$+ 2d_{45}(u_4u_5 + u_6u_7 + u_8u_9) \quad (11)$$

we get

$$d_{11} = d_{12} + d_{44} + d_{45} \quad (12)$$

Rotations about the x- and y-axes through any angle should also necessarily lead to the same equation (12). A general rotation about any axis can be effected by a superposition of rotations through different angles about the x, y and z axes. We have thus exhausted all the symmetry operations permissible for isotropic solids. It follows therefore that *the elastic behaviour of isotropic solids requires three independent constants for its description, which may be denoted by* d_{11}, d_{12} *and* d_{44}.

5. RELATIONS BETWEEN THE VARIOUS CONSTANTS

With the aid of the relations (6), (7), (8) and (12), the stress-strain relationships described by (12) can be rewritten. The expressions for the three stretches T_1, T_2 and T_3 become

$$T_1 = d_{11}u_1 + d_{12}(u_2 + u_3)$$
$$T_2 = d_{11}u_2 + d_{12}(u_3 + u_1) \quad (13)$$
$$T_3 = d_{11}u_3 + d_{12}(u_1 + u_2)$$

whereas the shearing stresses are given by

$$T_4 = d_{44}u_4 + (d_{11} - d_{12} - d_{44})u_5$$
$$T_5 = d_{44}u_5 + (d_{11} - d_{12} - d_{44})u_4 \quad (14)$$

and four similar equations.

We shall now evaluate some of the important elastic constants, *viz.*, the compressibility or bulk modulus, Young's modulus and Poisson's ratio in terms of these new constants. Consider first the case of a uniform hydrostatic pressure acting at all points on the surface of the body. The state of stress produced by such a compression of the solid is described by $T_1 = T_2 = T_3 = -p$; $T_4 = T_5 = \ldots = T_9 = 0$. Hence adding all the three equations in (13), we get

$$p = \tfrac{1}{3}(d_{11} + 2d_{12})\Delta \tag{15}$$

where Δ denotes the cubical compression $= -(u_1 + u_2 + u_3)$. The bulk modulus therefore is given by

$$k = \tfrac{1}{3}(d_{11} + 2d_{12}) \tag{16}$$

Similarly by considering the case of an isotropic body in the form of a cylindrical rod subjected to a tension T which is uniform over its plane ends, we could show that the Young's modulus E and Poisson's ratio σ are related to d_{11} and d_{12} in accordance with the equations

$$E = \frac{(d_{11} + 2d_{12})(d_{11} - d_{12})}{(d_{11} + d_{12})} \tag{17}$$

$$\sigma = \frac{d_{12}}{(d_{11} + d_{12})} \tag{18}$$

These expressions are in the same form as the corresponding ones for k, E and σ of the classical theory expressed in terms of the well-known con- stants c_{11} and c_{12}. The relations among the Young's modulus, bulk modulus, and Poisson's ratio are therefore the same both in the two-constant as well as in the three-constant theories. We emphasize the fact that all the three equations (16), (17) and (18) contain the constants d_{11} and d_{12} only, nd none of them involves d_{44} explicitly. This is because all these moduli re determinable from static homogeneous strains alone, whereas d_{44}, being constant involving rotations of the volume elements requires experiments ivolving twists for its evaluation.

It may be pointed out here that the relation (12) may be derived directly om very simple considerations. A cube which is subject to normal trac- ons on a pair of opposing faces and normal pressures of equal magnitude 1 an adjacent pair of faces would suffer no change of volume, but would pand and contract respectively in the direction of the two normals to the ces by an amount of which $(d_{11} - d_{12})$ is a measure. Likewise, if a pair opposing faces of a cube are subject to tangential tractions forming a uple and an adjacent pair also subject to tangential tractions which form balancing couple, the cube would suffer no change of volume but would dergo a change of shape without rotation of which $(d_{44} + d_{45})$ is readily seen to be a measure. The two systems of stresses and the resulting strains can readily be shown to be equivalent and it follows that $(d_{11} - d_{12}) = (d_{44} + d_{45})$.

6. VELOCITY OF PROPAGATION OF WAVES IN THE SOLID

In the absence of body forces, the general equations of motion of an elastic body are given by

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z};$$

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z}$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \qquad (19)$$

where ρ is the density of the material. For an isotropic solid, the stress-strain relations are given by equations (13) and (14). Adopting once again the primitive notation of writing differential coefficients $\partial u_x/\partial x, \ldots. \partial u_y/\partial z \ldots.$ for the strain components $u_1, \ldots. u_4 \ldots.$ etc., we get on substituting (13) and (14) in (19) that

$$\rho \frac{\partial^2 u_x}{dt^2} = (d_{11} - d_{12}) \nabla^2 u_x + d_{12} \frac{\partial \Delta}{dx}$$
$$+ (d_{11} - d_{12} - d_{44}) \left\{ \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)\right\} \qquad (20)$$

and two similar equations for the displacements in the y and z directions. In the above, Δ denotes the dilatation

$$\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

or simply divergence $\mathbf{u}$ where $\mathbf{u}$ is the vector whose components parallel to the axes are u_x, u_y and u_z respectively. The three equations in (20) can be combined together and written as a single equation in the form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (d_{11} - d_{12}) \nabla^2 \mathbf{u} + d_{12} \, \text{grad div } \mathbf{u}$$
$$+ (d_{11} - d_{12} - d_{44}) \text{ curl curl } \mathbf{u} \qquad (21)$$

Since curl curl $\mathbf{u}$ = grad div $\mathbf{u} - \nabla^2 \mathbf{u}$, (21) alternatively becomes

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = d_{44} \nabla^2 \mathbf{u} + (d_{11} - d_{44}) \text{ grad div } \mathbf{u} \qquad (22)$$

We shall now take the *divergence* of both sides of (22). This gives us

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = d_{11} \nabla^2 \Delta \tag{23}$$

The above is in fact the equation of wave propagation in the medium. *Compressional waves are therefore propagated in the solid with the velocity* $\sqrt{d_{11}/\rho}$.

Performing next the operation of *curl* on both sides of (22), and writing $\boldsymbol{\omega}$ for curl **u**, one gets

$$\rho \frac{\partial^2 \omega}{\partial t^2} = d_{44} \nabla^2 \omega \tag{24}$$

equation (24) therefore shows *that equivoluminal or distortional waves are propagated in the medium with the velocity* $\sqrt{d_{44}/\rho}$.

It will be noticed that the velocities of propagation of both the longitudinal and transverse waves determine the constants d_{11} and d_{44} only, and do not involve the constant d_{12} at all. On the other hand, d_{44} does not make its appearance in the modulii determinable by static homogeneous strains.

7. Summary

The notions regarding stresses and strains adopted in the classical theory of elasticity are critically examined. The neglect of rotations in the analysis of strain and of torques in the analysis of stress characteristic of that theory is shown to be unjustifiable. A reformulation of the stress-strain relationships taking account of these factors leads to the result that an isotropic solid has *three* independent elastic constants and not *two* as hitherto supposed. Two of these three constants determine the velocities of propagation respectively of longitudinal and transverse waves in the solid. The latter of them does not make its appearance in any observations involving only homogeneous strains nor does it appear in the formulae for the bulk modulus, Young's modulus and Poisson's ratio obtained in the present paper.

8. References

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