A GENERALISED THEORY OF THE CHRISTIANSEN EXPERIMENT

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1. Introduction

In a paper[1] published in these *Proceedings* nearly six years ago, the theory of the well-known Christiansen experiment was discussed on a wave-optical basis. The expression derived in that paper for the transmission coefficient of a Christiansen light filter showed it to be an exponential function involving five variables, namely the wave-length of the light, the thickness of the cell, the size of the individual particles of the powder, the difference between the refractive indices of the powder and the surrounding liquid and finally also the proportions of the volume of the cell occupied respectively by the liquid and by the particles of the powder. In the present paper, it is proposed to deal with the more general case in which the particles of the powder are *birefringent* and hence their refractive index varies with the orientation of the crystallites within the cell. The mathematical treatment adopted is on much the same lines as that followed by us in discussing the theory of the propagation of light in polycrystalline media.[2] The only difference, in fact, is that some of the cubical elements of volume each of edgelength Δ which we imagine the cell to consist of must now be considered as being filled either by the liquid of refractive index μ_l or by the crystallites. These latter are assumed to be of cubical shape and to have their edges parallel to the three optic directions for which the refractive indices are μ_1, μ_2 and μ_3 respectively. We also assume the incident light beam to be plane-polarised with its vibration direction parallel to one set of edges of the cubical blocks and that the operative refractive index of any one block may be either μ_1, μ_2 or μ_3 with equal probabilities if it is a crystallite or μ_l if it is filled with the liquid.

2. Mathematical Formulation

We shall denote by p and q the respective probabilities of a cubical block having anyone of the three principal refractive indices μ_1, μ_2, μ_3 of the crystallites and of the refractive index μ_l of the liquid. Then $3p$ and q would represent the proportion of solid and liquid elements in the Christiansen filter and therefore

$$3p + q = 1 \qquad (1)$$

As before, we consider a typical case in which the incident wave-train which we shall represent by

$$y = e^{\frac{2\pi i}{\lambda}(ct-Z)} \tag{2}$$

encounters in its passage through the Christiansen cell $(n-m)$ elementary cells of solid blocks and m cells of liquid elements. The probability of the occurrence of this event is obviously

$$\frac{n!}{(n-m)!\,m!}\,(3\,p)^{n-m}\,q^{m} \tag{3}$$

If further in any specification of the state of orientation of the crystallites inside the filter, k_1, k_2 and k_3 of the $(n-m)$ cubical blocks considered above have refractive indices μ_1, μ_2 abd μ_3 respectively, then the optical path retardation of the emergent wave for this configuration is $(k_1\mu_1+k_2\mu_2+k_3\mu_3)\,\Delta + m\mu_l\Delta$. We have in addition

$$(k_1+k_2+k_3) = (n-m) \tag{4}$$

Also, the probability of occurrence of a state in which k_1, k_2 and k_3 cells in a row of $(n-m)$ cells can be orientated so as to have refractive indices μ_1, μ_2 and μ_3 is

$$\frac{(n-m!)}{k_1!\,k_2!\,k_3!}\left(\frac{1}{3}\right)^{n-m} \tag{5}$$

Combining (3) and (5) we find that the proportion of area of the rear surface of the filter from which an emergent wave described by

$$e^{\frac{2\pi i}{\lambda}(ct-Z-\overline{k_1\mu_1+k_2\mu_2+k_3\mu_3+m\mu_l\Delta})} \tag{6}$$

proceeds is equal to

$$\frac{n!}{k_1!\,k_2!\,k_3!\,m!}\,p^{n-m}\,q^{m} \tag{7}$$

The disturbance emerging from the Christiansen cell can now be obtained by a superposition of all the different wave functions of the type (6) multiplied by suitable weight factors of which (7) is a typical example. Hence,

$$y = \sum_{k_1,k_2,k_3,m} \frac{n!}{k_1!\,k_2!\,k_3!\,m!}\,p^{n-m}\,q^{m}\,\exp\frac{2\pi i}{\lambda}(ct-Z-\overline{k_1\mu_1+k_2\mu_2+k_3\mu_3+m\mu_l\Delta})$$

$$= e^{\frac{2\pi i}{\lambda}(ct-Z)}\left\{p\left(e^{\frac{-2\pi i}{\lambda}\mu_1\Delta}+e^{\frac{-2\pi i}{\lambda}\mu_2\Delta}+e^{\frac{-2\pi i}{\lambda}\mu_3\Delta}\right)+qe^{\frac{-2\pi i}{\lambda}\mu_l\Delta}\right\}^{n} \tag{8}$$

