An Event Structure Semantics for General Petri Nets *

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November 1993

Abstract

In this paper we address the following question: What type of event structures are suitable for representing the behaviour of general Petri nets? As a partial answer to this question we define a new class of event structures called *local event structures* and identify a subclass called *UL-event structures*. We propose that UL-event structures are appropriate for capturing the behaviour of general Petri nets. Our answer is a partial one in that in the proposed event structure semantics, auto-concurrency is filtered out from the behaviour of Petri nets. It turns out that this limited event structure semantics for Petri nets is nevertheless a non-trivial and conservative extension of the (prime) event structure semantics of 1-safe Petri nets provided in [NPW]. We also show that the strong relationship between prime event structures and 1-safe Petri nets established in a categorical framework in [W3] can be extended to the present setting, provided we restrict our attention to the subclass of Petri nets whose behaviours do not exhibit any auto-concurrency. Finally, we show that Winskel's general and stable event structures can be smoothly related to local event structures and that similarly prime event structures can be related to UL-event structures.

Introduction

Prime event structures can be used to represent the behaviour of 1-safe Petri nets. This basic result was shown by Nielsen, Plotkin, and Winskel in [NPW]. The "universality" of their construction which associates a prime event structure with a 1-safe Petri net was later shown by Winskel [W3] in a categorical setting, and in the process provided strong evidence that the construction in [NPW] is not merely an ad hoc translation.

*An extended abstract of this paper has appeared in the proceedings of CONCUR'93 under the title: Local Event Structures and Petri Nets [HKT2].

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An obvious question that now arises is: when one moves up from 1-safe Petri nets to general Petri nets, what are the corresponding event structures that one should look for? The question is interesting because general Petri nets are a very natural generalization of 1-safe Petri nets. They seem to have a nice algebraic structure [W2, MM]. They are also a very simple kind of multiset rewrite systems. Some previous work in this area [E, MMS] has essentially proposed prime event structures as possible candidates for representing the behaviour of Petri nets. However, this entails having to view the tokens as "coloured" entities, which destroys the possibility of viewing Petri nets as simple *multiset* rewrite systems. It also leads to the counter-intuitive result that 1-safe Petri nets and general Petri nets give rise to the same set of behaviours in terms of event structures. Hence we are interested in finding a *proper* generalization of the event structure semantics for 1-safe Petri nets.

We propose here such a generalization with the help of a new class of event structures, called *local event structures*. These event structures are easy to define and require just a purely local concurrency axiom; no global order theoretic properties are demanded. It turns out that a subclass of the local event structures can be advocated as a *partial* answer to the question: what are the event structures that correspond to the behaviour of Petri nets? Our answer is partial in that in the event structure semantics for Petri nets that is being proposed here, auto-concurrency is filtered out from the behaviour of Petri nets. Auto-concurrency is the phenomenon by which multiple instances of a transition become enabled at a marking. This is impossible in a 1-safe Petri net.

To be more precise, we first define the class of local event structures. We then identify a subclass of these event structures that have a certain unique occurrence property. It turns out that this subclass is a proper and very generous generalization of the notion of prime event structures. We then show, as our first main result, how one can associate one member of this subclass of local event structures with each Petri net. In doing so we use the set of step firing sequences based on sets rather than the set of multiset firing sequences of a Petri net. It is in this sense that we filter out auto-concurrency, and hence the proposed event structure semantics is a restricted one. However, it is also the case that our event structure semantics for Petri nets is a strict extension of the prime event structure semantics for 1-safe Petri nets given in [NPW].

Next we turn to the problem of lifting the co-reflection between prime event structures and 1-safe Petri nets established by Winskel [W3]. It turns out that the category of Petri nets (under a reasonable choice of behaviour-preserving morphisms) is, due to auto-concurrency, too rich in terms of objects and arrows to let the desired co-reflection go through. Our second main result is that the desired co-reflection *does* go through if we restrict our attention to Petri nets that do not exhibit any auto-concurrency in their behaviour. Such Petri nets will be referred to as *co-safe* Petri nets here. It is worth pointing out that co-safe Petri nets constitute a non-trivial extension of the notion of 1-safe Petri nets. Hence through our second main result we have a complete event structure semantics for this large subclass of Petri nets.

In Section 1 we introduce local event structures. Then in Section 2, a unique occurrence property is defined using a new equivalence relation over prime intervals. This leads to the identification of the subclass of local event structures with the unique occurrence property. In Section 3, we introduce Petri nets and define the set of multiset firing sequences of a Petri net, and, as a derived notion, the set of step firing sequences. We then use the set of step firing sequences to construct a local event structure with the unique occurrence property.

In Section 4 we prepare the stage for discussing adjunctions by constructing a map from local event structures to Petri nets. Our map is such that the target of every local event structure will be a co-safe Petri net. In Section 5 we set up a category of Petri nets and argue with the help of an example why the co-reflection result of Winskel will not go through in the present setting. We then show that the desired co-reflection does go through if we restrict our attention to co-safe Petri nets.

In Section 6 it is shown that there exists a strong relationship between the local event structures introduced in this paper and Winskel's general event structures. To this end functors between the corresponding categories are constructed which constitute a reflection. Then we show that there is also a reflection between the category of local event structures with the unique occurrence property and the category of prime event structures.

Finally, the concluding section summarizes the results of the paper and discusses some related work.

1 Local Event Structures

In this section we introduce *local event structures* and structure-preserving morphisms between local event structures.

A local event structure is defined as a family of configurations. This is similar to the specification of Winskel's general event structures through families of configurations [W3]. However, in contrast to Winskel's event structures, here a family of configurations is equipped with an enabling relation which specifies *locally*, for each configuration, the possible concurrency of events at that configuration. This enabling relation satisfies some simple axioms.

For an arbitrary set X, we use $P_F(X)$ to denote the set of finite subsets of X. Furthermore, for $u \in P_F(X)$, the number of elements in u is denoted by $|u|$; if $|u| = 1$ then we notationally identify u with its only element.

Definition 1.1

A *local event structure* is a triple $ES = (E, C, \vdash)$ where E is a set of *events*, $C \subseteq P_F(E)$ is a non-empty set of *(finite) configurations*, and $\vdash \subseteq C \times P_F(E)$ is an *enabling relation* satisfying the following axioms. (In stating the axioms, and in what follows, we let c range over C and u range over $P_F(E)$.)

(A0) $\emptyset \neq c \Rightarrow \exists e \in c.\, c - e \vdash e$

(A1) $c \vdash \emptyset$

(A2) $c \vdash u \Rightarrow (c \cap u = \emptyset$ and $\forall v \subseteq u. (c \vdash v$ and $c \cup v \vdash u - v))$. $\square$

In the rest of this paper we refer to local event structures as L-event structures.

Note that (A0) implies that if $\emptyset \neq c \in C$ then there exists $e \in c$ such that $c - e \in C$. Hence $\emptyset \in C$, because C is non-empty. The axiom (A2) implies that if $c \vdash u$ then $c \cup v \in C$ for all $v \subseteq u$. Note also that the axiom (A1) could have been replaced by the condition that the enabling relation $\vdash$ is not empty.

Example 1.2

In Figure 1 three L-event structures $ES_i = (E_i, C_i, \vdash_i)$, $i = 1, 2, 3$, are depicted. In depicting an L-event structure $(E, C, \vdash)$ we use the following convention. If $c \vdash u$ then we draw a line between c and $c \cup u$ in case $|u| = 1$ and we draw a dotted line between c and $c \cup u$ in case $|u| \geq 2$. $\square$

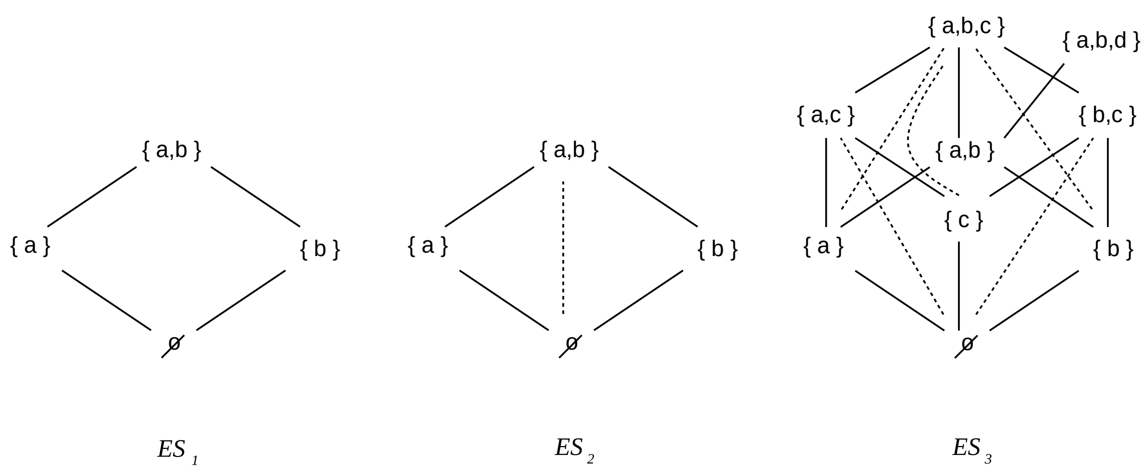


Figure 1: Three L-event structures

We would now like to establish some preliminary properties of L-event structures. Before doing so, we wish to emphasize that the inclusion relation between configurations in the present set-up does not carry much information. Consider the L-event structures depicted in Figure 2.

Clearly the sets of configurations of both these L-event structures (as well as those of the two L-event structures ES_1 and ES_2 shown in Figure 1) are identical. Thus the *reachability* relation between configurations of an L-event structure carries more useful information.

Let $ES = (E, C, \vdash)$ be an L-event structure. Then $\sqsubset\!\!\!\cdot_{ES} \subseteq C \times C$ is the least relation satisfying: if $c \vdash u$ then $c \sqsubset\!\!\!\cdot_{ES} c \cup u$. Let $\sqsubseteq_{ES} = (\sqsubset\!\!\!\cdot_{ES})^*$. Then it is easy to see that the relation $\sqsubseteq_{ES}$ is a partial ordering relation. In what follows we omit the subscript $_{ES}$ in $\sqsubset\!\!\!\cdot_{ES}$ and $\sqsubseteq_{ES}$ if ES is clear from the context.

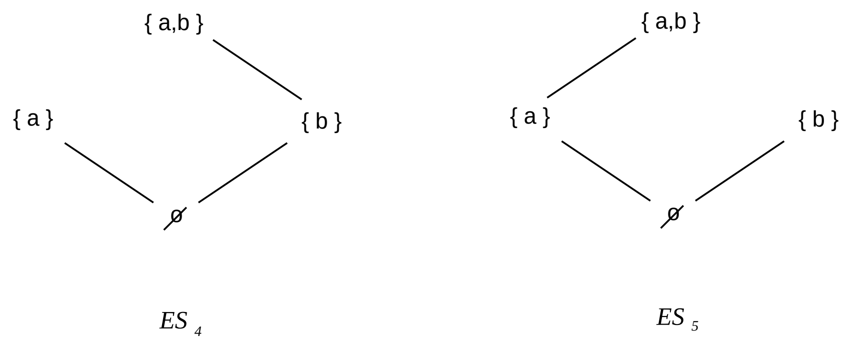


Figure 2: L-event structures with the same configurations

Lemma 1.3

Let $(E, C, \vdash)$ be an L-event structure and let $c \in C$ and $e_1, e_2 \in c$ be such that $e_1 \neq e_2$. Then

(1) $\exists c' \in C.\, c' \sqsubseteq c$ and $((e_1 \in c'$ and $c' \vdash e_2)$ or $(e_2 \in c'$ and $c' \vdash e_1))$

(2) $\exists c' \in C.\, c' \sqsubseteq c$ and $(e_1 \in c' \Leftrightarrow e_2 \notin c')$.

Proof.

In order to prove (1), we proceed by induction on $k = |c|$. If $k = 2$ then $c = \{e_1, e_2\}$ and by (A0), $c - e_1 \vdash e_1$ or $c - e_2 \vdash e_2$. In either case the required result follows.

If $k > 2$ then, again by (A0), there exists $e \in c$ such that $c - e \vdash e$. If $e = e_1$ or $e = e_2$ then let $c' = c - e$. Otherwise the required $c' \in C$ exists by the induction hypothesis applied to $c - e$.

(2) follows immediately from (1) and (A2). $\square$

Lemma 1.3(2) implies that, similar to Winskel's general event structures [W3], L-event structures satisfy a coincidence freeness property.

In formulating some other properties of L-event structures we will use the following notation and terminology.

For an arbitrary set X we let X^* denote the free monoid generated by X. The product operation is concatenation and the elements of X^* are called *words* or alternatively *sequences* (*over* X). The unit element of X^* is the empty word Λ and $X^+ = X^* - \{\Lambda\}$ is the set of non-empty words over X. Elements of $P_F(X)$ will be referred to as *steps* (*over* X) and elements of $(P_F(X))^+$ as *step sequences* (*over* X). We view $(P_F(X))^+$ as a (free) monoid: the unit element is $\emptyset \in P_F(X)$ and the product operation is the accordingly modified usual concatenation operation. Thus $\rho\emptyset = \emptyset\rho = \rho$ for all $\rho \in (P_F(X))^+$ where $\rho\emptyset$ denotes the product of ρ and $\emptyset$.

For $a \in X$ and $\rho \in (P_F(X))^+$, we let $num_a(\rho)$ denote the number of times a occurs in ρ. Thus $num_a(\emptyset) = 0$ and $num_a(\rho u) = num_a(\rho)+1$ if $a \in u$ and $num_a(\rho u) = num_a(\rho)$ if $a \notin u$.

We let $|\rho|$ denote the number of elements in ρ, that is $|\rho| = \sum_{a \in X} num_a(\rho)$, and $alph(\rho)$ denote the set of elements of X occurring in ρ, that is $alph(\rho) = \{a \in X \mid num_a(\rho) > 0\}$.

Let $ES = (E, C, \vdash)$ be an L-event structure. Then $SFS_{ES} \subseteq (P_F(E))^+$ is the set of *step firing sequences* of ES, and $cf_{ES} : SFS_{ES} \to P_F(E)$ is the function which associates with each step firing sequence the configuration it leads to. They are defined inductively as:

(1) $\emptyset \in SFS_{ES}$ and $cf_{ES}(\emptyset) = \emptyset$

(2) $(\rho \in SFS_{ES}$ and $cf_{ES}(\rho) \vdash u) \Rightarrow (\rho u \in SFS_{ES}$ and $cf_{ES}(\rho u) = cf_{ES}(\rho) \cup u)$.

If the L-event structure ES is clear from the context, then we may omit the subscript $_{ES}$ in SFS_{ES} and cf_{ES}.

The following lemma states some basic observations on the relationship between the step firing sequences and the configurations of an L-event structure. These observations will be frequently used in the sequel.

Lemma 1.4
Let $(E, C, \vdash)$ be an L-event structure. Then

(1) $\forall \rho \in SFS. (cf(\rho) \in C$ and $cf(\rho) = alph(\rho))$

(2) $C = \{alph(\rho) \mid \rho \in SFS\}$

(3) $\forall \rho, \rho' \in SFS. (alph(\rho) = alph(\rho') \Rightarrow (\rho u \in SFS \Leftrightarrow \rho' u \in SFS))$

(4) $\forall \rho \in SFS. \forall e \in E. num_e(\rho) \leq 1$.

Proof.

(1) Let $\rho \in SFS$. The proof is by induction on $k = |\rho|$. If $k = 0$ then $\rho = \emptyset$ and hence $cf(\rho) = \emptyset \in C$ and $cf(\rho) = \emptyset = alph(\rho)$. Now assume that $k > 0$. Then there exist $\rho' \in SFS$ and $\emptyset \neq u \in P_F(E)$ such that $cf(\rho') \vdash u$ and $\rho = \rho' u$. Hence $cf(\rho) = cf(\rho') \cup u \in C$ by (A2) and $cf(\rho) = alph(\rho)$ by the induction hypothesis applied to ρ'.

(2) If $\rho \in SFS$ then $alph(\rho) = cf(\rho) \in C$ by (1). Now let $c \in C$. We proceed by induction on $k = |c|$. If $k = 0$ then $c = \emptyset$ and hence $\rho = \emptyset \in SFS$ is such that $alph(\rho) = c$. Now assume that $k > 0$. Then by (A0) there exists $e \in c$ such that $c - e \vdash e$. By the induction hypothesis applied to $c - e$ there exists $\rho' \in SFS$ such that $alph(\rho') = cf(\rho') = c - e$. Then $\rho' e \in SFS$ by the definition of SFS and $alph(\rho' e) = c$.

(3) Let $\rho, \rho' \in SFS$ be such that $alph(\rho) = alph(\rho')$. If $u = \emptyset$ then $\rho u, \rho' u \in SFS$ by (A1). If $u \neq \emptyset$ then $cf(\rho) = cf(\rho')$ by (1) and hence $\rho u \in SFS$ iff $cf(\rho) \vdash u$ iff $\rho' u \in SFS$.

(4) Let $\rho \in SFS$. The proof is by induction on $k = |\rho|$. If $k = 0$ then the claim clearly holds. Now assume that $k > 0$. Then there exist $\rho' \in SFS$ and $\emptyset \neq u \in P_F(E)$ such that $\rho = \rho' u$ and $cf(\rho') \vdash u$. Then $num_e(\rho') \leq 1$ for all $e \in E$ by the induction hypothesis applied to ρ'. Because $cf(\rho') \cap u = \emptyset$ by (A2) and $alph(\rho') = cf(\rho')$ by (1) we can now conclude that also $num_e(\rho) \leq 1$ for all $e \in E$. $\square$

Finally in this section, we introduce structure-preserving morphisms between L-event structures.

Definition 1.5

An *LES-morphism* from an L-event structure $(E_1, C_1, \vdash_1)$ to an L-event structure $(E_2, C_2, \vdash_2)$ is a partial function $f : E_1 \rightarrow E_2$ such that:
$\forall c \in C_1. \forall u \in P_F(E_1). c \vdash_1 u \Rightarrow f(c) \vdash_2 f(u)$. $\square$

Here and in the sequel we adopt the convention that for a partial function $f : X_1 \rightarrow X_2$ and subsets $u_1 \subseteq X_1$ and $u_2 \subseteq X_2$, $f(u_1) = \{b \in X_2 \mid b = f(a) \text{ for some } a \in u_1\}$ and $f^{-1}(u_2) = \{a \in X_1 \mid f(a) = b \text{ for some } b \in u_2\}$.

This notion of morphism induces in a standard way a corresponding notion of isomorphism. Let, for an arbitrary L-event structure ES, id_{ES} denote the *identity LES-morphism* of ES which is the identity function on its events. Then an LES-morphism f from ES_1 to ES_2 is an *LES-isomorphism* iff there exists an LES-morphism g from ES_2 to ES_1 such that $g \circ f = id_{ES_1}$ and $f \circ g = id_{ES_2}$. It is easy to see that two L-event structures $ES_1 = (E_1, C_1, \vdash_1)$ and $ES_2 = (E_2, C_2, \vdash_2)$ are LES-isomorphic, denoted by $ES_1 \equiv ES_2$, iff there exists a bijection $f : E_1 \rightarrow E_2$ such that $c \vdash_1 u \Leftrightarrow f(c) \vdash_2 f(u)$.

We conclude with some properties of LES-morphisms which will be useful in later sections.

Lemma 1.6

Let f be an LES-morphism from $(E_1, C_1, \vdash_1)$ to $(E_2, C_2, \vdash_2)$ and let $c \in C_1$ and $e_1, e_2 \in c$ be such that $e_1 \neq e_2$ and both $f(e_1)$ and $f(e_2)$ are defined. Then $f(e_1) \neq f(e_2)$.

