THE DIFFRACTION OF LIGHT BY HIGH FREQUENCY SOUND WAVES: PART IV.

Generalised Theory.

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1. *Introduction.*

IN Part III[1] of this series of papers, we considered the Doppler effects and coherence phenomena among the diffracted components of light emerging from a rectangular cell of a medium traversed by supersonic waves perpendicular to the direction of the propagation of the incident plane wave of light. We showed, in the case of a progressive supersonic wave, that the nth order diffraction component which is inclined at an angle $\sin^{-1}(-n\lambda/\lambda^*)$ to the direction of propagation of the incident light has the frequency $\nu - n\nu^*$, where ν and λ denote the frequency and the wave-length of the incident light while ν^* and λ^* correspond to those of the sound wave. In the case of the diffraction of light by a standing sound wave, we got the interesting result that in any even order, radiations with frequencies $\nu \pm 2r\nu^*$, $(r = 0, 1, 2, \ldots)$, would be present while in any odd order, radiations with frequencies $\nu \pm \overline{2r+1}\,\nu^*$, $(r = 0, 1, 2, \ldots)$, would be present. These results give a satisfactory interpretation of the coherence phenomena among the diffraction components observed by Bär.[2] In the following, we show that our previous results remain valid even if we consider a *general* periodic supersonic wave and that they can be derived in a simple and direct fashion. We have also presented in the following, some general considerations of the problem on hand.

2. *Doppler effect and coherence phenomena.*

The partial differential equation governing the propagation of light in a medium with time-variation and space-variation in its refractive index is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \left[\frac{\mu(x, y, z, t)}{c}\right]^2 \frac{\partial^2\psi}{\partial t^2}$$

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[1] C. V. Raman and N. S. Nagendra Nath, *Proc. Ind. Acad. Sci. (A)*, 1936, **3**, 75.

[2] R. Bär, *Helv. Phy. Acta*, 1935, **8**, 591.

if the frequency of the time-variation of $\mu(x, y, z, t)$ is very slow compared to the time-variation of the wave-function of light. This would be so in the case of the propagation of light in a medium filled with sound waves for the frequency of the variation of $\mu(x, y, z, t)$ corresponds to the frequency of the sound waves present in the medium, which is negligible compared to the frequency of light.

If we choose our axes of reference such that the X-axis points to the direction of the propagation of the plane sound waves and the Z-axis points to the direction of the propagation of the incident plane wave of light, we could ignore the dependence of ψ on y and write the differential equation as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = \left[\frac{\mu(x, t)}{c}\right]^2 \frac{\partial^2\psi}{\partial t^2}$$

If $\mu(x, t)$ did not depend on time, ψ would have had the only time factor $\exp(2\pi i\nu t)$ where ν is the frequency of the incident light. If we consider the time variation of $\mu(x, t)$, we can write ψ as given by

$$\psi = \exp[2\pi i\nu t]\,\phi(x, z, t)$$

where $\phi(x, z, t)$ varies slowly in time compared to $\exp[2\pi i\nu t]$. On the consideration that $\nu^* << \nu$, we can show that

$$\left|4\pi\nu\frac{\partial\phi}{\partial t}\right| << \left|4\pi^2\nu^2\phi\right| \text{ and } \left|\frac{\partial^2\phi}{\partial t^2}\right| << \left|4\pi^2\nu^2\phi\right|.$$

With these considerations, we can consider the differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2}\{\mu(x, t)\}^2\phi$$

and obtain ψ by the equation

$$\psi = \exp[2\pi i\nu t]\,\phi(x, z, t).$$

As the sound waves which travel along the X-axis are periodic in space and time, we can regard $\mu(x, t)$ to be also periodic in x and t with the same periods in space and time. It should be noticed that we do not restrict $\mu(x, t)$ to be simply periodic in x and t but it may be a general periodic function of x and t, amenable to Fourier Analysis. Thus

$$\mu(x + p\lambda^*, t) = \mu(x, t)$$

and

$$\mu(x, t + p/\nu^*) = \mu(x, t)$$

where p is any integer.