The average refractive index of the medium is now

$$\mu = p\,(\mu_1 + \mu_2 + \mu_3) + q\mu_l \tag{9}$$

If the birefringence of the crystalline particles is small and if further the three refractive indices of the solid powder do not differ much from the index of the liquid, then an approximation for (8) for large values of n can be effected by proceeding exactly as in the preceding paper. Denoting by d the total thickness of the cell, and neglecting terms of order higher than two in the differences between the various refractive indices, we can rewrite (8) as

$$y = e^{\frac{2\pi i}{\lambda}(ct - Z - \mu d)} \left[1 - \frac{2\pi^2\Delta^2}{\lambda^2}\left\{\sum_{r=1}^{3} p_r (\mu_r - \mu)^2 + q\,(\mu_l - \mu)^2\right\}\right]^n \tag{10 a}$$

Now if $\mu_1, \mu_2 \ldots \mu_n$ are n quantities having the respective probabilities of occurrence $p_1, p_2 \ldots p_n$ in any observation, then

$$\sum_{\substack{r,s\\ r<s}} p_r p_s (\mu_r - \mu_s)^2 = \tfrac{1}{2} \sum_{r,s=1}^{n} p_r p_s (\mu_r - \mu_s)^2$$

$$= \sum_s p_s \left(\sum_r p_r \mu_r^2\right) - \left(\sum_r p_r \mu_r\right)^2$$

$$= \sum_r p_r \mu_r^2 - \mu^2$$

$$= \sum_r p_r (\mu_r - \mu)^2$$

where μ is the average of the n quantities $\mu_1, \mu_2, \ldots \mu_n$.

Applying the above result to (10) we find that for large values of n

$$y = \mathrm{R}e^{\frac{2\pi i}{\lambda}(ct - Z - \mu d)} \tag{10 b}$$

where

$$\mathrm{R} = exp\ \frac{-2\pi^2\Delta d}{\lambda^2}\{p^2 \Sigma (\mu_2 - \mu_3)^2 + pq\,\Sigma (\mu_1 - \mu_l)^2\} \tag{11}$$

The ratio of the intensity of the transmitted light to that of the incident radiation is therefore given by

$$\frac{\mathrm{I}}{\mathrm{I}_0} = \mathrm{R}^2 = exp\ \frac{-4\pi^2\Delta d}{\lambda^2}\{p^2 \Sigma (\mu_2 - \mu_3)^2 + pq\,\Sigma (\mu_1 - \mu_l)^2\} \tag{12}$$

Formula (12) expresses the extinction coefficient of light in its passage through the Christiansen cell in terms of several variables, namely, the wave-length of the light employed, the thickness of the cell, the size of the crystalline particles, the birefringence of the same and the three differences between the refractive index and the three principal indices of the birefringent material, and finally the proportion of the liquid and solid elements in the cell. By giving suitable values to p, q, μ_1, μ_2 and μ_3, several interesting cases of special importance can be deduced from (12). Thus we observe that the case of the polycrystalline aggregate considered earlier follows readily from (12) if $q = 0$ and $p = \frac{1}{3}$. Again, by writing $\mu_1 = \mu_2 = \mu_3$; $p = \frac{\sigma}{3}$ and $q = (1 - \sigma)$ which corresponds to the case of a Christiansen cell composed of isotropic particles mixed in a liquid in the proportion $\sigma : (1 - \sigma)$, we obtain the result

$$I = I_0 \, exp \, \frac{-4\sigma(1-\sigma)\pi^2 \Delta d}{\lambda^2} (\mu_1 - \mu_l)^2 \tag{13}$$

derived earlier by one of us[1] on different theoretical grounds. By writing $\mu_1 = \mu_2 = \mu_3$; $p = \frac{1}{6}$ and $q = \frac{1}{2}$ in (12) or $\sigma = \frac{1}{2}$ in (13) we get the expression

$$I = I_0 \, exp \, \frac{-\pi^2 \Delta d}{\lambda^2} (\mu_1 - \mu_l)^2 \tag{14}$$

for transmission by a cell composed of isotropic particles and a liquid of nearly the same refractive index mixed in equal proportions.