Proof.

By Lemma 1.3(1) we may assume without loss of generality that there exists $c' \sqsubseteq c$ such that $e_1 \in c'$ and $c' \vdash_1 e_2$. By the definition of an LES-morphism we then have $f(c') \vdash_2 f(e_2)$ and so $f(e_2) \notin f(c')$ by (A2), and $f(e_1) \in f(c')$. $\square$

Lemma 1.7

Let f be an LES-morphism from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$. Then $f(SFS_{ES_1}) \subseteq SFS_{ES_2}$ (where the homomorphic extension of f to step sequences is also denoted by f).

Proof.

Let $\rho \in SFS_{ES_1}$. We prove by induction on $|\rho|$ that $f(\rho) \in SFS_{ES_2}$. If $\rho = \emptyset$ then this is clear, so assume that there exist $\rho' \in SFS_{ES_1}$ and $\emptyset \neq u \in P_F(E_1)$ such that $\rho = \rho' u$. Then $alph(\rho') \vdash_1 u$. Hence $f(alph(\rho')) \vdash_2 f(u)$ because f is an LES-morphism. Since $f(\rho') \in SFS_{ES_2}$ by the induction hypothesis and $f(alph(\rho')) = alph(f(\rho'))$ this implies that $f(\rho')f(u) = f(\rho) \in SFS_{ES_2}$. $\square$

2 The Unique Occurrence Property

In this section we lift the unique occurrence property from the theory of prime event structures [NPW] to the more general framework of local event structures.

The definition of the unique occurrence property is based on an equivalence relation over *prime intervals*, that is, event occurrences. Rather than defining this equivalence relation directly in the context of local event structures, we define it in the more abstract setting of step sequences. Then the same idea of equivalence can be used in Section 3 to define a map from Petri nets to local event structures.

In order to define the equivalence relation and to establish some of its properties, we use an arbitrary but fixed set X, we let ρ range over $(P_F(X))^+$, a range over X, and u range over $P_F(X)$. Furthermore, we fix a set $L \subseteq (P_F(X))^+$ of step sequences satisfying the following two properties.

(L1) $\rho u \in L \Rightarrow \rho \in L$

(L2) $\rho u \in L \Rightarrow \forall v \subseteq u.\, \rho v(u - v) \in L$.

The set of *prime intervals* of L, denoted by PI_L, is given by: $PI_L = \{\rho a \mid \rho a \in L\}$. We sometimes write PI rather than PI_L if L is clear from the context.

Now let $R \subseteq PI \times PI$ be an equivalence relation. Then R is said to be *L-consistent* iff it satisfies the following conditions (C1) and (C2).

(C1) $(\rho u \in L$ and $a \in u) \Rightarrow \rho a\, R\, \rho(u - a)a$.

Note that (C1) is well-defined, because whenever $\rho u \in L$ and $a \in u$, then by (L2) $\rho a(u - a), \rho(u - a)a \in L$ and hence by (L1) also $\rho a \in L$.

The second condition demands that prime intervals $\rho a, \rho' a$ which have R-equivalent pasts in the sense that the same R-equivalent prime intervals occur in ρ and ρ' should in turn be R-equivalent. In order to formulate (C2) we adopt the following conventions.

$int_L : L \to P_F(PI)$, the function which maps each step sequence to the set of prime intervals in that sequence, is given inductively by: $int_L(\emptyset) = \emptyset$ and $int_L(\rho u) = int_L(\rho) \cup \{\rho a \mid a \in u\}$ for all $\rho u \in L$. Note that int_L is well-defined, because if $\rho u \in L$, then also $\rho \in L$ by (L1) and $\rho a \in L$ for all $a \in u$ by (L2). If L is clear from the context, then we may omit the subscript $_L$ in int_L.

For $\rho a \in PI$, $\langle \rho a \rangle_R$ is the equivalence class (under R) containing ρa, that is $\langle \rho a \rangle_R = \{\rho' a' \in PI \mid \rho' a'\, R\, \rho a\}$. Let $past_R : L \to P_F(PI/R)$ be given by: $past_R(\rho) = \{\langle \rho' a \rangle_R \mid \rho' a \in int(\rho)\}$.

(C2) $\rho a, \rho' a \in PI \Rightarrow (past_R(\rho) = past_R(\rho') \Rightarrow \rho a\, R\, \rho' a)$.

Note that in general there may be (infinitely) many equivalence relations which are L-consistent.

Lemma 2.1

Let $K = \{R \subseteq PI \times PI \mid R \text{ is an } L\text{-consistent equivalence relation}\}$. Then $K \neq \emptyset$ and $\bigcap K \in K$.

Proof.

Since $PI \times PI$ is clearly an equivalence relation which is L-consistent, we have that $K \neq \emptyset$.

Now let $\hat{R} = \bigcap K$. Then it is clear that $\hat{R}$ is an equivalence relation. Suppose $\rho u \in L$ and $a \in u$. Then $\rho a \, R \, \rho(u-a)a$ for all $R \in K$ because each $R \in K$ satisfies (C1). Hence also $\rho a \, \hat{R} \, \rho(u-a)a$.

In order to prove that $\hat{R}$ satisfies (C2), let $\rho a, \rho' a \in PI$ be such that $past_{\hat{R}}(\rho) = past_{\hat{R}}(\rho')$. It suffices to prove that $past_R(\rho) = past_R(\rho')$ for every $R \in K$. Because in that case $\rho a \, R \, \rho' a$ for every $R \in K$ and hence $\rho a \, \hat{R} \, \rho' a$.

So, let $R \in K$ and suppose $\langle \rho_1 a_1 \rangle_R \in past_R(\rho)$. Then there exists $\rho_2 a_2 \in int(\rho)$ such that $\langle \rho_1 a_1 \rangle_R = \langle \rho_2 a_2 \rangle_R$. We then also have that $\langle \rho_2 a_2 \rangle_{\hat{R}} \in past_{\hat{R}}(\rho) = past_{\hat{R}}(\rho')$. Then there exists $\rho_3 a_3 \in int(\rho')$ such that $\langle \rho_2 a_2 \rangle_{\hat{R}} = \langle \rho_3 a_3 \rangle_{\hat{R}}$. Hence also $\langle \rho_3 a_3 \rangle_R \in past_R(\rho')$. Moreover, $\langle \rho_2 a_2 \rangle_R = \langle \rho_3 a_3 \rangle_R$ because $\hat{R} \subseteq R$. This proves that $\langle \rho_1 a_1 \rangle_R \in past_R(\rho')$. Similarly it can be proved that $past_R(\rho') \subseteq past_R(\rho)$.

This proves that $past_R(\rho) = past_R(\rho')$ for all $R \in K$. $\square$

Hence there exists a least equivalence relation contained in $PI \times PI$ which is L-consistent. This equivalence relation (denoted as $\hat{R}$ in the proof of Lemma 2.1) will from now on be denoted as $\sim_L$.

In what follows we write $\langle \rho a \rangle_L$ and $past_L$ rather than $\langle \rho a \rangle_{\sim_L}$ and $past_{\sim_L}$ respectively. If $\sim_L$ is the only equivalence relation under consideration, then we may even omit the subscript $_L$.

Lemma 2.2

Let $\rho_1 a_1, \rho_2 a_2 \in PI$ be such that $\rho_1 a_1 \sim_L \rho_2 a_2$. Then

(1) $a_1 = a_2$ and $num_{a_1}(\rho_1) = num_{a_2}(\rho_2)$

(2) $\rho_1 a_1 \sim_{L'} \rho_2 a_2$ whenever $L' \subseteq (P_F(X))^+$ is such that L' satisfies (L1) and (L2) and $L \subseteq L'$.

Proof.

In order to prove (1), define the equivalence relation $R \subseteq PI \times PI$ by: $\rho a \, R \, \rho' a'$ iff $a = a'$ and $num_a(\rho) = num_{a'}(\rho')$. It is sufficient to prove that R is L-consistent. Then the required result would follow from the fact that $\sim_L \subseteq R$.

Clearly, R satisfies (C1). Let $\rho a, \rho' a \in PI$ be such that $past_R(\rho) = past_R(\rho')$. We first want to argue that $num_a(\rho') \geq num_a(\rho)$. If $num_a(\rho) = 0$ then this is trivial, so assume that $num_a(\rho) > 0$. Then there exists $\rho_1 a \in int(\rho)$ such that $num_a(\rho_1) = num_a(\rho) - 1$. Then

$\langle \rho_1 a \rangle_R \in past_R(\rho) = past_R(\rho')$. Hence there exists $\rho_2 a \in int(\rho')$ such that $\langle \rho_1 a \rangle_R = \langle \rho_2 a \rangle_R$ which implies that $num_a(\rho_1) = num_a(\rho_2)$. We now have $num_a(\rho') \geq num_a(\rho_2) + 1 = num_a(\rho_1) + 1 = num_a(\rho)$. Similarly we can prove that $num_a(\rho') \leq num_a(\rho)$ and thus $num_a(\rho) = num_a(\rho')$. Consequently $\rho a \, R \, \rho' a$ which implies that R satisfies (C2).

Now in order to prove (2), let $L' \subseteq (P_F(X))^+$ be such that $L \subseteq L'$ and L' satisfies (L1) and (L2).

Define the equivalence relation $R \subseteq PI_L \times PI_L$ by: $\rho a \, R \, \rho' a'$ iff $\rho a \sim_{L'} \rho' a'$. It is sufficient to prove that R is L-consistent because then $\sim_L \subseteq R$.

Clearly, R satisfies (C1). In order to prove (C2), let $\rho a, \rho' a \in PI_L$ be such that $past_R(\rho) = past_R(\rho')$. It is sufficient to show that $past_{L'}(\rho) = past_{L'}(\rho')$, because $\sim_{L'}$ satisfies (C2).

Let $\langle \rho_3 a_3 \rangle_{L'} \in past_{L'}(\rho)$. Then there exists $\rho_4 a_4 \in int_{L'}(\rho) = int_L(\rho)$ with $\langle \rho_3 a_3 \rangle_{L'} = \langle \rho_4 a_4 \rangle_{L'}$. Then also $\langle \rho_4 a_4 \rangle_R \in past_R(\rho) = past_R(\rho')$. Hence there exists $\rho_5 a_5 \in int_L(\rho') = int_{L'}(\rho')$ with $\langle \rho_4 a_4 \rangle_R = \langle \rho_5 a_5 \rangle_R$. Then $\rho_4 a_4 \sim_{L'} \rho_5 a_5$ by the definition of R. Moreover, $\langle \rho_5 a_5 \rangle_{L'} \in past_{L'}(\rho')$. This proves that $\langle \rho_3 a_3 \rangle_{L'} \in past_{L'}(\rho')$. Similarly it can be proved that $past_{L'}(\rho') \subseteq past_{L'}(\rho)$ and thus $past_{L'}(\rho) = past_{L'}(\rho')$. $\Box$

Note that for an L-event structure $ES = (E, C, \vdash)$, SFS is a subset of $(P_F(E))^+$ satisfying the conditions (L1) and (L2). Hence we have the equivalence relation $\sim_{SFS}$. In what follows we write PI_{ES}, int_{ES}, $\sim_{ES}$, $\langle \rho e \rangle_{ES}$, and $past_{ES}$ rather than PI_{SFS}, int_{SFS}, $\sim_{SFS}$, $\langle \rho e \rangle_{\sim_{ES}}$, and $past_{\sim_{ES}}$ respectively.

The unique occurrence property of local event structures is now defined in terms of the equivalence relation $\sim_{ES}$.

Definition 2.3
An L-event structure $ES = (E, C, \vdash)$ has the *unique occurrence property* if

(U1) $\forall e \in E. \exists \rho e \in PI_{ES}$

(U2) $\forall \rho_1 e, \rho_2 e \in PI_{ES}. \rho_1 e \sim_{ES} \rho_2 e$. $\Box$

From now on L-event structures satisfying the unique occurrence property will be referred to as *UL-event structures*.

Thus for an UL-event structure ES there exists a bijective correspondence between its events and the equivalence classes of its prime intervals under $\sim_{ES}$. Hence for each event all its occurrences are the same under $\sim_{ES}$

From the event structures from Example 1.2, ES_1 is not an UL-event structure. Both ES_2 and ES_3 are UL-event structures. In ES_3, $bc \sim_{ES_3} c$ and $cb \sim_{ES_3} b$ by (C1), and hence $past_{ES_3}(bc) = past_{ES_3}(cb)$. This implies that $bca \sim_{ES_3} cba$ by (C2). Then $a \sim_{ES_3} ca \sim_{ES_3} cba \sim_{ES_3} bca \sim_{ES_3} ba$ by (C1). Similarly, $b \sim_{ES_3} ab$, and hence $past_{ES_3}(ab) = past_{ES_3}(ba)$. Now $abd \sim_{ES_3} bad$ by (C2), even though $\{a, b\}$ is not enabled in $\emptyset$.

Next we show that there is a natural way to view *prime event structures* [NPW, W4] as UL-event structures. First we recall the definition of prime event structures from [W4].

Definition 2.4

A *prime event structure* is a triple $(E, \leq, \#)$ where E is a set of events, $\leq \subseteq E \times E$ is a partial order, the *causal dependency* relation, and $\# \subseteq E \times E$ is a symmetric, irreflexive relation, the *conflict* relation, satisfying

(P1) $e_0 \# e_1 \leq e_2 \Rightarrow e_0 \# e_2$

(P2) $\forall e \in E.\ \downarrow e$ is finite, where $\downarrow e = \{e' \in E \mid e' \leq e\}$. $\Box$

Let $P = (E, \leq, \#)$ be a prime event structure and $c \subseteq E$. We say that c is *downward-closed* iff $\forall e, e' \in E. ((e \in c \text{ and } e' \leq e) \Rightarrow e' \in c)$. We say that c is *#-free* iff $(c \times c) \cap \# = \emptyset$. If c is downward-closed and #-free, then c is called a *configuration*. In what follows we only deal with the *finite* configurations of a prime event structure. C_P denotes the set of finite configurations of the prime event structure P.

For a prime event structure $P = (E, \leq, \#)$, define $pu(P) = (E, C_P, \vdash)$ where $\vdash \subseteq C_P \times P_F(E)$ is given by: $c \vdash u$ iff $c \cap u = \emptyset$ and $\forall v \subseteq u.\, c \cup v \in C_P$.

Lemma 2.5

Let $P = (E, \leq, \#)$ be a prime event structure. Then $pu(P) = (E, C_P, \vdash)$ is an L-event structure.

Proof.

In order to prove that $pu(P)$ satisfies (A0), let $\emptyset \neq c \in C_P$. Let $e \in c$ be a maximal event in c in the sense that for all $e' \in c$, $e \leq e'$ implies that $e = e'$. Then $c - e \in C_P$ and hence $c - e \vdash e$. This proves that $pu(P)$ satisfies (A0). From the definition of $pu(P)$ it easily follows that $pu(P)$ satisfies (A1) and (A2). $\Box$

Our next aim is to prove that for each prime event structure P, the L-event structure $pu(P)$ has the unique occurrence property. The first step is to show that two step firing sequences of $pu(P)$ that lead to the same configuration have the same past (under $\sim_{pu(P)}$).

Lemma 2.6

Let $P = (E, \leq, \#)$ be a prime event structure with $pu(P) = (E, C_P, \vdash)$ and let $\rho_1, \rho_2 \in SFS$ be such that $alph(\rho_1) = alph(\rho_2)$. Then $past(\rho_1) = past(\rho_2)$.

Proof.

The proof is by induction on $k = |alph(\rho_1)|$. If $k = 0$ then $\rho_1 = \rho_2 = \emptyset$ and the claim clearly holds. Now assume that $k > 0$. Then there exist $\rho_1', \rho_2' \in SFS$ and $\emptyset \neq u_1, u_2 \in P_F(E)$ such that $\rho_1 = \rho_1' u_1$, $\rho_2 = \rho_2' u_2$, $cf(\rho_1') \vdash u_1$, and $cf(\rho_2') \vdash u_2$. Let $e_1 \in u_1$ and $e_2 \in u_2$. Then $\rho_1'(u_1 - e_1)e_1, \rho_2'(u_2 - e_2)e_2 \in SFS$ because $pu(P)$ satisfies (A2). Moreover, $past(\rho_1) = past(\rho_1'(u_1 - e_1)e_1)$ and $past(\rho_2) = past(\rho_2'(u_2 - e_2)e_2)$ because $\sim_{pu(P)}$ satisfies (C1).

If $e_1 = e_2$ then $alph(\rho_1'(u_1 - e_1)) = alph(\rho_2'(u_2 - e_2))$ and hence $past(\rho_1'(u_1 - e_1)) = past(\rho_2'(u_2 - e_2))$ by the induction hypothesis. This implies that $\rho_1'(u_1 - e_1)e_1 \sim_{pu(P)} \rho_2'(u_2 - e_2)e_2$, because $\sim_{pu(P)}$ satisfies (C2). Thus $past(\rho_1) = past(\rho_1'(u_1 - e_1)e_1) = past(\rho_1'(u_1 - e_1)) \cup \langle \rho_1'(u_1 - e_1)e_1 \rangle = past(\rho_2'(u_2 - e_2)) \cup \langle \rho_2'(u_2 - e_2)e_2 \rangle = past(\rho_2'(u_2 - e_2)e_2) = past(\rho_2)$.

Now assume that $e_1 \neq e_2$. Then it is easy to see that $alph(\rho_1) - \{e_1, e_2\} \in C_P$. By Lemma 2.5 and Lemma 1.4(2) there exists $\rho \in SFS$ such that $alph(\rho) = alph(\rho_1) - \{e_1, e_2\}$. Since $\rho e_1 \in SFS$ and $alph(\rho e_1) = alph(\rho'_2(u_2 - e_2))$, we have that $past(\rho e_1) = past(\rho'_2(u_2 - e_2))$ by the induction hypothesis. Similarly, $past(\rho e_2) = past(\rho'_1(u_1 - e_1))$. Hence $\rho e_1 e_2 \sim_{pu(P)} \rho'_2(u_2 - e_2)e_2$ and $\rho e_2 e_1 \sim_{pu(P)} \rho'_1(u_1 - e_1)e_1$ because $\sim_{pu(P)}$ satisfies (C2). Since $alph(\rho) \vdash \{e_1, e_2\}$ we also have that $\rho e_1 \sim_{pu(P)} \rho e_2 e_1$ and $\rho e_2 \sim_{pu(P)} \rho e_1 e_2$. Summarizing these results we can conclude that $past(\rho_1) = past(\rho'_1(u_1 - e_1)e_1) = past(\rho'_1(u_1 - e_1)) \cup \langle \rho'_1(u_1 - e_1)e_1 \rangle = past(\rho e_2) \cup \langle \rho e_2 e_1 \rangle = past(\rho) \cup \langle \rho e_2 \rangle \cup \langle \rho e_2 e_1 \rangle = past(\rho) \cup \langle \rho e_1 e_2 \rangle \cup \langle \rho e_1 \rangle = past(\rho e_1) \cup \langle \rho e_1 e_2 \rangle = past(\rho'_2(u_2 - e_2)) \cup \langle \rho'_2(u_2 - e_2)e_2 \rangle = past(\rho'_2(u_2 - e_2)e_2) = past(\rho_2)$.
□

Theorem 2.7

Let $P = (E, \leq, \#)$ be a prime event structure. Then $pu(P) = (E, C_P, \vdash)$ is an UL-event structure.

Proof.