If we consider the differential equation in which $\mu(x, t)$ has the above properties, we see that $\phi(x, z, t)$ should also be periodic in x and t with the same periods in the case we are considering. That is,

$$\phi(x + p\lambda^*, z, t) = \phi(x, z, t)$$

and

$$\phi(x, z, t + p/\nu^*) = \phi(x, z, t)$$

Hence we can write the double-Fourier expansion of $\phi\ (x, z, t)$ as

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{rs}\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i s \nu^* t}$$

Progressive Sound Waves.—In the case of the progressive waves travelling along the positive direction of the X-axis, we have the property that

$$\mu\ (x+\rho\lambda^*, t) = \mu\ (x, t-\rho/\nu^*)$$

where ρ is any number. Thus

$$\phi\ (x+\rho\lambda^*, z, t) = \phi\ (x, z, t-\rho/\nu^*) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Using the double-Fourier expansion, we can write (1) as

$$\Sigma\ \Sigma f_{rs}\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i s \nu^* t} e^{2\pi i r \rho}$$
$$= \Sigma\ \Sigma f_{rs}\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i s \nu^* t}\ e^{-2\pi i s \rho} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Comparing the Fourier coefficients on each side of (2), we get

$$f_{rs}\ (z)\ e^{2\pi i r \rho} = f_{rs}\ (z)\ e^{-2\pi i s \rho}$$

This could be true only if

$$f_{rs}\ (z) = 0 \quad \text{when } r \neq -s \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The condition (3) restricts the number of terms in the Fourier expansion of ϕ, so that

$$\phi\ (x, z, t) = \sum_{-\infty}^{\infty} f_r\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{-2\pi i r \nu^* t}$$

Thus

$$\psi\ (x, z, t) = \sum_{-\infty}^{\infty} f_r\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i(\nu - r\nu^*)t} \quad \ldots \quad \ldots \quad (4)$$

If one considers the diffraction effects of $\psi\ (x, z, t)$ given by (4), it is fairly obvious that the nth order diffraction component will be inclined at an angle $\sin^{-1}(-n\lambda/\lambda^*)$ with the incident beam of light and will have the frequency $\nu - n\nu^*$ and the relative intensity expression $|f_n\ (z)|^2$.

Standing Sound Waves.—In the case of standing waves, we have the property that

$$\mu\ (x + \frac{p\lambda^*}{2}, t) = \mu \left(x, t \pm \frac{p}{2\nu^*}\right), \quad p \text{ an integer,}$$

so that

$$\phi\ (x + \frac{p\lambda^*}{2}, z, t) = \phi \left(x, z, t \pm \frac{p}{2\nu^*}\right) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

If we use (5) in the double Fourier expansion of ϕ we get

$$\Sigma\ \Sigma\ f_{rs}\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i s \nu^* t}\ e^{\pi i r p}$$
$$= \Sigma\ \Sigma\ f_{rs}\ (z)\ e^{2\pi i r x/\lambda^*}\ e^{2\pi i s \nu^* t}\ e^{\pi i s p} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Comparing the Fourier coefficients in (6), we get

$$f_{rs}\ (z)\ e^{\pi i r p} = f_{rs}\ (z)\ e^{\pi i s p}$$

This means that $f_{rs}(z)$ is zero unless r and s are both even integers or odd integers.

Returning now to the Fourier expansion of ϕ, we could write it as

$$\phi(x, z, t) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r, 2s}(z)\, e^{2\pi i 2rx/\lambda^*}\, e^{2\pi i 2s\nu^* t}$$

$$+ \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r+1, 2s+1}(z)\, e^{2\pi i \overline{2r+1}x/\lambda^*}\, e^{2\pi i \overline{2s+1}\nu^* t}.$$

Thus

$$\psi(x, z, t) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r, 2s}(z)\, e^{2\pi i 2rx/\lambda^*}\, e^{2\pi i(\nu + 2s\nu^*)t}$$

$$+ \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r+1, 2s+1}(z)\, e^{2\pi i \overline{2r+1}x/\lambda^*}\, e^{2\pi i(\nu + \overline{2s+1}\nu^*)t} \quad .. \quad (7)$$

If one considers the diffraction effects of $\psi(x, z, t)$ given by (7), it will be quite easy to see that the diffraction orders could be classed into two groups, one containing the even ones and the other odd ones; any even order contains radiations with frequencies, ν, $\nu \pm 2\nu^*, \ldots,$ $\nu \pm 2r\nu^*, \ldots,$ and any odd order contains radiations with frequencies, $\nu \pm \nu^*$, $\nu \pm 3\nu^*, \ldots,$ $\nu \pm \overline{2r+1}\nu^*, \ldots$.

3. *The case when the disturbance in the medium is simple harmonic.*

If we suppose that the variation in the refractive index of the medium is simple harmonic along the X-axis, it can be represented as

$$\mu(x, t) = \mu_0 + \mu \sin 2\pi(\nu^* t - x/\lambda^*)$$

in the case of a progressive wave, while it will be of the form

$$\mu(x, t) = \mu_0 - \mu \sin(2\pi x/\lambda^*) \sin(2\pi\nu^* t)$$

in the case of a standing wave, where $\mu(x, t)$ is the refractive index of the medium at height x and at time t, μ_0 is the constant refractive index of the medium when there is no sound wave and μ is the *maximum variation* of the refractive index from μ_0.