3. SOME FURTHER REMARKS

Considering once again the general formula (12), we may draw attention to certain features which we may expect to observe in the Christiansen experiment with birefringent powders differing from those noticeable when isotropic powders are employed. In the latter case, the transmission would be complete for the particular wave-length for which the solid and liquid have equal refractive indices and would fall off rapidly on either side of such wave-length. The light not transmitted by the cell would appear as a diffusion halo surrounding the direction of the optical image of the source as seen through the cell. Brilliant chromatic effects are accordingly to be expected and are indeed observed with isotropic powders in the experiment.

Formula (12) shows clearly that the effect of the birefriengence of the powder is to diminish the intensity of the transmitted light for all wave-lengths, and we cannot therefore, expect any observable transmission through the cell with strongly birefringent powders, unless the size of the particles be very small and the thickness of the cell be reduced to a minimum. In such cases the colour of the transmitted light would be determined predominantly by the factor $1/\lambda^2$ appearing in the argument of the exponential. Hence, it would be reddish in colour and the chromatic effects observed with isotropic powders would be absent. In these circumstances, the liquid in the cell serves only to secure optical continuity between the discrete particles contained in the cell. On the other hand, if the birefringence be small, one may expect to observe chromatic effects similar to those observed with isotropic powders. It would be necessary, however, to work with fairly fine powders and moderate cell thickness for noticeable transmission to occur. The formula also shows that the maximum transmission in these circumstances would be exhibited for those wave-lengths for which the refractive index of the liquid is most nearly equal to a species of average of the three indices of the crystal. But this is not the average index in the ordinary sense of the word.

Another important consequence of the formula is that even if the birefringence of the powder be not very small, chromatic effects would be observable when the proportion of the volume in the cell occupied by the powder is sufficiently small. In the usual form of the Christiansen experiment, the particles are allowed to settle down and form a compact aggregate at the bottom of the cell. To observe the effects now contemplated, the contents of the cell should be stirred up; alternatively the particles should be so small that they remain suspended for a long time within the liquid. In other words, dilute suspensions of strongly birefringent powders may be expected to give brilliant chromatic effects in a Christiansen cell. However, our theory could hardly be expected to give more than a qualitative indication of the phenomena then noticeable.

Finally, we come to the question of the state of polarisation of the transmitted light as also of the diffracted light when observed with birefringent powders. The present theory indicates that if the light incident on the cell be plane-polarised, both the transmitted and the diffracted light should also be perfectly plane-polarised. So far as the transmitted light is concerned, there can be no doubt that the theoretical result is correct. For, any ellipticity consequent on the passage of light through an arbitrarily orientated crystallite would give rise to a component perpendicular to the original vibration direction in the diffracted light. But such components cannot appear in the light transmitted by the cell in the true optical sense. Indeed, provided the birefringence is small and the size of the particles and the thickness of the cell are moderate we may expect also to find that the diffraction halo is itself strongly polarised. In other circumstances, however, especially when the particles are strongly birefringent, the diffracted light would exhibit a marked imperfection of polarisation by reason of the ellipticity effects which have dropped out of consideration in the present treatment of the problem.

SUMMARY

A formula is derived for the transmission coefficient of a Christiansen cell containing particles of a birefringent material whose interstices are filled up by a liquid of suitably adjusted refractive index. The consequences of the formula and especially the influence of the birefringence on the spectral character of the transmitted light are discussed.

REFERENCES

1. Raman, Sir, C. V. .. *Proc. Ind. Acad. Sci.*, 1949, **29 A**, 381.
2. —— and Viswanathan, K. S. *Ibid.*, 1955, **41 A**, 37.