By Lemma 2.5, $pu(P)$ is an L-event structure. We must show that $pu(P)$ has the unique occurrence property as stated in Definition 2.3.

Let $e \in E$. Then $\downarrow e - e, \downarrow e \in C_P$ and hence $\downarrow e - e \vdash e$. By Lemma 2.5 and Lemma 1.4(2), there exists $\rho \in SFS$ such that $alph(\rho) = \downarrow e - e$.

Then $\rho e \in PI$ and hence condition (U1) is satisfied. In order to prove that condition (U2) is satisfied, we first show that $\rho e \sim_{pu(P)} \rho' e$ for all $\rho' e \in PI$. Then by the transitivity of $\sim_{pu(P)}$ we have that also $\rho' e \sim_{pu(P)} \rho'' e$ for all $\rho' e, \rho'' e \in PI$.

So let $\rho' e \in PI$. Then $alph(\rho' e) \in C_P$ and hence $alph(\rho) \subseteq alph(\rho')$. We prove that $\rho e \sim_{pu(P)} \rho' e$ by induction on $|alph(\rho')|$. If $alph(\rho') = alph(\rho)$ then $past(\rho) = past(\rho')$ by Lemma 2.6. Hence $\rho e \sim_{pu(P)} \rho' e$ because $\sim_{pu(P)}$ satisfies (C2). Now assume that $|alph(\rho')| > |alph(\rho)|$. Then there exists $e' \in alph(\rho') - alph(\rho)$ such that e' is a maximal element in $alph(\rho')$ under $<$. Such an e' must exist because $alph(\rho')$ is a finite set and $<$ is a partial ordering relation. Then $alph(\rho') - e' \in C_P$ and $(alph(\rho') - e') \cup e \in C_P$. Let $\rho'' \in SFS$ be such that $alph(\rho'') = alph(\rho') - e'$. Then $\rho'' e \in PI$. Because $|alph(\rho'')| < |alph(\rho')|$, $\rho'' e \sim_{pu(P)} \rho e$ by the induction hypothesis. Now $alph(\rho'' e') = alph(\rho')$ and hence $past(\rho'' e') = past(\rho')$ by Lemma 2.6. Hence $\rho'' e' e \sim_{pu(P)} \rho' e$ because $\sim_{pu(P)}$ satisfies (C2). Since $alph(\rho'') \vdash \{e, e'\}$ and $\sim_{pu(P)}$ satisfies (C1), we also have that $\rho'' e' e \sim_{pu(P)} \rho'' e$. We can now conclude that $\rho e \sim_{pu(P)} \rho'' e \sim_{pu(P)} \rho'' e' e \sim_{pu(P)} \rho' e$. This proves condition (U2).
□

As to be expected, not every UL-event structure arises in this fashion. For instance, the UL-event structure ES_3 in Example 1.2 can not be the UL-event structure associated with any prime event structure. In Section 6 we will say more about the relationship between prime event structures and UL-event structures.

3 An Event Structure Semantics for Petri Nets

In [NPW] it has been shown how to associate a prime event structure with every 1-safe Petri net. Here we show how to associate an UL-event structure with every Petri net. It turns out that for 1-safe Petri nets both constructions agree (upto isomorphism) via the correspondence between prime event structures and UL-event structures given in the previous section.

Definition 3.1
A *Petri net* is a quadruple $N = (S, T, W, M_{in})$ where

(1) S is a set of *places* and T is a set of *transitions* such that $S \cap T = \emptyset$

(2) $W : (S \times T) \cup (T \times S) \to \mathbf{N}$ is a *weight function*

(3) $M_{in} : S \to \mathbf{N}$ is the *initial marking* of N. $\square$

Given a Petri net $N = (S, T, W, M_{in})$ and $x \in S \cup T$, let ${}^\bullet x = \{y \mid W(y, x) > 0\}$ be the set of *pre-elements* of x and $x^\bullet = \{y \mid W(x, y) > 0\}$ be the set of *post-elements* of x.

Observe that the initial marking of a Petri net can be seen as a multiset of places. Also in defining the dynamics of a Petri net we use multisets. Here, a multiset (over some given set X) is a function $u : X \to \mathbf{N}$. A multiset u is *finite* if $\sum_{a \in X} u(a) < \infty$. The set of finite multisets over X is denoted by $M_F(X)$. Note that $M_F(X)$ contains the empty multiset, denoted by $\underline{0}$, where $\underline{0}(a) = 0$ for all $a \in X$. A multiset u over X with the property that $u(a) \leq 1$ for all $a \in X$, may be identified with the subset $\{a \in X \mid u(a) = 1\}$ of X. In particular, if u is such that there is precisely one element $a \in X$ with $u(a) = 1$ and $u(b) = 0$ for all $b \in X$ with $b \neq a$, then we simply write a for u.

We view $(M_F(X))^+$ as a (free) monoid: the unit element is $\underline{0} \in M_F(X)$ and the product operation is the accordingly modified usual concatenation operation. Thus $\rho\underline{0} = \underline{0}\rho = \rho$ for all $\rho \in (M_F(X))^+$.

Definition 3.2
Let $N = (S, T, W, M_{in})$ be a Petri net. The set $MFS_N \subseteq (M_F(T))^+$ of *multiset firing sequences* of N, the set RM_N of *reachable markings* of N, and the *multiset transition relation* $\Longrightarrow_N \subseteq \{M_{in}\} \times MFS_N \times RM_N$ are the least sets satisfying the following two conditions.

(1) $\underline{0} \in MFS_N$, $M_{in} \in RM_N$, and $M_{in} \overset{\underline{0}}{\Longrightarrow}_N M_{in}$

(2) Suppose $\rho \in MFS_N$ and $M_{in} \overset{\rho}{\Longrightarrow}_N M$. Furthermore, suppose $u \in M_F(T)$ is such that $\forall s \in S.\, M(s) \geq \sum_{t \in T} u(t) \cdot W(s, t)$. Then $\rho u \in MFS_N$, $M' \in RM_N$, and $M_{in} \overset{\rho u}{\Longrightarrow}_N M'$ where $\forall s \in S.\, M'(s) = M(s) + \sum_{t \in T} u(t) \cdot (W(t, s) - W(s, t))$. $\square$

Given a Petri net $N = (S, T, W, M_{in})$, let $SFS_N = MFS_N \cap (P_F(T))^+$. We refer to SFS_N as the set of *step firing sequences* of N.

Now we will use SFS_N rather than MFS_N to associate an UL-event structure with every Petri net. It is in this sense that our event structure semantics "filters" out auto-concurrency.

The construction from Petri nets to UL-event structures is based on the equivalence relation $\sim_{SFS_N}$ over the prime intervals $PI_{SFS_N} = \{\rho t \mid \rho t \in SFS_N \text{ and } t \in T\}$ associated with SFS_N. That is, we follow the approach outlined in Section 2. Note that SFS_N satisfies the conditions (L1) and (L2) from Section 2 which implies that $\sim_{SFS_N}$ can be defined. In what follows we write PI_N, int_N, $\sim_N$, $\langle \rho t \rangle_N$, and $past_N$ rather than PI_{SFS_N}, int_{SFS_N}, $\sim_{SFS_N}$, $\langle \rho t \rangle_{\sim_N}$, and $past_{\sim_N}$ respectively.

Using these notions we can now associate with each Petri net N an L-event structure $nu(N)$. Then we prove that $nu(N)$ is even an UL-event structure.

Definition 3.3

Let $N = (S, T, W, M_{in})$ be a Petri net. Then $nu(N) = (E, C, \vdash)$ where

$E = \{\langle \rho t \rangle_N \mid \rho t \in PI_N\}$

$C = \{past_N(\rho) \mid \rho \in SFS_N\}$

$\vdash \subseteq C \times P_F(E)$ is given by: $c \vdash u$ iff there exists $\rho v \in SFS_N$ such that $past_N(\rho) = c$, and $u = \{\langle \rho t \rangle_N \mid t \in v\}$. $\square$

Lemma 3.4

Let $N = (S, T, W, M_{in})$ be a Petri net. Then $nu(N) = (E, C, \vdash)$ is an L-event structure.

Proof.

Let $\emptyset \neq \hat{c} \in C$. Then there exists $\rho u \in SFS_N$ such that $u \neq \emptyset$ and $\hat{c} = past_N(\rho u)$. Let $t \in u$. Then $\rho(u - t)t \in SFS_N$. Hence $past_N(\rho(u - t)) \vdash \langle \rho(u - t)t \rangle_N$. By condition (C1) we have that $\rho t \sim_N \rho(u - t)t$. Since $num_t(\rho_1) < num_t(\rho)$ for all $\rho_1 t \in int(\rho(u - t))$, we must have that $\langle \rho t \rangle_N \notin past_N(\rho(u - t))$ by Lemma 2.2(1). Hence $past_N(\rho(u - t)) = past_N(\rho u) - \langle \rho t \rangle_N$ and thus $\hat{c} - \langle \rho t \rangle_N \vdash \langle \rho t \rangle_N$. This proves that $nu(N)$ satisfies (A0).

Since $\rho\emptyset \in SFS_N$ for all $\rho \in SFS_N$, we have that $\hat{c} \vdash \emptyset$, for all $\hat{c} \in C$, and so $nu(N)$ also satisfies (A1).

Let $\hat{c} \in C$ and $\hat{u} \in P_F(E)$ be such that $\hat{c} \vdash \hat{u}$. Let $\rho u \in SFS_N$ be such that $past_N(\rho) = \hat{c}$ and $\hat{u} = \{\langle \rho t \rangle_N \mid t \in u\}$. First we must prove that $\hat{c} \cap \hat{u} = \emptyset$. If $\langle \rho_1 t_1 \rangle_N \in \hat{c} = past_N(\rho)$, then $num_{t_1}(\rho_1) < num_{t_1}(\rho)$ by Lemma 2.2(1). On the other hand, $\langle \rho_1 t_1 \rangle_N \in \hat{u}$ implies that $num_{t_1}(\rho_1) = num_{t_1}(\rho)$ by Lemma 2.2(1). Hence $\hat{c} \cap \hat{u} = \emptyset$. Now let $\hat{v} \subseteq \hat{u}$. Let $v \subseteq u$ be such that $\hat{v} = \{\langle \rho t \rangle_N \mid t \in v\}$. Then $\rho v(u - v) \in SFS_N$. Hence $\hat{c} \vdash \hat{v}$ and $\hat{c} \cup \hat{v} \vdash \{\langle \rho v t \rangle_N \mid t \in u - v\}$. For all $t \in u - v$, $\rho(v \cup t) \in SFS_N$ and so by condition (C1), $\rho t \sim_N \rho v t$. Therefore $\{\langle \rho v t \rangle_N \mid t \in u - v\} = \hat{u} - \hat{v}$. This proves that $nu(N)$ satisfies (A2). $\square$

Example 3.5

Let N_1 be the Petri net depicted in Figure 3 with its associated L-event structure $nu(N_1)$.

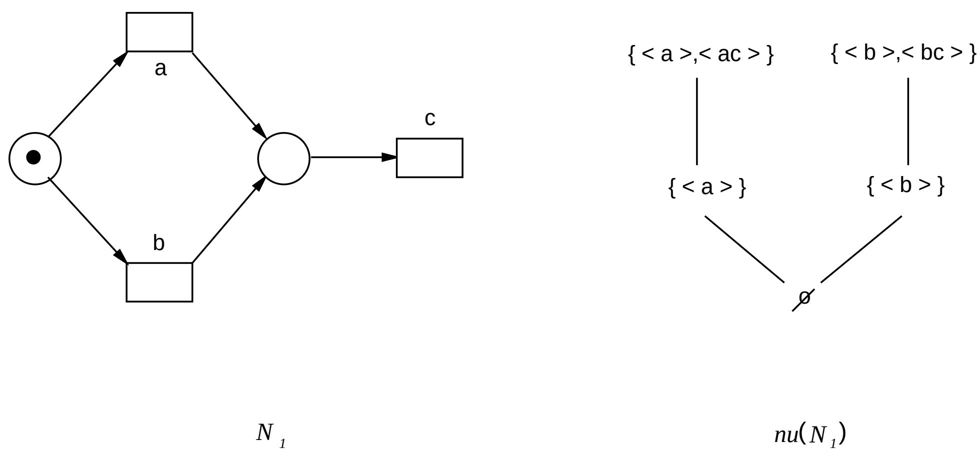


Figure 3: A Petri net and its associated L-event structure

For the transition c of N_1 there are two different events in $nu(N_1)$: $\langle ac\rangle_{N_1}$ and $\langle bc\rangle_{N_1}$. The L-event structure $nu(N_1)$ has four events and also four different equivalence classes of prime intervals (under $\sim_{nu(N_1)}$). Hence $nu(N_1)$ has the unique occurrence property.

Let N_2 be the Petri net depicted in Figure 4. In N_2, a and b can only occur concurrently if c occurs first. The transition d can only occur if both a and b have occurred, but c has not yet occurred. The L-event structure $nu(N_2)$ is ES_3 from Example 1.2 (where the unique equivalence class corresponding to each transition has been replaced by the transition itself). Thus, also $nu(N_2)$ has the unique occurrence property. $\square$

We now wish to prove that, given an arbitrary Petri net $N = (S, T, W, M_{in})$, the L-event structure $nu(N) = (E, C, \vdash)$ always has the unique occurrence property. To this end we first show how the set of step firing sequences of $nu(N)$ can be derived from the set of step firing sequences of N by means of a function seq_N which associates with every step firing sequence of N a step sequence over E.

Define the function $seq_N : SFS_N \to (P_F(E))^+$ inductively by: $seq_N(\emptyset) = \emptyset$ and $seq_N(\rho u) = seq_N(\rho)\{\langle \rho t\rangle_N \mid t \in u\}$. If the Petri net N is clear from the context, then we may omit the subscript $_N$ in seq_N.

Lemma 3.6

Let $N = (S, T, W, M_{in})$ be a Petri net. Then $seq(SFS_N) = SFS_{nu(N)}$.

Proof.

Let $nu(N) = (E, C, \vdash)$. Let $\rho \in SFS_N$. We prove that $seq(\rho) \in SFS_{nu(N)}$ and $cf(seq(\rho)) = past_N(\rho)$ by induction on $|\rho|$. If $\rho = \emptyset$ then this is clear, so assume

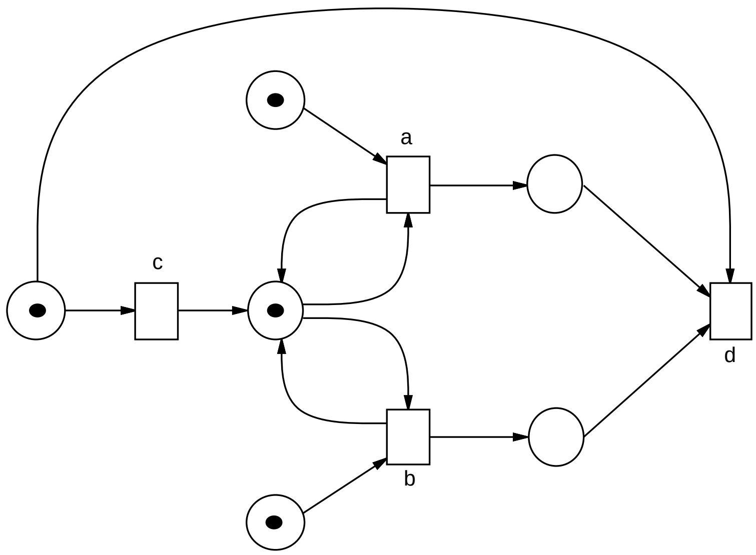


Figure 4: The Petri net N_2

that $\rho = \rho' u$ with $\rho' \in \mathit{SFS}_N$ and $\emptyset \neq u \in P_F(T)$. By the induction hypothesis $\mathit{seq}(\rho') \in \mathit{SFS}_{nu(N)}$ and $\mathit{cf}(\mathit{seq}(\rho')) = \mathit{past}_N(\rho')$. We also have, by the definition of $\vdash$, that $\mathit{past}_N(\rho') \vdash \hat{u}$ where $\hat{u} = \{\langle \rho' t \rangle_N \mid t \in u\}$. Hence $\mathit{seq}(\rho')\hat{u} \in \mathit{SFS}_{nu(N)}$ and $\mathit{cf}(\mathit{seq}(\rho')\hat{u}) = \mathit{past}_N(\rho') \cup \hat{u}$. Since $\mathit{seq}(\rho')\hat{u} = \mathit{seq}(\rho)$ and $\mathit{past}_N(\rho') \cup \hat{u} = \mathit{past}_N(\rho)$, we can now conclude that $\mathit{seq}(\rho) \in \mathit{SFS}_{nu(N)}$ and $\mathit{cf}(\mathit{seq}(\rho)) = \mathit{past}_N(\rho)$.

Now let $\hat{\rho} \in \mathit{SFS}_{nu(N)}$. We prove by induction on $|\hat{\rho}|$ that there exists $\rho \in \mathit{SFS}_N$ with $\mathit{seq}(\rho) = \hat{\rho}$ and $\mathit{past}_N(\rho) = \mathit{alph}(\hat{\rho})$. If $\hat{\rho} = \emptyset$ then $\rho = \emptyset$ is as required, so assume that $\hat{\rho} = \hat{\rho}'\hat{u}$ with $\hat{\rho}' \in \mathit{SFS}_{nu(N)}$ and $\emptyset \neq \hat{u} \in P_F(E)$. By the induction hypothesis there exists $\rho' \in \mathit{SFS}_N$ such that $\mathit{seq}(\rho') = \hat{\rho}'$ and $\mathit{past}_N(\rho') = \mathit{alph}(\hat{\rho}')$. Since $\mathit{past}_N(\rho') \vdash \hat{u}$ there exist $\rho_1 \in \mathit{SFS}_N$ and $u \in P_F(T)$ such that $\rho_1 u \in \mathit{SFS}_N$, $\mathit{past}_N(\rho_1) = \mathit{past}_N(\rho')$, and $\hat{u} = \{\langle \rho_1 t \rangle_N \mid t \in u\}$. From $\mathit{past}_N(\rho_1) = \mathit{past}_N(\rho')$ and Lemma 2.2(1) it easily follows that $\mathit{num}_t(\rho_1) = \mathit{num}_t(\rho')$ for all $t \in T$ and hence ρ_1 and ρ' lead to the same marking. Then we know from $\rho_1 u \in \mathit{SFS}_N$ that also $\rho' u \in \mathit{SFS}_N$. Moreover, $\langle \rho_1 t \rangle_N = \langle \rho' t \rangle_N$ for all $t \in u$ by condition (C2). Hence $\mathit{seq}(\rho' u) = \mathit{seq}(\rho')\{\langle \rho' t \rangle_N \mid t \in u\} = \hat{\rho}'\hat{u}$ and $\mathit{past}_N(\rho' u) = \mathit{past}_N(\rho') \cup \{\langle \rho' t \rangle_N \mid t \in u\} = \mathit{alph}(\hat{\rho}') \cup \hat{u} = \mathit{alph}(\hat{\rho}'\hat{u})$. $\Box$

The above lemma allows us to characterize $\mathit{int}_{nu(N)}$ as follows.

Lemma 3.7

Let $N = (S, T, W, M_{in})$ be a Petri net and let $\rho \in \mathit{SFS}_N$. Then $\mathit{int}_{nu(N)}(\mathit{seq}(\rho)) = \{\mathit{seq}(\rho')\langle \rho' t \rangle_N \mid \rho' t \in \mathit{int}_N(\rho)\}$.

Proof.