Progressive Wave.—To obtain the wave function for the emerging wave-front of light, we have to solve the differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2}\{\mu(x, t)\}^2\phi$$

$$= [A + \frac{B}{2i}\{e^{i(bx-\epsilon)} - e^{-i(bx-\epsilon)}\}]\,\phi \quad .. \quad .. \quad (8)$$

where $b = 2\pi/\lambda^*$, $\epsilon = 2\pi\nu^* t$, $A = -4\pi^2\mu_0^2/\lambda^2$ and $B = 8\pi^2\mu_0\,\mu/\lambda^2$ omitting the second order term with coefficient μ^2.

We have shown in the previous section that ϕ can be developed as a Fourier series in x and t as

$$\sum_{-\infty}^{\infty} f_r(z)\, e^{2\pi i r x/\lambda^*}\, e^{-2\pi i r \nu^* t}$$

or
$$\sum_{-\infty}^{\infty} f_r(z)\, e^{irbx}\, e^{-ir\epsilon} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Substituting the Fourier series (9) in the differential equation (8) and comparing the Fourier coefficients we obtain the equation

$$\frac{d^2 f_n}{dz^2} - (\mathrm{A} + b^2 n^2)\, f_n = \frac{\mathrm{B}}{2i}\,(f_{n-1} - f_{n+1})$$

Putting $f_n(z) = \exp(-iu\mu_0 z)\, \phi_n(z)$ where $u = 2\pi/\lambda$
we obtain

$$\frac{d^2\phi_n}{dz^2} - 2iu\mu_0 \frac{d\phi_n}{dz} - b^2 n^2 \phi_n = -\frac{\mathrm{B}i}{2}\,(\phi_{n-1} - \phi_{n+1})$$

Putting $z = (2\pi\mu)^{-1}\lambda\xi$, we obtain

$$\mu^2 \frac{d^2\phi_n}{d\xi^2} - 2i\mu_0\mu \frac{d\phi_n}{d\xi} - \frac{n^2\lambda^2}{\lambda^{*2}}\,\phi_n = -\mu_0\mu i\,(\phi_{n-1} - \phi_{n+1})$$

As μ_0, being the refractive index of the medium, is in the neighbourhood of unity and μ is in the neighbourhood of 10^{-5}, we can omit the first term on the left hand side and consider the differential equation

$$2\,\frac{d\phi_n}{d\xi} - (\phi_{n-1} - \phi_{n+1}) = \frac{in^2\lambda^2}{\mu_0\mu\lambda^{*2}}\,\phi_n.$$

If there were no term on the right hand side, ϕ_n, would be the Bessel Function $\mathrm{J}_n(\xi)$ or $\mathrm{J}_n(2\pi\mu z/\lambda)$ satisfying the required boundary conditions. This follows as a consequence of Sonine's[3] theorem which gives that if

$$2\,\frac{d\phi_n}{d\xi} - (\phi_{n-1} - \phi_{n+1}) = 0,$$

then ϕ_n could be developed as a series in Bessel Functions as

$$\phi_n(\xi) = \phi_n(0)\,\mathrm{J}_0(\xi) + \sum_{1}^{\infty} [\phi_{n-s}(0) + (-)^s \phi_{n+s}(0)]\,\mathrm{J}_s(\xi)$$

Setting the boundary conditions that

$$\phi_0(0) = 1 \text{ and } \phi_s(0) = 0, \qquad s \neq 0$$

we get

$$\phi_n(\xi) = \mathrm{J}_n(\xi).$$

If n is not too great and $\lambda^2/\lambda^{*2}\mu$ is small, we can approximate

$$\phi_n(\xi) \backsimeq \mathrm{J}_n(\xi) = \mathrm{J}_n\left(\frac{2\pi\mu z}{\lambda}\right)$$

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[3] N. Nielsen, *Handbuch der theorie der Cylinderfunktionen*, p. 286 (1904 edition).

If the cell is bound by $z = L$ at the emerging face, it will be easy to see that the relative intensity of the nth order diffraction component would be $J_n^2\ (2\pi\mu L/\lambda)$.