If $\rho = \emptyset$ then the claim trivially holds, so assume that $\rho = \rho_1 u$ with $\rho_1 \in \mathit{SFS}_N$ and $\emptyset \neq u \in P_F(T)$ and suppose that $\mathit{int}_{nu(N)}(\mathit{seq}(\rho_1)) = \{\mathit{seq}(\rho')\langle \rho' t \rangle_N \mid \rho' t \in \mathit{int}_N(\rho_1)\}$. Then

$int_{nu(N)}(seq(\rho)) = int_{nu(N)}(seq(\rho_1)) \cup \{seq(\rho_1)\hat{t} \mid \hat{t} \in \{\langle \rho_1 t\rangle_N \mid t \in u\}\} = \{seq(\rho')\langle \rho' t\rangle_N \mid \rho' t \in int_N(\rho)\}$. □

Lemma 3.6 implies a close relationship between the prime intervals of a Petri net N and the prime intervals of $nu(N) : PI_{nu(N)} = \{seq(\rho)\langle \rho t\rangle_N \mid \rho t \in PI_N\}$. Using Lemma 3.6 and Lemma 3.7 it is shown next that there is also a strong correspondence between the equivalence classes of prime intervals under $\sim_N$ and $\sim_{nu(N)}$.

Lemma 3.8

Let $N = (S, T, W, M_{in})$ be a Petri net and let $\rho_1 t_1, \rho_2 t_2 \in PI_N$. Then $\rho_1 t_1 \sim_N \rho_2 t_2$ iff $seq(\rho_1)\langle \rho_1 t_1\rangle_N \sim_{nu(N)} seq(\rho_2)\langle \rho_2 t_2\rangle_N$.

Proof.

If $seq(\rho_1)\langle \rho_1 t_1\rangle_N \sim_{nu(N)} seq(\rho_2)\langle \rho_2 t_2\rangle_N$, then by Lemma 2.2(1) $\langle \rho_1 t_1\rangle_N = \langle \rho_2 t_2\rangle_N$.

In order to prove the implication in the other direction, assume that $\langle \rho_1 t_1\rangle_N = \langle \rho_2 t_2\rangle_N$. Define the equivalence relation $R \subseteq PI_N \times PI_N$ by: $\rho t \, R \, \rho' t'$ iff $seq(\rho)\langle \rho t\rangle_N \sim_{nu(N)} seq(\rho')\langle \rho' t'\rangle_N$. Suppose that R is SFS_N-consistent. Since $\sim_N$ is the least equivalence relation which is SFS_N-consistent it follows that $\sim_N \subseteq R$. Hence $\rho_1 t_1 \, R \, \rho_2 t_2$ and thus, by the definition of R, $seq(\rho_1)\langle \rho_1 t_1\rangle_N \sim_{nu(N)} seq(\rho_2)\langle \rho_2 t_2\rangle_N$.

In order to prove that R satisfies (C1), suppose $\rho u \in SFS_N$ and $t \in u$. Since $\sim_N$ satisfies (C1), we have $\langle \rho t\rangle_N = \langle \rho(u-t)t\rangle_N$. We also have, by Lemma 3.6, that $seq(\rho u) \in SFS_{nu(N)}$. Combining this with $\sim_{nu(N)}$ satisfies (C1) leads to $seq(\rho)\langle \rho t\rangle_N \sim_{nu(N)} seq(\rho)(\hat{u} - \langle \rho t\rangle_N)\langle \rho t\rangle_N$ where $\hat{u} = \{\langle \rho t'\rangle_N \mid t' \in u\}$, because . Since $seq(\rho)(\hat{u} - \langle \rho t\rangle_N) = seq(\rho(u-t))$, we can now conclude by the definition of R that $\rho t R \rho(u-t)t$. This proves that R satisfies (C1).

Now suppose $\rho t, \rho' t \in PI_N$ are such that $past_R(\rho) = past_R(\rho')$. In order to prove that $\rho t \, R \, \rho' t$, we must show that $seq(\rho)\langle \rho t\rangle_N \sim_{nu(N)} seq(\rho')\langle \rho' t\rangle_N$. Because $\sim_{nu(N)}$ satisfies (C2), it suffices to prove that $past_{nu(N)}(seq(\rho)) = past_{nu(N)}(seq(\rho'))$ and $\langle \rho t\rangle_N = \langle \rho' t\rangle_N$.

In order to prove that $past_{nu(N)}(seq(\rho)) = past_{nu(N)}(seq(\rho'))$, let $\langle \hat{\rho}_1 \hat{t}_1\rangle_{nu(N)} \in past_{nu(N)}(seq(\rho))$. Then there exists $\hat{\rho}_3 \hat{t}_3 \in int(seq(\rho))$ such that $\langle \hat{\rho}_1 \hat{t}_1\rangle_{nu(N)} = \langle \hat{\rho}_3 \hat{t}_3\rangle_{nu(N)}$. By Lemma 3.7 there exists $\rho_3 t_3 \in int(\rho)$ such that $\hat{\rho}_3 \hat{t}_3 = seq(\rho_3)\langle \rho_3 t_3\rangle_N$. Then $\langle \rho_3 t_3\rangle_R \in past_R(\rho) = past_R(\rho')$. Hence there exists $\rho_4 t_4 \in int(\rho')$ such that $\langle \rho_3 t_3\rangle_R = \langle \rho_4 t_4\rangle_R$. Then, again by Lemma 3.7, $seq(\rho_4)\langle \rho_4 t_4\rangle_N \in int(seq(\rho'))$. Moreover, $\hat{\rho}_3 \hat{t}_3 \sim_{nu(N)} seq(\rho_4)\langle \rho_4 t_4\rangle_N$ by the definition of R. Hence $\langle \hat{\rho}_1 \hat{t}_1\rangle_{nu(N)} = \langle seq(\rho_4)\langle \rho_4 t_4\rangle_N\rangle_{nu(N)} \in past_{nu(N)}(seq(\rho'))$. This proves that $past_{nu(N)}(seq(\rho)) \subseteq past_{nu(N)}(seq(\rho'))$. By a symmetric argument we can show that $past_{nu(N)}(seq(\rho')) \subseteq past_{nu(N)}(seq(\rho))$ and thus $past_{nu(N)}(seq(\rho)) = past_{nu(N)}(seq(\rho'))$.

In order to prove that $\langle \rho t\rangle_N = \langle \rho' t\rangle_N$, it suffices to prove that $past_N(\rho) = past_N(\rho')$ because $\sim_N$ satisfies (C2). Let $\langle \rho_3 t_3\rangle_N \in past_N(\rho)$. Then there exists $\rho_4 t_4 \in int(\rho)$ such that $\langle \rho_3 t_3\rangle_N = \langle \rho_4 t_4\rangle_N$. By Lemma 3.7 we now have that $\hat{\rho}_4 \hat{t}_4 \in int(seq(\rho))$ where $\hat{\rho}_4 = seq(\rho_4)$ and $\hat{t}_4 = \langle \rho_4 t_4\rangle_N$. Hence $\langle \hat{\rho}_4 \hat{t}_4\rangle_{nu(N)} \in past_{nu(N)}(seq(\rho)) = past_{nu(N)}(seq(\rho'))$. Then there exists $\hat{\rho}_5 \hat{t}_5 \in int(seq(\rho'))$ such that $\langle \hat{\rho}_4 \hat{t}_4\rangle_{nu(N)} = \langle \hat{\rho}_5 \hat{t}_5\rangle_{nu(N)}$. By Lemma 2.2(1), $\hat{t}_4 = \hat{t}_5$. By Lemma 3.7 there exists $\rho_5 t_5 \in int(\rho')$ such that $\hat{\rho}_5 = seq(\rho_5)$ and $\hat{t}_5 = \langle \rho_5 t_5\rangle_N$. Then $\hat{t}_5 \in past_N(\rho')$, and so $\langle \rho_3 t_3\rangle_N = \hat{t}_4 = \hat{t}_5 \in past_N(\rho')$. This proves that $past_N(\rho) \subseteq past_N(\rho')$. Similarly we have that $past_N(\rho') \subseteq past_N(\rho)$ and thus $past_N(\rho) = past_N(\rho')$.

This finishes the proof that R satisfies (C2). Now we can conclude that $seq(\rho_1)\langle\rho_1 t_1\rangle_N \sim_{nu(N)} seq(\rho_2)\langle\rho_2 t_2\rangle_N$. $\square$

One of the main results of this paper can now be stated.

Theorem 3.9
Let $N = (S, T, W, M_{in})$ be a Petri net. Then $nu(N)$ is an UL-event structure.
Proof.
By Lemma 3.4, $nu(N)$ is an L-event structure. We must verify that $nu(N)$ satisfies the conditions (U1) and (U2) specified in the definition of the unique occurrence property.

Let $nu(N) = (E, C, \vdash)$. If $\langle\rho t\rangle_N \in E$ then $\rho t \in SFS_N$ and hence $past_N(\rho) \vdash \langle\rho t\rangle_N$. Hence $nu(N)$ satisfies (U1). Now in order to prove (U2), let $\hat{\rho}_1\hat{t}_1, \hat{\rho}_2\hat{t}_2 \in PI_{nu(N)}$ be such that $\hat{t}_1 = \hat{t}_2$. By Lemma 3.6 there exist $\rho_1, \rho_2 \in SFS_N$ and $t_1, t_2 \in T$ such that $\rho_1 t_1, \rho_2 t_2 \in SFS_N$, $\hat{\rho}_1 = seq(\rho_1)$, $\hat{\rho}_2 = seq(\rho_2)$, $\hat{t}_1 = \langle\rho_1 t_1\rangle_N$, and $\hat{t}_2 = \langle\rho_2 t_2\rangle_N$. Since $\hat{t}_1 = \hat{t}_2$ we then have by Lemma 3.8, that $\hat{\rho}_1\hat{t}_1 \sim_{nu(N)} \hat{\rho}_2\hat{t}_2$. $\square$

In [NPW] a map from 1-safe Petri nets to prime event structures is defined, which associates a prime event structure $npw(N)$ with each 1-safe Petri net N. In the present setting, a 1-safe Petri net is a Petri net N in which for every $M \in RM_N$ and every s of N, $M(s) \leq 1$. In addition we require, similar to [NPW], that a 1-safe Petri net does not have isolated transitions, that is transitions t with ${}^\bullet t \cup t^\bullet = \emptyset$.

Now let $NPW = pu \circ npw$, where pu is the map from prime event structures to UL-event structures defined in Section 1. Then we have the following result.

Theorem 3.10
Let N be a 1-safe Petri net. Then $nu(N) \equiv NPW(N)$. $\square$

The proof of this result is tedious, but straightforward to obtain by basically using arguments available in the literature. In particular, [WN] contains a representation result linking prime event structures to the Mazurkiewicz trace languages. The proof of this representation result given in [WN] can be easily adapted to serve as the backbone of the proof of Theorem 3.10.

Thus our event structure semantics for Petri nets, when restricted to 1-safe Petri nets, agrees completely (upto isomorphism) with the event structure semantics of [NPW] for 1-safe Petri nets. Clearly, the class of 1-safe Petri nets is properly included in the class of Petri nets. Note that the class of prime event structures (under the map pu) is properly included in the class of UL-event structures. Hence Theorem 3.9, Theorem 3.10, and Example 3.5 together assure us that our event structure semantics for Petri nets (even with auto-concurrency filtered out) is a *strictly* conservative extension of the basic result in [NPW].

To conclude this section, we identify the subclass of Petri nets which do not exhibit any auto-concurrency in their behaviours. This subclass of *co-safe* Petri nets will play a role in Section 5.

Definition 3.11

A Petri net N is *co-safe* if $MFS_N = SFS_N$. $\square$

Note that every 1-safe Petri net is co-safe. The class of co-safe Petri nets is however a non-trivial extension of the class of 1-safe Petri nets. The Petri net N_2 depicted in Figure 4 is co-safe, but not 1-safe. Interestingly enough, co-safe Petri nets also arise as the targets of the net semantics constructed for the process algebra called Petri Box Calculus [BDH]. This follows from the work of [De].

4 From Local Event Structures to Petri Nets

In [NPW] it is not only shown how to associate a prime event structure with each 1-safe Petri net, but also a map from prime event structures to 1-safe Petri nets is given. Our aim is to lift this construction also here; in other words, set up a map from UL-event structures to Petri nets. It turns out that the construction we have in mind works for *all* L-event structures. Hence we will construct a map from L-event structures to Petri nets. As a consequence, we will be able to show later that every L-event structure can in fact be represented as an UL-event structure.

Given a prime event structure $(E, \leq, \#)$, the causality relation $\leq$, the conflict relation $\#$, and the fact that each event occurs at most once makes it possible in [NPW] to quickly manufacture a suitable set of conditions. It is then easy to associate, in a canonical way, a 1-safe Petri net with each prime event structure. In the present setting, it is far from clear what causality, concurrency, and conflict could mean. Fortunately, there is a fairly well-understood construction, the so-called "regional" construction, by which one can manufacture places (of a Petri net) out of concurrency models which have a natural transition relation associated with them. (See, e.g., [ER], [NRT], [WN], [HKT1], [M]).

Definition 4.1

Let $ES = (E, C, \vdash)$ be an L-event structure. A *region* of ES is a function $r : C \cup E \to \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ satisfying the following conditions.

(1) $\forall c \in C.\, r(c) \in \mathbf{N}$ and $\forall e \in E.\, r(e) \in \mathbf{N} \times \mathbf{N}$.

For $e \in E$ we write $r(e) = ({}^{r}e, e^{r})$.

(2) $c \vdash u \Rightarrow (r(c) \geq \sum_{e \in u} {}^{r}e$ and $r(c \cup u) = r(c) + \sum_{e \in u}(e^{r} - {}^{r}e))$.

A region r of ES is *non-trivial* if $\exists e \in E.\, r(e) \neq (0, 0)$.
The set of non-trivial regions of ES is denoted by R_{ES}. $\square$

The map en from L-event structures to Petri nets is defined as follows. Let $ES = (E, C, \vdash)$ be an L-event structure. Then $en(ES) = (R_{ES}, E, W, M_{in})$ where

(1) $W : (R_{ES} \times E) \cup (E \times R_{ES}) \to \mathbf{N}$ is such that $\forall r \in R_{ES}. \forall e \in E.\, W(r, e) = {}^{r}e$ and $W(e, r) = e^{r}$

(2) $M_{in} : R_{ES} \rightarrow \mathbf{N}$ is such that $\forall r \in R_{ES}.\, M_{in}(r) = r(\emptyset)$.

The Petri net $en(ES)$ is "saturated" in the sense that no new places can be added without changing its behaviour or duplicating places.

For the L-event structure ES_3 from Example 1.2 the Petri net $en(ES_3)$ is depicted in Figure 5 where only some of the infinite number of places of $en(ES_3)$ have been drawn.

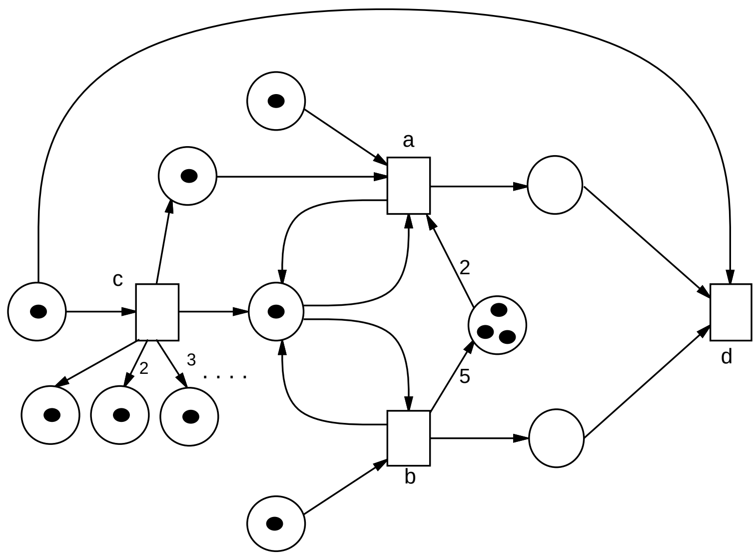


Figure 5: The Petri net $en(ES_3)$

The following lemma shows that $en(ES)$ has the same step firing sequences as ES. Moreover, it turns out that $MFS_{en(ES)} = SFS_{en(ES)}$ and so $en(ES)$ is a co-safe Petri net. While it is fairly straightforward to prove that $SFS_{ES} \subseteq SFS_{en(ES)}$, the converse inclusion requires a more complicated proof showing that ES has enough regions to prevent the existence of "wrong" step firing sequences in $SFS_{en(ES)}$.

Lemma 4.2
Let $ES = (E, C, \vdash)$ be an L-event structure. Then $SFS_{ES} = MFS_{en(ES)} = SFS_{en(ES)}$.
Proof.
Let $en(ES) = (R_{ES}, E, W, M_{in})$. Let for each $e \in E$ the function $r_e : C \cup E \rightarrow \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ be given by:

(1) $\forall e' \in E.\, r_e(e') = \begin{cases} (1,1) & \text{if } e' = e \\ (0,0) & \text{otherwise} \end{cases}$

(2) $\forall c \in C.\, r_e(c) = 1$.

Then each r_e is a non-trivial region of ES, and so it is clear that $MFS_{en(ES)} = SFS_{en(ES)}$.

Now suppose $\rho \in SFS_{ES}$. We prove by induction on $|\rho|$ that $\rho \in SFS_{en(ES)}$ and $r(alph(\rho)) = M(r)$ for all $r \in R_{ES}$ where $M \in RM_{en(ES)}$ is such that $M_{in} \stackrel{\rho}{\Longrightarrow}_{en(ES)} M$. If $\rho = \emptyset$ then this follows immediately, so assume that $\rho = \rho' u$ with $u \neq \emptyset$. Then $alph(\rho') \vdash u$. By the induction hypothesis $\rho' \in SFS_{en(ES)}$ and $r(alph(\rho')) = M'(r)$ for all $r \in R_{ES}$ where $M_{in} \stackrel{\rho'}{\Longrightarrow}_{en(ES)} M'$. By the definition of a region and the definition of $en(ES)$, $M'(r) = r(alph(\rho')) \geq \sum_{e \in u} {}^{r}e = \sum_{e \in u} W(r, e)$ for all $r \in R_{ES}$. This proves that $\rho' u \in SFS_{en(ES)}$. Moreover, if $M_{in} \stackrel{\rho}{\Longrightarrow}_{en(ES)} M$ then $r(alph(\rho)) = r(alph(\rho')) + \sum_{e \in u}(e^{r} - {}^{r}e) = M'(r) + \sum_{e \in u}(W(e, r) - W(r, e)) = M(r)$ for all $r \in R_{ES}$.

Conversely, suppose that $\rho \in SFS_{en(ES)}$. We prove by induction on $|\rho|$ that $\rho \in SFS_{ES}$ and, for all $r \in R_{ES}$, $M(r) = r(alph(\rho))$ where $M \in RM_{en(ES)}$ is such that $M_{in} \stackrel{\rho}{\Longrightarrow}_{en(ES)} M$. If $\rho = \emptyset$ then this is clear, so assume that $\rho = \rho' u$ with $\rho' \in SFS_{en(ES)}$ and $\emptyset \neq u \in P_F(E)$. Let $M' \in RM_{en(ES)}$ be such that $M_{in} \stackrel{\rho'}{\Longrightarrow}_{en(ES)} M'$. By the induction hypothesis $\rho' \in SFS_{ES}$ and, for all $r \in R_{ES}$, $M'(r) = r(alph(\rho'))$. We first prove that $alph(\rho') \cap u = \emptyset$.

Suppose $e \in alph(\rho')$. Then define $r\langle e \rangle : C \cup E \to \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ as follows.

(1) $\forall e' \in E.\, r\langle e \rangle(e') = \begin{cases} (1,0) & \text{if } e' = e \\ (0,0) & \text{otherwise.} \end{cases}$

(2) $\forall c \in C.\, r\langle e \rangle(c) = \begin{cases} 0 & \text{if } e \in c \\ 1 & \text{otherwise.} \end{cases}$

Claim 1. $r\langle e \rangle \in R_{ES}$.

Let us assume that Claim 1 holds. Then we have $M'(r\langle e \rangle) = r\langle e \rangle(alph(\rho')) = 0$. In addition we know that $W(r\langle e \rangle, e) = 1$ and, because $\rho' u \in SFS_{en(ES)}$, we also know that $M'(r\langle e \rangle) \geq \sum_{e' \in u} W(r\langle e \rangle, e')$. All this leads to the conclusion that $e \notin u$. This proves that $alph(\rho') \cap u = \emptyset$.

Now we observe that $\rho = \rho' u \in SFS_{ES}$ if $alph(\rho') \vdash u$. So denote $c = alph(\rho')$ and assume that $c \vdash u$ does not hold. This leads to a contradiction as we show next.

Define $r\langle u, c \rangle : C \cup E \to \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ as follows.

(1) $\forall e \in E.\, r\langle u, c \rangle(e) = \begin{cases} (1,0) & \text{if } e \in c \\ (1,1) & \text{if } e \in u \\ (0,1) & \text{otherwise.} \end{cases}$

(2) $\forall c' \in C.\, r\langle u, c \rangle(c') = |c| + |u| - 1 + \sum_{e \in c'}(e^{r\langle u,c \rangle} - {}^{r\langle u,c \rangle}e)$.

Claim 2. $r\langle u, c \rangle \in R_{ES}$.

If Claim 2 holds, then $M'(r\langle u, c \rangle) = r\langle u, c \rangle(c) = |u| - 1 < |u| = \sum_{e \in u} {}^{r\langle u,c \rangle}e =$

$\sum_{e \in u} W(r\langle u, c\rangle, e)$, a contradiction with $\rho' u \in SFS_{en(ES)}$. Thus $c \vdash u$ and hence $\rho = \rho' u \in SFS_{ES}$. Moreover, $r(alph(\rho)) = r(c \cup u) = r(c) + \sum_{e \in u}(e^r - {}^r e) = M'(r) + \sum_{e \in u}(W(e, r) - W(r, e)) = M(r)$ for all $r \in R_{ES}$.

Thus if we prove Claim 1 and Claim 2 then we can conclude that $SFS_{ES} = SFS_{en(ES)}$.

Proof of Claim 1.

To simplify the notation we write r instead of $r\langle e\rangle$. Suppose $c' \vdash v$. Since $c' \cap v = \emptyset$ by (A2) we then have that $r(c' \cup v) = r(c') - |v \cap e| = r(c') + \sum_{e' \in v}(e'^r - {}^r e')$ and $r(c') = r(c' \cup v) + |v \cap e| \geq |v \cap e| = \sum_{e' \in v} {}^r e'$. Hence r is a region of ES which is clearly non-trivial. This proves Claim 1.

Proof of Claim 2.

In order to simplify the notation, we write r instead of $r\langle u, c\rangle$ in this proof.

Suppose $c' \in C$ and $v \in P_F(E)$ are such that $c' \vdash v$. Since $c' \cap v = \emptyset$ by (A2) we immediately have that $r(c' \cup v) = r(c') + \sum_{e \in v}(e^r - {}^r e)$. Now we must prove that $r(c') \geq \sum_{e \in v} {}^r e$.

Let $n = |v \cap (c \cup u)| = \sum_{e \in v} {}^r e$. Then we must prove that $r(c') \geq n$. Set $k = |c' \cap u|$ and $j = |c' \cap c|$ and $m = |c' \cap (E - (c \cup u))|$. Since $c \cap u = \emptyset$ and $c' \cap v = \emptyset$ it follows that $n \leq |c| + |u| - k - j$. Moreover, by the definition of r, it is clear that $r(c') = |c| + |u| - 1 + k + m - k - j = |c| + |u| - 1 + m - j$. Hence if $m + k \geq 1$ we are done. Therefore we assume in the rest of the proof that $m = k = 0$. In other words, we assume that $c' \subseteq c$. This leads to the equation $r(c') = |c| + |u| - 1 - |c'|$. On the other hand, $n \leq |c| + |u| - |c'|$. If $n < |c| + |u| - |c'|$ then we at once get $r(c') \geq n$. We now wish to argue that $n = |c| + |u| - |c'|$ leads to a contradiction.

To see this, suppose that $n = |c| + |u| - |c'|$. Let $v_1 = v \cap c$ and $v_2 = v \cap u$. Then from $c' \cap v = \emptyset$ and $c' \subseteq c$ it follows that $v_1 = c - c'$ and $v_2 = u$. Since $c' \vdash v$ we also have that $c' \vdash (v_1 \cup v_2)$ by (A2). Again by (A2) we now know that $(c' \cup v_1) \vdash v_2$. Since $c' \cup v_1 = c$ and $v_2 = u$ this leads to a contradiction. This proves that $n = |c| + |u| - |c'|$ is not possible, so $r(c') \geq n$.

This proves that r is a region of ES. Since $u \neq \emptyset$, r is also non-trivial. This finishes the proof of Claim 2. $\square$

From the proof of the above lemma it follows that $en(ES)$ is not just a co-safe Petri net. In fact $en(ES)$ has enough places to ensure that it is a *locally sequential* Petri net.

A locally sequential Petri net is a Petri net $N = (S, T, W, M_{in})$ where for each $t \in T$ there exists a "private" place $s_t \in S$ such that $M_{in}(s_t) = 1$ and, for each $x \in T$, $W(s_t, x) = W(x, s_t) = 1$ if $x = t$ and $W(s_t, x) = W(x, s_t) = 0$ otherwise.

Thus in a locally sequential Petri net co-safety is guaranteed by purely structural means.

Recall that our main aim is to associate a Petri net with every UL-event structure. It turns out that our map en (which acts on all L-event structures), when restricted to UL-event structures, fits in very well with the map nu from Petri nets to UL-event structures given in Section 3.

Let $ES = (E, C, \vdash)$ be an UL-event structure with $nu(en(ES)) = (\hat{E}, \hat{C}, \hat{\vdash})$. Define $v_{ES} : E \to \hat{E}$ as follows. Let $e \in E$. By the unique occurrence property there exists a unique equivalence class $\langle \rho e \rangle_{ES}$. Now let $v_{ES}(e) = \langle \rho e \rangle_{en(ES)}$. Note that by Lemma 4.2, $SFS_{ES} \subseteq SFS_{en(ES)}$, and so $v_{ES}(e)$ is well-defined by Lemma 2.2(2).

Theorem 4.3

Let ES be an UL-event structure. Then v_{ES} an LES-isomorphism from ES to $nu(en(ES))$ and so $ES \equiv nu(en(ES))$.

Proof.

Let $ES = (E, C, \vdash)$ and $nu(en(ES)) = (\hat{E}, \hat{C}, \hat{\vdash})$ and let $c \in C$ and $u \in P_F(E)$.

Suppose $c \vdash u$. Let $\rho \in SFS_{ES}$ be such that $alph(\rho) = c$. Then $\rho u \in SFS_{ES}$ and hence $\rho u \in SFS_{en(ES)}$ by Lemma 4.2. This implies by the definition of nu that $past_{en(ES)}(\rho) \hat{\vdash} \hat{u}$ where $\hat{u} = \{\langle \rho e \rangle_{en(ES)} \mid e \in u\}$. In order to prove that $v_{ES}(c) \hat{\vdash} v_{ES}(u)$ we must prove that $v_{ES}(c) = past_{en(ES)}(\rho)$ and $v_{ES}(u) = \hat{u}$.

Suppose $e_1 \in c$ with $\rho_1 e_1 \in PI_{ES}$ such that $v_{ES}(e_1) = \langle \rho_1 e_1 \rangle_{en(ES)}$. From $e_1 \in alph(\rho)$ it follows that there exists $\rho_1' e_1 \in int_{ES}(\rho) = int_{en(ES)}(\rho)$. Moreover, by the unique occurrence property $\langle \rho_1 e_1 \rangle_{ES} = \langle \rho_1' e_1 \rangle_{ES}$ and hence, by Lemma 2.2(1) and Lemma 4.2, also $\langle \rho_1 e_1 \rangle_{en(ES)} = \langle \rho_1' e_1 \rangle_{en(ES)}$. Since $\langle \rho_1' e_1 \rangle_{en(ES)} \in past_{en(ES)}(\rho)$, this proves that $v_{ES}(e_1) \in past_{en(ES)}(\rho)$.

Now suppose $\langle \rho_1 e_1 \rangle_{en(ES)} \in past_{en(ES)}(\rho)$. Then there exists $\rho_1' e_1 \in int_{en(ES)}(\rho) = int_{ES}(\rho)$ such that $\langle \rho_1 e_1 \rangle_{en(ES)} = \langle \rho_1' e_1 \rangle_{en(ES)}$. Hence $e_1 \in alph(\rho) = c$ and $v_{ES}(e_1) = \langle \rho_1' e_1 \rangle_{en(ES)}$. This proves that $past_{en(ES)}(\rho) \subseteq v_{ES}(c)$ and hence $v_{ES}(c) = past_{en(ES)}(\rho)$. It easily follows that $v_{ES}(u) = \hat{u}$. Hence $v_{ES}(c) \hat{\vdash} v_{ES}(u)$. This proves that v_{ES} is an LES-morphism from ES to $nu(en(ES))$.

In order to prove that v_{ES} is an LES-isomorphism, suppose $v_{ES}(c) \hat{\vdash} v_{ES}(u)$. Then there exists $\rho v \in SFS_{en(ES)}$ such that $v_{ES}(c) = past_{en(ES)}(\rho)$ and $v_{ES}(u) = \{\langle \rho e \rangle_{en(ES)} \mid e \in v\}$. This implies that $c = alph(\rho)$ and $u = v$. Moreover, $\rho v \in SFS_{ES}$ by Lemma 4.2 and hence $c \vdash u$. Since v_{ES} is a bijection, we can conclude that v_{ES} is an LES-isomorphism. $\square$

Once again this result mirrors a property established for prime event structures in [NPW].

5 Universality of the Constructions

The back-and-forth constructions established in [NPW] between 1-safe Petri nets and prime event structures were later proved by Winskel [W3] to be the "right" ones. He achieved this by equipping both classes of objects with suitable behaviour-preserving morphisms and showed that the constructions of [NPW] smoothly lift to a pair of functors which constitute a co-reflection. Our aim here is to explore to what extent we can mimic this categorical result in the present, much richer setting. We show that due to auto-concurrency we can not obtain a co-reflection between the categories of UL-event structures and Petri nets defined in this section. We do however get a co-reflection for the subcategory of co-safe Petri nets. This is the main result of this section. A consequence of this result is that the category of UL-event structures is a full co-reflective subcategory of the category of L-event structures.

Let us first introduce the various categories. We have already defined morphisms for L-event structures, which leads to the following definition.

Definition 5.1
Let $\mathcal{LES}$ be the category which has L-event structures as its objects and LES-morphisms as its arrows. The identity morphism associated with an object is the identity function on its events; composition of LES-morphisms is composition of partial functions.
Let $\mathcal{ULES}$ be the full subcategory of $\mathcal{LES}$ the objects of which are UL-event structures. $\square$

As for Petri nets, previous research [W2, M] shows that the notion of morphism for Petri nets formulated in the next definition is the appropriate one in the present context.

Definition 5.2
$\mathcal{PN}$ is the category which has Petri nets as its objects and *PN-morphisms* as its arrows. A PN-morphism $(\beta, \eta) : (S_1, T_1, W_1, M_1) \to (S_2, T_2, W_2, M_2)$ consists of partial functions $\beta : S_2 \to S_1$ and $\eta : T_1 \to T_2$ such that

(1) $\forall s_2 \in S_2. (\beta(s_2) \text{ is defined } \Rightarrow M_2(s_2) = M_1(\beta(s_2)))$

(2) $\forall t_1 \in T_1. (\eta(t_1) \text{ is undefined} \Rightarrow \beta^{-1}({}^\bullet t_1) = \beta^{-1}(t_1{}^\bullet) = \emptyset)$

(3) $\forall t_1 \in T_1. (\eta(t_1) \text{ is defined} \Rightarrow$

(3a) $\beta^{-1}({}^\bullet t_1) = {}^\bullet\eta(t_1)$ and $\beta^{-1}(t_1{}^\bullet) = \eta(t_1)^\bullet$ and

(3b) $\forall s_2 \in {}^\bullet\eta(t_1). W_2(s_2, \eta(t_1)) = W_1(\beta(s_2), t_1)$ and

(3c) $\forall s_2 \in \eta(t_1)^\bullet. W_2(\eta(t_1), s_2) = W_1(t_1, \beta(s_2)))$.

The identity morphism associated with an object is the pair of identity functions on places and transitions; composition of PN-morphisms (β_1, η_1) from N_1 to N_2 and (β_2, η_2) from N_2 to N_3 is the PN-morphism $(\beta_1 \circ \beta_2, \eta_2 \circ \eta_1)$ from N_1 to N_3 (where $\circ$ denotes composition of partial functions). $\square$

Example 5.3
The pair of functions (β, η) indicated in Figure 6 is a PN-morphism from N_3 to N_4. $\square$

PN-morphisms are behaviour-preserving in the following sense [M].

Lemma 5.4
Let N_i, $i = 1, 2$, be Petri nets and let (β, η) be a PN-morphism from N_1 to N_2. Then $\eta(\rho) \in MFS_{N_2}$ for all $\rho \in MFS_{N_1}$. $\square$

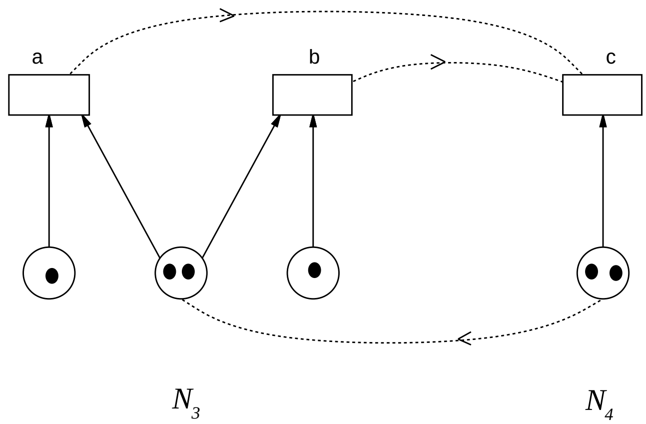


Figure 6: A PN-morphism (β, η)

In a later part of this section we will use the fact that the Petri net $en(ES)$ associated with an L-event structure ES in Section 4 has no isolated places and is *S-simple*.

A Petri net (S, T, W, M_{in}) is S-simple if $\forall s_1, s_2 \in S. (M_{in}(s_1) = M_{in}(s_2)$ and $\forall t \in T. (W(t, s_1) = W(t, s_2)$ and $W(s_1, t) = W(s_2, t)) \Rightarrow s_1 = s_2)$.

For such a Petri net, a PN-morphism is completely determined by its transition function, which follows from another result by [M].

Lemma 5.5

Let (β_1, η) and (β_2, η) be a pair of PN-morphisms from N_1 to N_2 where N_1 has no isolated places and is S-simple. Then $\beta_1 = \beta_2$. $\square$

We are looking for a co-reflection between $\mathcal{ULES}$ and $\mathcal{PN}$ in which the left adjoint would act as en on the objects of $\mathcal{ULES}$ and the right adjoint would act as nu on the objects of $\mathcal{PN}$.

To achieve this, we would like to extend the map nu to become a functor from $\mathcal{PN}$ to $\mathcal{ULES}$ in such a way that prime intervals are preserved. This means that whenever (β, η) is a PN-morphism from N to N' and $\langle \rho t \rangle_N$ is an event of $nu(N)$, then $\eta(t)$ is defined iff $nu((\beta, \eta))(\langle \rho t \rangle_N)$ is defined. Unfortunately, this is not possible. Consider, e.g., the PN-morphism (β, η) from N_3 to N_4 in Example 5.3. The UL-event structure $nu(N_3)$ has two events, $\langle a \rangle_{N_3} = \langle ba \rangle_{N_3}$ and $\langle b \rangle_{N_3} = \langle ab \rangle_{N_3}$. Also the UL-event structure $nu(N_4)$ has two events, $\langle c \rangle_{N_4}$ and $\langle cc \rangle_{N_4}$. Even though both $\eta(a)$ and $\eta(b)$ are defined, there exists however no LES-morphism f from $nu(N_3)$ to $nu(N_4)$ in which both $f(\langle a \rangle_{N_3})$ and $f(\langle b \rangle_{N_3})$ are defined.

The problem is that in a PN-morphism transitions which can occur concurrently, may be mapped to the same transition, leading to auto-concurrency. As a consequence, *step* firing sequences of the first Petri net may be mapped to *multiset* firing sequences of the second Petri net. For this reason we restrict our attention to co-safe Petri nets in the rest of this section.

Definition 5.6

Let $\mathcal{PNS}$ be the full subcategory of $\mathcal{PN}$ the objects of which are co-safe Petri nets. $\square$

In what follows the map *nu* defined in Section 3, when restricted to co-safe Petri nets, is extended to a functor from $\mathcal{PNS}$ to $\mathcal{ULES}$. Then the map *en* defined in Section 4 is extended to a functor from $\mathcal{LES}$ to $\mathcal{PNS}$. Once these functors are defined we can prove the desired co-reflection between $\mathcal{ULES}$ and $\mathcal{PNS}$.

From Lemma 5.4 we already know that for co-safe Petri nets prime intervals are preserved under PN-morphisms. In the following lemma it is proved that for co-safe Petri nets also equivalence of prime intervals is preserved under PN-morphisms.

Lemma 5.7

Let $N_i = (S_i, T_i, W_i, M_i)$, $i = 1, 2$, be co-safe Petri nets and let (β, η) be a PN-morphism from N_1 to N_2. Let $t \in T$ be such that $\eta(t)$ is defined and let $\rho t, \rho' t \in PI_{N_1}$. Then $\rho t \sim_{N_1} \rho' t$ implies $\eta(\rho)\eta(t) \sim_{N_2} \eta(\rho')\eta(t)$.

Proof.

Define $R \subseteq PI_{N_1} \times PI_{N_1}$ by: $\rho_1 t_1 \, R \, \rho_2 t_2$ iff ($t_1 = t_2$ and $\eta(t_1)$ is undefined) or ($\eta(t_1)$ and $\eta(t_2)$ are defined and $\eta(\rho_1)\eta(t_1) \sim_{N_2} \eta(\rho_2)\eta(t_2)$). Note that R is an equivalence relation. Suppose R is SFS_{N_1}-consistent. Then since $\sim_{N_1}$ is the least equivalence relation which is SFS_{N_1}-consistent, it follows that $\sim_{N_1} \subseteq R$. Hence $\rho t \sim_{N_1} \rho' t$ implies $\rho t \, R \, \rho' t$ and thus, by the definition of R, $\eta(\rho)\eta(t) \sim_{N_2} \eta(\rho')\eta(t)$. Thus it is sufficient to prove that R satisfies the conditions (C1) and (C2).