The case of the standing wave.—In this case we have to write $\phi\ (x, z, t)$ as given by

$$\phi\ (x, z, t) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r,\ 2s}\ e^{2\pi i 2rx/\lambda^*}\ e^{2\pi i 2s\nu^* t}$$

$$+ \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r+1,\ 2s+1}\ e^{2\pi i \overline{2r+1}x/\lambda^*}\ e^{2\pi i \overline{2s+1}\nu^* t}$$

$$= \sum_{-\infty}^{\infty} g_r\ (z, t)\ e^{2\pi i rx/\lambda^*} \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Substituting (10) the differential equation for ϕ and comparing the coefficients, we obtain

$$\frac{\partial^2 g_n}{\partial z^2} - \frac{4\pi i\mu_0}{\lambda}\frac{\partial g_n}{\partial z} - \frac{4\pi^2 n^2}{\lambda^{*2}} g_n = \frac{4\pi^2\mu_0\mu \sin\epsilon}{\lambda^2 i}\ (g_{n-1} - g_{n+1})$$

Putting $z = (2\pi\mu)^{-1}\lambda\xi$ we obtain

$$\mu^2 \frac{\partial^2 g_n}{\partial \xi^2} - 2i\mu_0\mu\frac{\partial g_n}{\partial \xi} - \frac{n^2\lambda^2}{\lambda^{*2}} g_n = -\mu_0\mu i \sin\epsilon\ (g_{n-1} - g_{n+1}).$$

Under the same considerations as in the previous paragraph, we will have to solve the equation

$$2\frac{\partial g_n}{\partial \xi} - \sin\epsilon\ (g_{n-1} - g_{n+1}) = \frac{in^2\lambda^2}{\mu_0\mu\lambda^{*2}} g_n.$$

If n is not too great and $\lambda^2/\lambda^{*2}\mu$ is small we can approximate

$$g_n\ (\xi, \epsilon) \backsimeq J_n\ (\xi \sin\epsilon) = J_n\left(\frac{2\pi\mu z}{\lambda} \sin\ \overline{2\pi\nu^* t}\right).$$

But we have shown in Part III,[1] that

$$J_{2n}\ (v \sin\epsilon) = \sum_{-\infty}^{\infty}(-)^r J_{n-r}\ (v/2)\ J_{n+r}\ (v/2)\ e^{i2r\epsilon}$$

$$J_{2n+1}\ (v \sin\epsilon) = -i\sum_{-\infty}^{\infty}(-)^r J_{n-r}\ (v/2)\ J_{n+r+1}\ (v/2)\ e^{i\overline{2r+1}\epsilon}.$$

Hence,

$$\psi\ (x, z, t) \backsimeq \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} (-)^r J_{r-s}\ (v/2)\ J_{r+s}\ (v/2)\ e^{2\pi i 2rx/\lambda^*}\ e^{2\pi i(\nu+2s\nu^*)t}$$

$$- i\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(-)^r J_{r-s}\ (v/2)\ J_{r+s+1}\ (v/2)\ e^{2\pi i\overline{2r+1}x/\lambda^*}\ e^{2\pi i(\nu+\overline{2s+1}\nu^*)t}$$

If one considers now the diffraction effects due to this emerging wave-front at $z = L$, it can be seen that an even order, say $2n$, contains radiations with frequencies $\nu \pm 2r\nu^*$, $(r = 0, 1, 2, \ldots)$, the relative intensity of the $\nu \pm 2r\nu^*$

sub-component being $J^2_{n-r}(\pi\mu L/\lambda)\ J^2_{n+r}(\pi\mu L/\lambda)$ and an odd order, say $2n+1$, contains radiations with frequencies $\nu \pm \overline{2r+1}\nu^*$, ($r = 0, 1, 2, \ldots$), the relative intensity of the $\nu \pm \overline{2r+1}\nu^*$ sub-component being $J^2_{n-r}(\pi\mu L/\lambda)\ J^2_{n+r+1}(\pi\mu L/\lambda)$.

4. *Summary.*

The essential idea that the phenomenen of the diffraction of light by high frequency sound waves depends on the corrugated nature of the transmitted wave-front of light, pointed out by the authors in their first paper, has been developed on general considerations in this paper. The results in this paper can be summarised as follows :—

(1) If progressive sound-waves travel in a rectangular medium normal to two faces and the direction of propagation of a plane beam of incident light, the incident light will be diffracted at the angles given by $\sin^{-1}(-n\lambda/\lambda^*)$ and the light belonging to the nth order will have the frequency $\nu - n\nu^*$.

(2) If the sound waves are stationary, the incident light will be diffracted at the angles given by $\sin^{-1}(-n\lambda/\lambda^*)$, an even order would contain radiations with frequencies, ν, $\nu \pm 2\nu^*$, $\nu \pm 4\nu^*, \ldots$, $\nu \pm 2r\nu^*, \ldots$, and an odd order would contain radiations with frequencies $\nu \pm \nu^*$, $\nu \pm 3\nu^*$, $\nu \pm 5\nu^*, \ldots$, $\nu \pm \overline{2r+1}\ \nu^*, \ldots$

(3) A differential-difference equation has been obtained for the amplitude function of the diffracted orders whose approximate solution is satisfied by the Bessel Functions already obtained by the authors in their previous papers.