Suppose $\rho_1 u \in SFS_{N_1}$ and $t_1 \in u$. If $\eta(t_1)$ is undefined then we immediately have that $\rho_1 t_1 R \rho_1 (u - t_1) t_1$, so assume that $\eta(t_1)$ is defined. Then $\eta(\rho_1 u) \in SFS_{N_2}$ by Lemma 5.4 and $\eta(t_1) \in \eta(u)$. Since $\sim_{N_2}$ satisfies (C1), it then follows that $\eta(\rho_1)\eta(t_1) \sim_{N_2} \eta(\rho_1)(\eta(u) - \eta(t_1))\eta(t_1)$. Moreover, by Lemma 5.4 and the fact that N_2 is co-safe we have that $\eta(\rho_1)(\eta(u) - \eta(t_1)) = \eta(\rho_1(u - t_1))$. This yields $\rho_1 t_1 \, R \, \rho_1(u - t_1)t_1$ by the definition of R. Thus R satisfies (C1).

Now suppose $\sigma t', \sigma' t' \in PI_{N_1}$ are such that $past_R(\sigma) = past_R(\sigma')$. If $\eta(t')$ is undefined then we immediately have that $\sigma t' \, R \, \sigma' t'$, so assume that $\eta(t')$ is defined. Suppose $past_{N_2}(\eta(\sigma)) = past_{N_2}(\eta(\sigma'))$. Then since $\sim_{N_2}$ satisfies (C2) we know that $\eta(\sigma)\eta(t') \sim_{N_2} \eta(\sigma')\eta(t')$ and hence $\sigma t' \, R \, \sigma' t'$. Thus in order to prove that R satisfies (C2), it is sufficient to prove that $past_{N_2}(\eta(\sigma)) = past_{N_2}(\eta(\sigma'))$.

Let $\langle \rho_1 t_1 \rangle_{N_2} \in past_{N_2}(\eta(\sigma))$. Then there exists $\rho_2 t_2 \in int(\sigma)$ such that $\eta(t_2)$ is defined and $\langle \rho_1 t_1 \rangle_{N_2} = \langle \eta(\rho_2)\eta(t_2) \rangle_{N_2}$. Then also $\langle \rho_2 t_2 \rangle_R \in past_R(\sigma) = past_R(\sigma')$. Hence there exists $\rho_3 t_3 \in int(\sigma')$ such that $\langle \rho_2 t_2 \rangle_R = \langle \rho_3 t_3 \rangle_R$. Since $\eta(t_2)$ is defined this implies that $\eta(t_3)$ is also defined and $\langle \eta(\rho_2)\eta(t_2) \rangle_{N_2} = \langle \eta(\rho_3)\eta(t_3) \rangle_{N_2}$. Moreover, $\langle \eta(\rho_3)\eta(t_3) \rangle_{N_2} \in past_{N_2}(\eta(\sigma'))$ by the definition of *past*. Hence $\langle \rho_1 t_1 \rangle_{N_2} \in past_{N_2}(\eta(\sigma'))$. This proves that $past_{N_2}(\eta(\sigma)) \subseteq past_{N_2}(\eta(\sigma'))$. Similarly we have $past_{N_2}(\eta(\sigma')) \subseteq past_{N_2}(\eta(\sigma))$ and thus $past_{N_2}(\eta(\sigma)) = past_{N_2}(\eta(\sigma'))$. $\square$

Now we can extend the map *nu* to a functor, also denoted by *nu*, from $\mathcal{PNS}$ to $\mathcal{ULES}$.

Let N_1 and N_2 be a pair of co-safe Petri nets and let (β, η) be a PN-morphism from N_1 to N_2. Suppose $nu(N_1) = (E_1, C_1, \vdash_1)$ and $nu(N_2) = (E_2, C_2, \vdash_2)$. Then we define $nu((\beta, \eta))$ to be the partial function from E_1 to E_2 given by:

$$\forall \langle \rho t \rangle_{N_1} \in E_1.\, nu((\beta,\eta))(\langle \rho t \rangle_{N_1}) = \begin{cases} \text{undefined} & \text{if } \eta(t) \text{ is undefined} \\ \langle \eta(\rho)\eta(t) \rangle_{N_2} & \text{otherwise.} \end{cases}$$

Note that by Lemma 5.7 $nu((\beta, \eta))$ is well-defined.

Lemma 5.8

Let N_1 and N_2, be co-safe Petri nets and let (β, η) be a PN-morphism from N_1 to N_2. Then $nu((\beta, \eta))$ is an LES-morphism from $nu(N_1)$ to $nu(N_2)$.

Proof.

Let $nu(N_1) = (E_1, C_1, \vdash_1)$ and $nu(N_2) = (E_2, C_2, \vdash_2)$. Let $nu((\beta, \eta))$ be denoted by f. Given $\hat{c} \vdash_1 \hat{u}$ we have to prove that $f(\hat{c}) \vdash_2 f(\hat{u})$. So suppose $\hat{c} \vdash_1 \hat{u}$. Then there exists $\rho u \in SFS_{N_1}$ such that $\hat{c} = past_{N_1}(\rho)$ and $\hat{u} = \{\langle \rho t \rangle_{N_1} \mid t \in u\}$. By Lemma 5.4 we have that $\eta(\rho), \eta(\rho u) \in SFS_{N_2}$. Hence by the definition of $\vdash_2$ $past_{N_2}(\eta(\rho)) \vdash_2 \{\langle \eta(\rho) t' \rangle_{N_2} \mid t' \in \eta(u)\}$. Now $past_{N_2}(\eta(\rho)) = \{\langle \rho_2 t_2 \rangle_{N_2} \mid \rho_2 t_2 \in int(\eta(\rho))\} = \{\langle \eta(\rho_1)\eta(t_1) \rangle_{N_2} \mid \rho_1 t_1 \in int(\rho)$ with $\eta(t_1)$ defined $\} = f(past_{N_1}(\rho)) = f(\hat{c})$. Furthermore, $\{\langle \eta(\rho) t' \rangle_{N_2} \mid t' \in \eta(u)\} = \{\langle \eta(\rho)\eta(t) \rangle_{N_2} \mid t \in u$ with $\eta(t)$ defined$\} = f(\hat{u})$. And so $f(\hat{c}) \vdash_2 f(\hat{u})$ as required. $\square$

From the definition of nu it easily follows that nu preserves identities and respects composition. Hence the following result follows from Theorem 3.9 and Lemma 5.8.

Theorem 5.9

nu is a functor from $\mathcal{PNS}$ to $\mathcal{ULES}$. $\square$

Next the map en is extended to a functor - also denoted by en - from $\mathcal{LES}$ to $\mathcal{PNS}$. Then we show that this functor is in fact full and faithful.

In order to define en on arrows, we first need the following notion of the inverse image of a region. Given an LES-morphism f from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$ and a region r of ES_2, define $f^{-1}(r) : C_1 \cup E_1 \to \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ by:

(1) $\forall c \in C_1.\, f^{-1}(r)(c) = r(f(c))$

(2) $\forall e \in E_1.\, f^{-1}(r)(e) = \begin{cases} r(f(e)) & \text{if } f(e) \text{ is defined} \\ (0,0) & \text{otherwise.} \end{cases}$

Lemma 5.10

Let f be an LES-morphism from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$ and let r be a region of ES_2. Then $f^{-1}(r)$ is a region of ES_1.

Proof.

Suppose $c \vdash_1 u$. By the definition of an LES-morphism we have that $f(c) \vdash_2 f(u)$. Since r is a region of ES_2 this implies that $r(f(c)) \geq \sum_{e \in f(u)} {}^r e$ and $r(f(c) \cup f(u)) = r(f(c)) + \sum_{e \in f(u)} (e^r - {}^r e)$. Hence by Lemma 1.6, $f^{-1}(r)(c) = r(f(c)) \geq \sum_{e \in f(u)} {}^r e = \sum_{e \in u} {}^{f^{-1}(r)} e$ and $f^{-1}(r)(c \cup u) = r(f(c \cup u)) = r(f(c)) + \sum_{e \in f(u)} (e^r - {}^r e) = f^{-1}(r)(c) + \sum_{e \in u} (e^{f^{-1}(r)} - {}^{f^{-1}(r)} e)$.
$\square$

Note that in general, $f^{-1}(r)$ as defined above need not be a *non-trivial* region of ES_1.

The arrow-part of en is now defined as follows. Let $ES_1 = (E_1, C_1, \vdash_1)$ and $ES_2 = (E_2, C_2, \vdash_2)$ be a pair of L-event structures and let f be an LES-morphism from ES_1 to ES_2. Then $en(f) = (\beta_f, \eta_f)$ where $\eta_f = f$ and $\beta_f : R_{ES_2} \to R_{ES_1}$ is given by:

$$\forall r \in R_{ES_2}.\, \beta_f(r) = \begin{cases} f^{-1}(r) & \text{if } f^{-1}(r) \text{ is non-trivial} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Lemma 5.11

Let f be an LES-morphism from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$. Then $en(f) = (\beta_f, \eta_f)$ is a PN-morphism from $en(ES_1) = (R_{ES_1}, E_1, W_1, M_1)$ to $en(ES_2) = (R_{ES_2}, E_2, W_2, M_2)$.

Proof.

Let $r \in R_{ES_2}$ be such that $\beta_f(r)$ is defined. Then $M_2(r) = r(\emptyset) = f^{-1}(r)(\emptyset) = M_1(f^{-1}(r))$. This proves condition (1) in the definition of a PN-morphism.

If $t_1 \in E_1$ is such that $\eta_f(t_1)$ is undefined, then $f(t_1)$ is undefined, and therefore $f^{-1}(r)(t_1) = (0,0)$ for all $r \in R_{ES_2}$. Assume $r_2 \in \beta_f^{-1}({}^\bullet t_1) \cup \beta_f^{-1}(t_1{}^\bullet)$. Then $\beta_f(r_2) = f^{-1}(r_2) \in R_{ES_1}$ and $f^{-1}(r_2)(t_1) = \beta_f(r_2)(t_1) = (W_1(\beta_f(r_2), t_1), W_1(t_1, \beta_f(r_2))) \neq (0,0)$, a contradiction. This implies that $\beta_f^{-1}({}^\bullet t_1) = \beta_f^{-1}(t_1{}^\bullet) = \emptyset$, so (β_f, η_f) satisfies condition (2) in the definition of a PN-morphism.

Finally, assume that $t_1 \in E_1$ is such that $\eta_f(t_1) = f(t_1)$ is defined with $\eta_f(t_1) = t_2$. Then $f^{-1}(r)(t_1) = r(f(t_1)) = ({}^r t_2, t_2^r)$ for all $r \in R_{ES_2}$. Hence $r \in {}^\bullet t_2$ if and only if $f^{-1}(r) \in {}^\bullet t_1$, that is $r \in \beta_f^{-1}({}^\bullet t_1)$. Similarly it can be proved that $\beta_f^{-1}(t_1{}^\bullet) = t_2{}^\bullet$. Moreover, for all $r \in {}^\bullet t_2$, $W_1(\beta_f(r), t_1) = W_2(r, t_2)$ and, for all $r \in t_2{}^\bullet$, $W_1(t_1, \beta_f(r)) = W_2(t_2, r)$. This proves condition (3) in the definition of a PN-morphism. $\square$

Now we are ready to prove that en is a functor, which is full and faithful. That en is full means that for any two $\mathcal{LES}$-objects ES_1 and ES_2 and for any arrow (β, η) from $en(ES_1)$ to $en(ES_2)$, there exists an arrow f from ES_1 to ES_2 such that $en(f) = (\beta, \eta)$. That en is faithful means that different arrows between $\mathcal{LES}$-objects are mapped to different arrows between their images.

Theorem 5.12

en is a full and faithful functor from $\mathcal{LES}$ to $\mathcal{PNS}$.

Proof.

In order to prove that en is a functor from $\mathcal{LES}$ to $\mathcal{PNS}$, it is by Lemma 4.2 and Lemma 5.11 sufficient to prove that en preserves identities and respects composition. Clearly en preserves identities. Assume that f_1 is an LES-morphism from ES_1 to ES_2 and f_2 is an LES-morphism from ES_2 to ES_3. We have that $\eta_{f_2 \circ f_1} = f_2 \circ f_1 = \eta_{f_2} \circ \eta_{f_1}$. Because $en(ES)$ is S-simple we have by Lemma 5.5 that $en(f_2 \circ f_1) = (\beta_{f_2 \circ f_1}, \eta_{f_2 \circ f_1}) = (\beta_{f_1} \circ \beta_{f_2}, \eta_{f_2} \circ \eta_{f_1}) = (\beta_{f_2}, \eta_{f_2}) \circ (\beta_{f_1}, \eta_{f_1}) = en(f_2) \circ en(f_1)$.

In order to prove that en is full, let $ES_1 = (E_1, C_1, \vdash_1)$ and $ES_2 = (E_2, C_2, \vdash_2)$ be L-event structures and let (β, η) be a PN-morphism from $en(ES_1)$ to $en(ES_2)$. We first prove that η is an LES-morphism from ES_1 to ES_2. Suppose $c \vdash_1 u$. Let $\rho \in SFS_{ES_1}$ be such that $alph(\rho) = c$. Then $\rho u \in SFS_{ES_1}$ and hence we also have, by Lemma 4.2, that $\rho u \in SFS_{en(ES_1)}$. By Lemma 5.4 we then have that $\eta(\rho u) \in SFS_{en(ES_2)}$. Again by Lemma 4.2 we now have that $\eta(\rho u) \in SFS_{ES_2}$. Hence $alph(\eta(\rho)) \vdash_2 \eta(u)$. Because $alph(\eta(\rho)) = \eta(c)$ we can now conclude that $\eta(c) \vdash_2 \eta(u)$. This proves that η is an LES-morphism from ES_1 to ES_2. Since $en(ES_1)$ is S-simple Lemma 5.5 can be applied and so $en(\eta) = (\beta, \eta)$. This proves that en is full.

Finally, if f and g are LES-morphisms from ES_1 to ES_2 such that $f \neq g$ then also $en(f) \neq en(g)$ by the definition of en. Hence en is faithful. $\square$

Next we show that $en \circ i$ and nu form a co-reflection with $en \circ i$ as the left adjoint, where i is the inclusion functor from $\mathcal{ULES}$ to $\mathcal{LES}$. In what follows we write ES and f rather than $i(ES)$ and $i(f)$ for $\mathcal{ULES}$-objects ES and $\mathcal{ULES}$-arrows f respectively.

In order to facilitate the proof of this result we first define the PN-morphisms which turn out to form the co-unit of the adjunction. To do this the following regions of the L-event structure associated with a co-safe Petri net are defined.

Let $N = (S, T, W, M_{in})$ be a co-safe Petri net with $nu(N) = (E, C, \vdash)$ and let $s \in S$. Define $r_s : C \cup E \to \mathbf{N} \cup (\mathbf{N} \times \mathbf{N})$ by:

(1) $\forall \rho \in SFS_N.\, r_s(past_N(\rho)) = M(s)$ where $M \in RM_N$ is such that $M_{in} \overset{\rho}{\Longrightarrow}_N M$

(2) $\forall \langle \rho t \rangle_N \in E.\, r_s(\langle \rho t \rangle_N) = (W(s,t), W(t,s))$.

From Lemma 2.2(1) it easily follows that part (1) in the definition of r_s is well-defined.

Lemma 5.13
Let $N = (S, T, W, M_{in})$ be a co-safe Petri net and let $s \in S$. Then r_s is a region of $nu(N)$.
Proof.
Let $nu(N) = (E, C, \vdash)$. Suppose $\hat{c} \vdash \hat{u}$. Then there is $\rho u \in SFS_N$ is such that $\hat{c} = past_N(\rho)$ and $\hat{u} = \{\langle \rho t \rangle_N \mid t \in u\}$. Let $M, M' \in RM_N$ be such that $M_{in} \overset{\rho}{\Longrightarrow}_N M$ and $M_{in} \overset{\rho u}{\Longrightarrow}_N M'$. Then $r_s(\hat{c}) = M(s) \geq \sum_{t \in u} W(s,t) = \sum_{t \in u} {}^{r_s}\langle \rho t \rangle_N$ and $r_s(\hat{c} \cup \hat{u}) = M'(s) = M(s) + \sum_{t \in u}(W(t,s) - W(s,t)) = r_s(\hat{c}) + \sum_{t \in u}(\langle \rho t \rangle_N{}^{r_s} - {}^{r_s}\langle \rho t \rangle_N)$. $\square$

Given a co-safe Petri net $N = (S, T, W, M_{in})$ with $nu(N) = (E, C, \vdash)$ and $en(nu(N)) = (R_{nu(N)}, E, \hat{W}, \hat{M}_{in})$, we define $fold_S : S \to R_{nu(N)}$ and $fold_T : E \to T$ by:

(1) $\forall s \in S.\, fold_S(s) = \begin{cases} r_s & \text{if } r_s \text{ is non-trivial} \\ \text{undefined} & \text{otherwise.} \end{cases}$

(2) $\forall \langle \rho t \rangle_N \in E.\, fold_T(\langle \rho t \rangle_N) = t$.

Lemma 5.14

Let $N = (S, T, W, M_{in})$ be a co-safe Petri net with $nu(N) = (E, C, \vdash)$ and $en(nu(N)) = (R_{nu(N)}, E, \hat{W}, \hat{M}_{in})$. Then $(fold_S, fold_T)$ is a PN-morphism from $en(nu(N))$ to N.

Proof.

Suppose $s \in S$ is such that $fold_S(s)$ is defined. Then $\hat{M}_{in}(fold_S(s)) = \hat{M}_{in}(r_s) = r_s(\emptyset) = M_{in}(s)$ which proves condition (1) in the definition of PN-morphism.

Because $fold_T$ is a total function, condition (2) in the definition of PN-morphism trivially holds.

In order to prove condition (3), suppose $\langle \rho t \rangle_N \in E$. If $s \in fold_S^{-1}({}^\bullet\langle \rho t \rangle_N)$ then we must have that $r_s \in {}^\bullet\langle \rho t \rangle_N$, that is $\hat{W}(r_s, \langle \rho t \rangle_N) > 0$. This implies that ${}^{r_s}\langle \rho t \rangle_N > 0$ and hence $W(s,t) > 0$. This proves that $s \in {}^\bullet t = {}^\bullet fold_T(\langle \rho t \rangle_N)$. On the other hand, if $s \in {}^\bullet fold_T(\langle \rho t \rangle_N) = {}^\bullet t$, then ${}^{r_s}\langle \rho t \rangle_N = W(s,t) > 0$. Thus r_s is non-trivial and $\hat{W}(r_s, \langle \rho t \rangle_N) = {}^{r_s}\langle \rho t \rangle_N > 0$. Then $r_s \in {}^\bullet\langle \rho t \rangle_N$ and hence $s \in fold_S^{-1}({}^\bullet\langle \rho t \rangle_N)$. Moreover, $W(s, fold_T(\langle \rho t \rangle_N)) = W(s,t) = \hat{W}(r_s, \langle \rho t \rangle_N) = \hat{W}(fold_S(s), \langle \rho t \rangle_N)$. Similarly it can be proved that $fold_S^{-1}(\langle \rho t \rangle_N{}^\bullet) = fold_T(\langle \rho t \rangle_N)^\bullet$ and $W(fold_T(\langle \rho t \rangle_N), s) = \hat{W}(\langle \rho t \rangle_N, fold_S(s))$. This proves condition (3) in the definition of PN-morphism. $\Box$

Now we can prove the main result of this section.

Theorem 5.15

$en \circ i : \mathcal{ULES} \to \mathcal{PNS}$ and $nu : \mathcal{PNS} \to \mathcal{ULES}$ form a co-reflection with $en \circ i$ the left adjoint and the arrows υ_{ES} as unit.

Proof.

Let $ES = (E, C, \vdash)$ be an UL-event structure, let $N = (S, T, W, M_{in})$ be a co-safe Petri net, and let f be an LES-morphism from ES to $nu(N) = (\hat{E}, \hat{C}, \hat{\vdash})$. We must show that there is a unique PN-morphism (β, η) from $en(ES) = (R_{ES}, E, W_{ES}, M)$ to N such that the following diagram commutes.

$$\begin{tikzcd}
ES \arrow[r, "\upsilon_{ES}"] \arrow[d, "f"'] & nu(en(ES)) \arrow[dl, dotted, "nu((\beta, \eta))"] \\
nu(N) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
en(ES) \arrow[d, dotted, "!(\beta, \eta)"] \\
N
\end{tikzcd}$$

Define (β, η) by $(\beta, \eta) = (fold_S, fold_T) \circ en(f)$. Hence $\beta : S \to R_{ES}$ is such that for all $s \in S$, $\beta(s) = f^{-1}(r_s)$ if $f^{-1}(r_s)$ is non-trivial and $\beta(s)$ is undefined otherwise. The function $\eta : E \to T$ is such that for all $e \in E$, $\eta(e) =$ undefined if $f(e)$ is undefined and $\eta(e) = t$ if $f(e)$ is defined with $f(e) = \langle \rho t \rangle_N$. Because $(fold_S, fold_T)$ and $en(f)$ are

PN-morphisms by Lemma 5.14 and Lemma 5.11 respectively, and because the composition of PN-morphisms is again a PN-morphism, the pair (β, η) is a PN-morphism.

The next thing to prove is that $nu((\beta,\eta)) \circ v_{ES} = f$. Let $e \in E$. Then $f(e)$ is undefined iff $\eta(e)$ is undefined iff $(nu((\beta,\eta)) \circ v_{ES})(e)$ is undefined. So assume that $f(e)$ is defined. Let $\rho \in SFS_{ES}$ be such that $\rho e \in SFS_{ES}$. By the unique occurrence property ρ exists. By Lemma 4.2 we then have that also $\rho, \rho e \in SFS_{en(ES)}$ and hence Lemma 5.4 implies that $\eta(\rho), \eta(\rho e) \in SFS_N$. Furthermore, by Lemma 1.7, $f(\rho), f(\rho e) \in SFS_{nu(N)}$.

We first prove, by induction on $|\rho|$, that $alph(f(\rho)) = past_N(\eta(\rho))$. If $\rho = \emptyset$ then this is clear, so assume that $\rho = \rho' u$ with $\rho' \in SFS_{ES}$ and $\emptyset \neq u \in P_F(E)$.

Then $alph(f(\rho)) = alph(f(\rho')) \cup f(u)$ and $past_N(\eta(\rho)) = past_N(\eta(\rho')) \cup \hat{u}$ where $\hat{u} = \{\langle \eta(\rho')\eta(e')\rangle_N \mid e' \in u \text{ with } \eta(e') \text{ defined } \}$. By the induction hypothesis, $alph(f(\rho')) = past_N(\eta(\rho'))$. From $f(\rho' u) \in SFS_{nu(N)}$ we have that $alph(f(\rho'))\hat{\vdash} f(u)$. On the other hand, from $\eta(\rho' u) \in SFS_N$ we have that $past_N(\eta(\rho'))\hat{\vdash}\hat{u}$. It is now sufficient to prove that $f(u) = \hat{u}$. By the definition of $\hat{\vdash}$, $alph(f(\rho'))\hat{\vdash} f(u)$ implies that there exists $\rho_1 u_1 \in SFS_N$ such that $alph(f(\rho')) = past_N(\rho_1)$ and $f(u) = \{\langle \rho_1 e_1\rangle_N \mid e_1 \in u_1\}$. Let $e' \in u$ be such that $f(e')$ is defined. Then there exists $e_1 \in u_1$ such that $f(e') = \langle \rho_1 e_1\rangle_N$. Then $e_1 = \eta(e')$ by the definition of η. Since $past_N(\rho_1) = alph(f(\rho')) = past_N(\eta(\rho'))$ and $\sim_N$ satisfies (C2), we must now have that $\langle \eta(\rho')\eta(e')\rangle_N = \langle \rho_1 e_1\rangle_N$. This proves that $f(u) = \hat{u}$ and we can conclude that $alph(f(\rho)) = past_N(\eta(\rho))$.

From $f(\rho e) \in SFS_{nu(N)}$ we know that $alph(f(\rho))\hat{\vdash} f(e)$. Then there exists $\rho_2 e_2 \in SFS_N$ such that $alph(f(\rho)) = past_N(\rho_2)$ and $f(e) = \langle \rho_2 e_2\rangle_N$. Then $e_2 = \eta(e)$ by the definition of η. Since $past_N(\rho_2) = alph(f(\rho)) = past_N(\eta(\rho))$ and $\sim_N$ satisfies (C2), we now have that $\langle \rho_2 e_2\rangle_N = \langle \eta(\rho)\eta(e)\rangle_N$. This implies that $(nu((\beta,\eta)) \circ v_{ES})(e) = nu((\beta,\eta))(\langle \rho e\rangle_{en(ES)}) = \langle \eta(\rho)\eta(e)\rangle_N = \langle \rho_2 e_2\rangle_N = f(e)$ what had to be proved.

Finally, in order to prove that (β,η) is the unique PN-morphism from $en(ES)$ to N such that $nu((\beta,\eta)) \circ v_{ES} = f$, assume that (β', η') is any PN-morphism from $en(ES)$ to N such that $nu((\beta',\eta')) \circ v_{ES} = f$. Then for all $e \in E$, $\eta(e)$ is undefined iff $f(e)$ is undefined iff $\eta'(e)$ is undefined. Now let $e \in E$ be such that $\eta'(e)$ is defined. Let $\rho \in SFS_{en(ES)}$ be such that $v_{ES}(e) = \langle \rho e\rangle_{en(ES)}$.

Then $\langle \eta(\rho)\eta(e)\rangle_N = nu((\beta,\eta)) \circ v_{ES}(e) = f(e) = nu((\beta',\eta')) \circ v_{ES}(e) = \langle \eta'(\rho)\eta'(e)\rangle_N$. Now Lemma 2.2(1) guarantees that $\eta(e) = \eta'(e)$. This proves that $\eta = \eta'$. We can now conclude by Lemma 5.5 that $\beta = \beta'$ because $en(ES)$ is S-simple.

This proves that $en \circ i$ and nu form an adjunction with $en \circ i$ as the left adjoint and the arrows v_{ES} as unit. By Theorem 4.3 the arrows v_{ES} are LES-isomorphisms and so the adjunction is even a co-reflection. $\square$

It is easy to verify that the arrows $(fold_S,\, fold_T)$ form the co-unit of the adjunction between $\mathcal{ULES}$ and $\mathcal{PNS}$. Each UL-event structure ES is isomorphic to the UL-event structure $nu(en(ES))$ by Theorem 4.3. Hence for each co-safe Petri net N, $en(nu(N))$ yields an UL-event structure which is isomorphic to the UL-event structure yielded by N. The Petri net $en(nu(N))$ has a number of other interesting properties. It is saturated with respect to the places and each transition can occur exactly once. Hence the Petri net $en(nu(N))$ may be viewed as a "behavioural unfolding" of N. The associated "fold morphism" is $(fold_S, fold_T)$.

As a consequence of Theorem 5.15 each L-event structure can in fact be represented as an UL-event structure in a canonical way.

Corollary 5.16

$i : \mathcal{ULES} \to \mathcal{LES}$ and $nu \circ en : \mathcal{LES} \to \mathcal{ULES}$ form a co-reflection with i the left adjoint and the arrows υ_{ES} as unit.

Proof.

Let ES be an UL-event structure, let ES' be an L-event structure, and let f be an LES-morphism from ES to $nu(en(ES'))$. It must be proved that there is a unique LES-morphism g from ES to ES' such that the following diagram commutes.

$$\begin{tikzcd}
ES \arrow[r, "\upsilon_{ES}"] \arrow[d, "f"'] & nu(en(ES)) \arrow[dl, dashed, "nu(en(g))"] \\
nu(en(ES')) &
\end{tikzcd}
\qquad
\begin{tikzcd}
en(ES) \arrow[d, dashed, "en(g)"] \\
en(ES')
\end{tikzcd}
\qquad
\begin{tikzcd}
ES \arrow[d, dashed, "!\, g"] \\
ES'
\end{tikzcd}$$

By Theorem 5.15 there exists a unique PN-morphism (β, η) from $en(ES)$ to $en(ES')$ such that $nu((\beta, \eta)) \circ \upsilon_{ES} = f$. Then because en is full and faithful there exists a unique LES-morphism g from ES to ES' such that $en(g) = (\beta, \eta)$ and hence $nu \circ en(g) \circ \upsilon_{ES} = f$.
$\square$

In the beginning of this section we argued that it is not possible to obtain a co-reflection between $\mathcal{ULES}$ and $\mathcal{PN}$. Hence we restricted the category $\mathcal{PN}$ by cutting down on the *objects*. Another possibility is to cut down on the *arrows* of $\mathcal{PN}$.

Definition 5.17

(1) Let $N = (S, T, W, W_{in})$ be a Petri net. Then $co_N \subseteq T \times T$ is given by: $t\, co_N\, t' \Leftrightarrow t \neq t'$ and $\exists \rho u \in MFS_N. (u(t) > 0$ and $u(t') > 0)$.

(2) Let (β, η) be a PN-morphism from $N_1 = (S_1, T_1, W_1, M_1)$ to $N_2 = (S_2, T_2, W_2, M_2)$. Then (β, η) is *co-injective* if for all $t, t' \in T_1$, if $\eta(t)$ and $\eta(t')$ are both defined and $t\, co_{N_1}\, t'$, then $\eta(t) \neq \eta(t')$. $\square$

Definition 5.18

Let $\mathcal{PNC}$ be the subcategory of $\mathcal{PN}$ the objects of which are Petri nets and the arrows of which are co-injective PN-morphisms. $\square$

From Lemma 5.4 we immediately have that if (β, η) is a co-injective PN-morphism from N_1 to N_2, then $\eta(\rho) \in SFS_{N_2}$ for all $\rho \in SFS_{N_1}$.

Note also that by Lemma 5.4 $\mathcal{PNS}$ is a subcategory of $\mathcal{PNC}$.

It is easy to see that the proof of the co-reflection between $\mathcal{ULES}$ and $\mathcal{PNS}$ still goes through with $\mathcal{PNC}$ instead of $\mathcal{PNS}$ (where nu is extended to a functor from $\mathcal{PNC}$ to $\mathcal{ULES}$ in the obvious way). Hence we also have the following result.

Theorem 5.19

$en \circ i : \mathcal{ULES} \rightarrow \mathcal{PNC}$ and $nu : \mathcal{PNC} \rightarrow \mathcal{ULES}$ form a co-reflection with $en \circ i$ the left adjoint and the arrows v_{ES} as unit. $\square$

6 Relationship to other Classes of Event Structures

In this section we study the relationship between the event structures introduced in this paper and some of the well-known classes of event structures that have appeared in the literature. The motivation is to show that though our event structures have been formulated mainly in order to capture the behaviour of Petri nets, they might be of some independent interest. In particular, they appear to be smooth generalizations of some well-understood classes of event structures.

We will first consider the class of event structures formulated by Winskel in [W3] in the spirit of Information Systems. This class of event structures will be referred to here as *W-event structures*. We will first exhibit a natural functor from W-event structures to L-event structures and then show that this functor has a left adjoint. In fact this adjunction turns out to be a reflection. We then show that this reflection can be further extended to be a reflection between L-event structures and an important subclass of W-event structures, called *stable* W-event structures. Finally, we show that a similar reflective relationship can also be established between UL-event structures and prime event structures. The corresponding functor from prime event structures to UL-event structures is an extension of the map pu defined in Section 2.

First the category of (general) event structures from [W3] is defined.

Definition 6.1

$\mathcal{WES}$ is the category of *W-event structures* specified as follows.
An object of $\mathcal{WES}$ is a W-event structure $W = (E, C)$ where E is a set of events and $C \subseteq P_F(E)$ is a non-empty set of (finite) configurations such that

(W1) $\emptyset \neq c \Rightarrow \exists e \in c.\, c - e \in C$

(W2) $c \uparrow c' \Rightarrow c \cup c' \in C$ (where $c \uparrow c'$ iff there exists $c'' \in C$ such that $c \subseteq c''$ and $c' \subseteq c''$).

An arrow of $\mathcal{WES}$ is a *WES-morphism* $f : (E_1, C_1) \rightarrow (E_2, C_2)$ which is a partial function $f : E_1 \rightarrow E_2$ such that

(1) $\forall c \in C_1.\, f(c) \in C_2$

(2) $\forall c \in C_1. \forall e_1, e_2 \in c.$ if $e_1 \neq e_2$ and $f(e_1)$ and $f(e_2)$ are both defined, then $f(e_1) \neq f(e_2)$.

The identity morphism associated with an object is the identity function on its events and composition of arrows is composition of partial functions. $\square$

For a W-event structure $W = (E, C)$, define $we(W) = (E, C, \vdash)$ where $\vdash \subseteq C \times P_F(E)$ is given by: $c \vdash u$ iff $c \cap u = \emptyset$ and $\forall v \subseteq u.\, c \cup v \in C$.

For a WES-morphism f, define $we(f) = f$.

Lemma 6.2

Let W be a W-event structure. Then $we(W)$ is an L-event structure.

Proof.

Follows easily from the definitions. $\square$

Note that not every L-event structure arises in this fashion (see, for instance, the L-event structures ES_1 and ES_3 depicted in Figure 1).

Lemma 6.3

Let f be a WES-morphism from $W_1 = (E_1, C_1)$ to $W_2 = (E_2, C_2)$. Then $we(f)$ is an LES-morphism from $we(W_1) = (E_1, C_1, \vdash_1)$ to $we(W_2) = (E_2, C_2, \vdash_2)$.

Proof.

Suppose that $c \vdash_1 u$. Then $c \cap u = \emptyset$ and $c \cup u \in C$. Hence $f(c) \cap f(u) = \emptyset$ by condition (2) in the definition of WES-morphism. Moreover, $c \cup v \in C_1$ for all $v \subseteq u$ and so by condition (1), $f(c \cup v) = f(c) \cup f(v) \in C_2$ for all $v \subseteq u$. Hence $f(c) \vdash_2 f(u)$. $\square$

Lemma 6.2 and Lemma 6.3 now lead to the following result.

Theorem 6.4

we is a functor from $\mathcal{WES}$ to $\mathcal{LES}$. $\square$

The map ew from $\mathcal{LES}$ to $\mathcal{WES}$ is defined as follows. For an L-event structure $ES = (E, C, \vdash)$, define $ew(ES) = (E, \hat{C})$ where $\hat{C}$ is the least subset of $P_F(E)$ containing C which satisfies (W2).

Note that $ew(ES)$ is well-defined, because both $P_F(E)$ and $\bigcap\{C' \subseteq P_F(E) \mid C \subseteq C'$ and C' satisfies (W2)$\}$ satisfy (W2).

For an LES-morphism f, define $ew(f) = f$.

Lemma 6.5

Let $ES = (E, C, \vdash)$ be an L-event structure. Then $ew(ES) = (E, \hat{C})$ is a W-event structure.

Proof.

In order to prove that $ew(ES)$ satisfies (W1), let $\emptyset \neq c \in \hat{C}$. If $c \in C$, then there exists $e \in E$ such that $c - e \vdash e$ because ES satisfies (A0). Hence $c - e \in C \subseteq \hat{C}$. So assume that $c \notin C$. Then by the minimality of $\hat{C}$ there exist $c_1, c_2 \in \hat{C}$ with $c_1 \uparrow c_2$ such that $c = c_1 \cup c_2$, $|c_1| < |c|$, and $|c_2| < |c|$. Thus $|c| \geq 2$. Assume that for all $\hat{c} \in \hat{C}$ with $1 \leq |\hat{c}| < |c|$, there exists an $e \in E$ such that $\hat{c} - e \in \hat{C}$. Then there exist $e_1, \ldots, e_n \in E$ with $n = |c_1|$ such that $c_1 = \{e_1, \ldots, e_n\}$, and $\{e_1, \ldots, e_i\} \in \hat{C}$ for all $0 \leq i \leq n$. Because $|c_1| < |c|$ and $|c_2| < |c|$ there exists a largest integer k such that $k \in \{1, \ldots, n\}$ and $e_k \notin c_2$. Hence $e_{k+1}, \ldots, e_n \in c_2$. Then, by the definition of $\hat{C}$, $\{e_1, \ldots, e_{k-1}\} \cup c_2 = c - e_k \in \hat{C}$. This proves that $ew(ES)$ satisfies (W1).

From the definition of $ew(ES)$ we immediately have that $ew(ES)$ satisfies (W2). $\square$

The following lemma is used in proving in Lemma 6.7 that arrows of $\mathcal{LES}$ are mapped by ew to arrows of $\mathcal{WES}$.

Lemma 6.6

Let $ES = (E, C, \vdash)$ be an L-event structure with $ew(ES) = (E, \hat{C})$. Then $\hat{c} \in \hat{C}$ implies that there exists $c \in C$ such that $\hat{c} \subseteq c$.

Proof.

Let $\hat{c} \in \hat{C}$. If $\hat{c} \in C$ then the claim holds trivially, so suppose that $\hat{c} \in \hat{C} - C$. Now assume to the contrary that there exists no $c \in C$ such that $\hat{c} \subseteq c$. Let $C' = \hat{C} - \{c' \in \hat{C} \mid \hat{c} \subseteq c'\}$. Then $C \subseteq C'$ because $C \subseteq \hat{C}$ and $\{c' \in \hat{C} \mid \hat{c} \subseteq c'\} \cap C = \emptyset$. Suppose $c_0, c_1, c_2 \in C'$ are such that $c_1 \subseteq c_0$ and $c_2 \subseteq c_0$. $\hat{C}$ satisfies (W2) and so $c_1 \cup c_2 \in \hat{C}$. By $c_1 \cup c_2 \subseteq c_0 \in C'$ and $\hat{c} \not\subseteq c_0$ we have $\hat{c} \not\subseteq c_1 \cup c_2$. Hence $c_1 \cup c_2 \in C'$. This leads to the conclusion that C' satisfies (W2), a contradiction with the minimality of $\hat{C}$. Thus there exists $c \in C$ such that $\hat{c} \subseteq c$. $\square$

Lemma 6.7

Let f be an LES-morphism from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$. Then $ew(f)$ is a WES-morphism from $ew(ES_1) = (E_1, \hat{C}_1)$ to $ew(ES_2) = (E_2, \hat{C}_2)$.

Proof.

Let $c \in \hat{C}_1$. By condition (1) in the definition of WES-morphism, $f(c) \in \hat{C}_2$ should hold. We prove this by induction on $|c|$. If $c \in C_1$, then by (A1) $c \vdash_1 \emptyset$. Since f is an LES-morphism, we have in this case $f(c) \vdash_2 \emptyset$ and so $f(c) \in C_2 \subseteq \hat{C}_2$. Now assume that $|c| > 1$ with $c \in \hat{C}_1 - C_1$. Then by the minimality of $\hat{C}_1$ there exist $c_1, c_2 \in \hat{C}_1$ such that $c = c_1 \cup c_2$, $|c_1| < |c|$, and $|c_2| < |c|$. Hence $f(c_1), f(c_2) \in \hat{C}_2$ by the induction hypothesis. By Lemma 6.6 there exists a $c' \in C_1$ such that $c \subseteq c'$. We then have as above that $f(c') \in C_2 \subseteq \hat{C}_2$. Thus $f(c_1), f(c_2), f(c') \in \hat{C}_2$ and $f(c_1) \subseteq f(c')$ and $f(c_2) \subseteq f(c')$. Then $f(c_1) \cup f(c_2) = f(c) \in \hat{C}_2$ because $\hat{C}_2$ satisfies (W2).

That condition (2) in the definition of a WES-morphism is satisfied by f can be seen as follows: let $c \in \hat{C}_1$ and $e_1, e_2 \in c$ be such that $e_1 \neq e_2$ and $f(e_1)$ and $f(e_2)$ are both defined. Again Lemma 6.6 guarantees the existence of a $c' \in C_1$ such that $c \subseteq c'$. Then Lemma 1.3(1) gives $f(e_1) \neq f(e_2)$. $\square$

Lemma 6.5 and Lemma 6.7 yield the following result.

Theorem 6.8

ew is a functor from $\mathcal{LES}$ to $\mathcal{WES}$. $\square$

Now we prove that ew and we form an adjunction. The co-unit of this adjunction is given by the identity arrows id_W for each W-event structure W. Hence the adjunction is a reflection. Note that the co-unit is well-defined because $ew(we(W)) = W$.

Theorem 6.9

$ew : \mathcal{LES} \to \mathcal{WES}$ and $we : \mathcal{WES} \to \mathcal{LES}$ form a reflection with ew the left adjoint and the identity arrows id_W as co-unit.

Proof.

Let $ES = (E, C, \vdash)$ be an L-event structure, let $W = (E', C')$ be a W-event structure, and let g be a WES-morphism from $ew(ES) = (E, \hat{C})$ to W. Then we must prove that there exists a unique LES-morphism f from ES to $we(W) = (E', C', \vdash')$ such that the following diagram commutes.

$$\begin{tikzcd} ES \arrow[dd, dotted, "!f"] \\ \\ we(W) \end{tikzcd} \qquad \begin{tikzcd} ew(ES) \arrow[dd, "g"'] \arrow[dr, dotted, "ew(f)"] & \\ & ew(we(W)) \arrow[dl, "id_W"] \\ W & \end{tikzcd}$$

Since ew is the identity on arrows, it is sufficient to prove that g is an LES-morphism from ES to $we(W)$. Suppose $c \vdash u$. Then $c \cap u = \emptyset$ and $c \cup v \in C$, for all $v \subseteq u$ by (A2). Since g is a WES-morphism from $ew(ES)$ to W we now have that $c \cup v \in C \subseteq \hat{C}$ implies $g(c) \cup g(v) \in C'$, for all $v \subseteq u$, and $g(c) \cap g(u) = \emptyset$. Hence $g(c) \vdash' g(u)$. $\square$

Our next aim is to prove that there is also a reflection between $\mathcal{LES}$ and the category of *stable* W-event structures [W3].

Definition 6.10

$\mathcal{SWES}$, the category of *stable W-event structures*, is the full subcategory of $\mathcal{WES}$ the objects (E, C) of which satisfy

(W3) $c \uparrow c' \Rightarrow c \cap c' \in C$. $\square$

In order to prove the desired reflection between $\mathcal{LES}$ and $\mathcal{SWES}$, we first show that there is a reflection between $\mathcal{WES}$ and $\mathcal{SWES}$.

First a map ws from $\mathcal{WES}$ to $\mathcal{SWES}$ is defined.

Given a W-event structure $W = (E, C)$, define $C^{(i)} \subseteq P_F(E)$ with $i \geq 0$ inductively by: $C^{(0)} = C$ and, for $i \geq 1$, $C^{(i)} = C^{(i-1)} \cup \{c \cup c',\ c \cap c' \mid c, c' \in C^{(i-1)}$ with $c \uparrow c'$ in $C^{(i-1)}\}$. Now define $ws(W) = (E, \hat{C})$ where $\hat{C} = \bigcup_{i \geq 0} C^{(i)}$.

For a WES-morphism f, define $ws(f) = f$.

As the following example illustrates it is not sufficient to simply add in a given W-event structure W configurations to ensure that (W3) is satisfied. Whereas W already satisfies (W1) and (W2), adding configurations to ensure that (W3) is satisfied may destroy the condition (W2).

Example 6.11

Let $W = (E, C)$ be the non-stable W-event structure depicted in Figure 7.

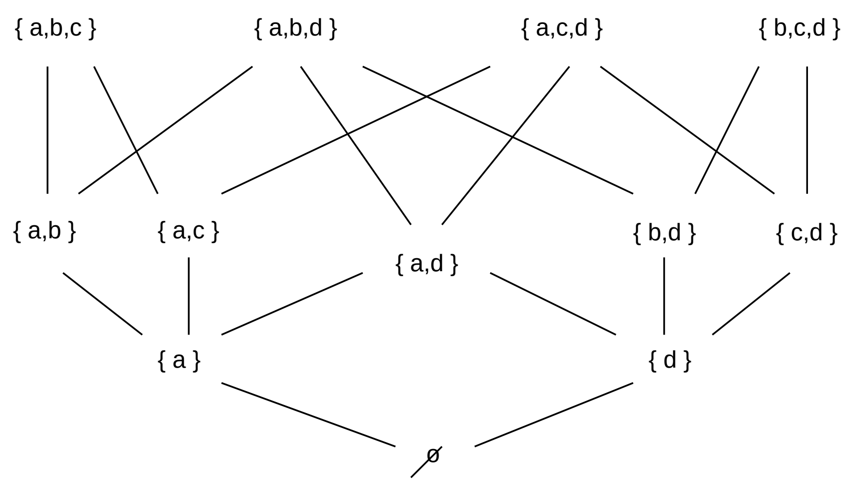


Figure 7: A non-stable W-event structure

For this W-event structure $\{b\} \in C^{(1)}$ because $\{a, b\} \uparrow \{b, d\}$. Similarly $\{a, c\} \uparrow \{c, d\}$ implies that $\{c\} \in C^{(1)}$. Now $C^{(1)} = C \cup \{\{b\}, \{c\}\}$ satisfies (W3), but it does not satisfy (W2) anymore. Since $\{b\} \uparrow \{c\}$ we have to add $\{b, c\}$, thus obtaining $C^{(2)} = C^{(1)} \cup \{\{b, c\}\}$. $C^{(2)}$ satisfies (W2) and (W3) and so $C^{(i)} = C^{(i-1)}$ for all $i \geq 3$. Hence $\hat{C} = C \cup \{\{b\}, \{c\}, \{b, c\}\}$. $\square$

Lemma 6.12

Let $W = (E, C)$ be a W-event structure. Then $ws(W) = (E, \hat{C})$ is a stable W-event structure.

Proof.

In order to prove that $ws(W)$ satisfies (W1), let $\emptyset \neq c \in \hat{C}$. Let $k \geq 0$ be minimal such that $c \in C^{(k)}$. We prove by induction on k that there exists $e \in c$ such that $c - e \in C^{(k)} \subseteq \hat{C}$.

If $k = 0$ then $c \in C$ and since W satisfies (W1), there exists $e \in c$ such that $c - e \in C = C^{(0)}$. Now suppose that $k \geq 1$. Then by the minimality of k there exist $c_1, c_2 \in C^{(k-1)}$ with $c_1 \uparrow c_2$ such that $c = c_1 \cup c_2$ or $c = c_1 \cap c_2$. By the induction hypothesis there exist $e_1, \ldots, e_n \in E$ with $n = |c_1|$ such that $c_1 = \{e_1, \ldots, e_n\}$ and $\{e_1, \ldots, e_i\} \in C^{(k-1)}$ for all $0 \leq i \leq n$. By the minimality of k, $c_1 \neq c$ and $c_2 \neq c$.

First assume that $c = c_1 \cup c_2$. Let m be the largest integer such that $m \in \{1, \ldots, n\}$ and $e_m \notin c_2$. Hence $e_{m+1}, \ldots, e_n \in c_2$. Then, by the definition of $C^{(k)}$, $\{e_1, \ldots, e_{m-1}\} \cup c_2 = c - e_m \in C^{(k)}$.

Now assume that $c = c_1 \cap c_2$. Let m be the largest integer such that $m \in \{1, \ldots, n\}$ and $e_m \in c_2$. Hence $e_{m+1}, \ldots, e_n \notin c_2$. Then, by the definition of $C^{(k)}$, $\{e_1, \ldots, e_{m-1}\} \cap c_2 = c - e_m \in C^{(k)}$.

This proves that $ws(W)$ satisfies (W1). From the definition of $ws(W)$ we immediately have that $ws(W)$ satisfies (W2) and (W3). $\square$

Lemma 6.13

Let f be a WES-morphism from $W_1 = (E_1, C_1)$ to $W_2 = (E_2, C_2)$. Then $ws(f)$ is a WES-morphism from $ws(W_1) = (E_1, \hat{C}_1)$ to $ws(W_2) = (E_2, \hat{C}_2)$.

Proof.

Let $c \in \hat{C}_1$. It must be proved that $f(c) \in \hat{C}_2$ and that f is injective on c.

Let $k \geq 0$ be minimal such that $c \in C_1^{(k)}$. We prove by induction on k that $f(c) \in C_2^{(k)} \subseteq \hat{C}_2$ and that f is injective on c. If $k = 0$ then $c \in C_1$ and hence $f(c) \in C_2 = C_2^{(0)}$. Since f is a WES-morphism from W_1 to W_2, f is injective on c. Now assume that $k \geq 1$. Then there exist $c_0, c_1, c_2 \in C_1^{(k-1)}$ with $c_1 \subseteq c_0$ and $c_2 \subseteq c_0$ such that $c = c_1 \cup c_2$ or $c = c_1 \cap c_2$. By the induction hypothesis $f(c_0), f(c_1), f(c_2) \in C_2^{(k-1)}$ and f is injective on c_0. Hence f is also injective on c. Now $f(c_1) \subseteq f(c_0)$ and $f(c_2) \subseteq f(c_0)$ and so by the definition of $C_2^{(k)}$ it follows that $f(c_1 \cup c_2) = f(c_1) \cup f(c_2) \in C_2^{(k)}$ and $f(c_1 \cap c_2) = f(c_1) \cap f(c_2) \in C_2^{(k)}$. This proves that $f(c) \in C_2^{(k)}$. $\square$

Lemma 6.12 and Lemma 6.13 yield the following result.

Theorem 6.14

ws is a functor from $\mathcal{WES}$ to $\mathcal{SWES}$. $\square$

As the next theorem shows ws is the left adjoint to the inclusion functor i from $\mathcal{SWES}$ to $\mathcal{WES}$. The co-unit of this adjunction is given by the identity arrows id_W for each stable W-event structure W. Hence the adjunction is a reflection. Note that the co-unit is well-defined because $ws(W) = W$ for each stable W-event structure W.

Theorem 6.15

$ws : \mathcal{WES} \to \mathcal{SWES}$ and $i : \mathcal{SWES} \to \mathcal{WES}$ form a reflection with ws the left adjoint and the identity arrows id_W as co-unit.

Proof.

Let $W = (E, C)$ be a W-event structure, let $W' = (E', C')$ be a stable W-event structure, and let g be a WES-morphism from $ws(W) = (E, \hat{C})$ to W'. Then we must prove that there exists a unique WES-morphism f from W to W' such that the following diagram commutes.

$$\begin{tikzcd}
W \arrow[dd, dotted, "!f"] & & ws(W) \arrow[dd, "g"'] \arrow[dr, dotted, "ws(f)"] & \\
 & & & ws(W') \arrow[dl, "id_{W'}"] \\
W' & & W' &
\end{tikzcd}$$

Since ws is the identity on arrows, it is sufficient to prove that g is a WES-morphism from W to W'. This however follows immediately from the observation that $C \subseteq \hat{C}$. $\square$

The reflections from Theorem 6.9 and Theorem 6.15 can now be composed which yields the following result.

Theorem 6.16

$ws \circ ew : \mathcal{LES} \to \mathcal{SWES}$ and $we \circ i : \mathcal{SWES} \to \mathcal{LES}$ form a reflection with $ws \circ ew$ the left adjoint and the identity arrows id_W as co-unit. $\square$

Finally in this section, we show that the relationship between UL-event structures and prime event structures can also be expressed as a reflection between the corresponding categories.

It is easy to show that prime event structures have the following property.

Lemma 6.17

Let $P = (E, \leq, \#)$ be a prime event structure. Then the following statements are equivalent:

(1) $\neg(e_1 \# e_2)$

(2) $\downarrow e_1 \cup \downarrow e_2 \in C_P$

(3) $\exists c \in C_P . \{e_1, e_2\} \subseteq c$. $\square$

Definition 6.18

$\mathcal{PES}$ is the category which has prime event structures as its objects and *PES-morphisms* as its arrows.
A PES-morphism $f : (E_1, \leq_1, \#_1) \to (E_2, \leq_2, \#_2)$ is a partial function $f : E_1 \to E_2$ such that

(1) $f(e)$ is defined $\Rightarrow \downarrow f(e) \subseteq f(\downarrow e)$

(2) ($f(e_1)$ and $f(e_2)$ are defined and $f(e_1)\#_2 f(e_2)) \Rightarrow e_1 \#_1 e_2$

(3) ($f(e_1)$ and $f(e_2)$ are defined and $f(e_1) = f(e_2)) \Rightarrow (e_1 \#_1 e_2$ or $e_1 = e_2)$.

The identity morphism associated with an object is the identity function on its events; composition of PES-morphisms is composition of partial functions. □

An alternative characterization of PES-morphisms is stated in the next lemma, which is straightforward to prove (see also [WN]). This characterization in terms of the finite configurations is used as a definition for PES-morphisms in, e.g., [W1, WN].

Lemma 6.19

Let $P_1 = (E_1, \leq_1, \#_1)$ and $P_2 = (E_2, \leq_2, \#_2)$ be prime event structures and let $f : E_1 \to E_2$ be a partial function. Then f is a PES-morphism iff

(1') $\forall c \in C_{P_1}.\, f(c) \in C_{P_2}$

(2') $\forall c \in C_{P_1}. \forall e_1, e_2 \in c.$ if $e_1 \neq e_2$ and $f(e_1)$ and $f(e_2)$ are both defined, then $f(e_1) \neq f(e_2)$. □

In Section 1 the map pu is defined which maps each prime event structure to an UL-event structure. In order to extend this map to a functor, define for a given PES-morphism f, $pu(f) = f$.

Lemma 6.20

Let f be a PES-morphism from $P_1 = (E_1, \leq_1, \#_1)$ to $P_2 = (E_2, \leq_2, \#_2)$. Then $pu(f)$ is an LES-morphism from $pu(P_1) = (E_1, C_{P_1}, \vdash_1)$ to $pu(P_2) = (E_2, C_{P_2}, \vdash_2)$.

Proof.

Suppose that $c \vdash_1 u$. Then $c \cap u = \emptyset$ and $c \cup u \in C_{P_1}$. So by condition (2') in Lemma 6.19, $f(c) \cap f(u) = \emptyset$. We also have that $c \cup v \in C_{P_1}$ for all $v \subseteq u$. Thus by condition (1') in Lemma 6.19, $f(c \cup v) = f(c) \cup f(v) \in C_{P_2}$ for all $v \subseteq u$. Consequently, $f(c) \vdash_2 f(u)$. □

The following result now follows immediately from Theorem 2.7 and Lemma 6.20.

Theorem 6.21

pu is a functor from $\mathcal{PES}$ to $\mathcal{ULES}$. □

For an L-event structure $ES = (E, C, \vdash)$, define $up(ES) = (E, \leq, \#)$ where $\leq \subseteq E \times E$ is such that $e_1 \leq e_2$ iff $\forall c \in C.\, (e_2 \in c \Rightarrow e_1 \in c)$ and $\# \subseteq E \times E$ is such that $e_1 \# e_2$ iff $\forall c \in C.\, (e_1 \in c \Rightarrow e_2 \notin c)$.

For an LES-morphism f, define $up(f) = f$.

The map up thus defined is a functor from $\mathcal{ULES}$ to $\mathcal{PES}$ as we show in the following lemmas.

Lemma 6.22

Let $ES = (E, C, \vdash)$ be an L-event structure which satisfies condition (U1) in the definition of the unique occurrence property. Then $up(ES) = (E, \leq, \#)$ is a prime event structure.

Proof.

Clearly, $\#$ is irreflexive and symmetric and $\leq$ is reflexive and transitive. In order to prove that $\leq$ is anti-symmetric, suppose $e_1, e_2 \in E$ are such that $e_1 \leq e_2$ and $e_2 \leq e_1$. Then for all $c \in C$, $e_1 \in c$ iff $e_2 \in c$. By condition (U1) in the definition of the unique occurrence property there exists $c \in C$ such that $e_1 \in c$ and hence by Lemma 1.3(2) $e_1 = e_2$. This proves that $\leq$ is a partial order.

In order to prove that $up(ES)$ satisfies (P1), suppose $e_0 \# e_1 \leq e_2$. If $c \in C$ is such that $e_0 \in c$, then $e_1 \notin c$ by the definition of $\#$ and hence $e_2 \notin c$ by the definition of $\leq$. Thus $e_0 \# e_2$.

Now in order to prove that $up(ES)$ satisfies (P2), let $e \in E$. Then by condition (U1) in the definition of the unique occurrence property, there exists $c \in C$ such that $e \in c$. Then $\downarrow e \subseteq c$ and hence $\downarrow e$ is finite. $\square$

Example 6.23

Let ES_6 and ES_7 be the L-event structures depicted in Figure 8.

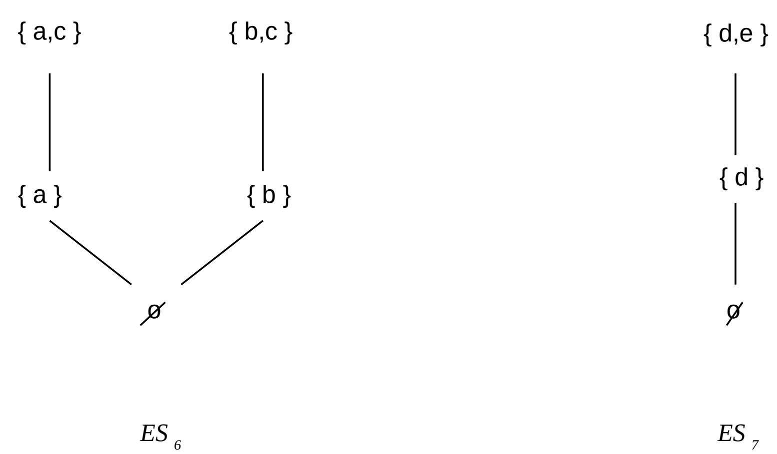


Figure 8: L-event structures ES_6 and ES_7

Define f by $f(a) = f(b) = d$ and $f(c) = e$. Then f is an LES-morphism from ES_6 to ES_7. Since $\{c\} \in C_{up(ES_6)}$ while $f(\{c\}) = \{e\} \notin C_{up(ES_7)}$, Lemma 6.19 implies that $up(f)$ is not a PES-morphism from $up(ES_6)$ to $up(ES_7)$. $\square$

As this example shows, arbitrary LES-morphisms are not preserved under up. LES-morphisms between L-event structures with the unique occurrence property are however preserved under up.

Lemma 6.24

Let f be an LES-morphism from $ES_1 = (E_1, C_1, \vdash_1)$ to $ES_2 = (E_2, C_2, \vdash_2)$ where ES_1 and ES_2 are UL-event structures. Then $up(f)$ is a PES-morphism from $up(ES_1) = (E_1, \leq_1, \#_1)$ to $up(ES_2) = (E_2, \leq_2, \#_2)$.

Proof.

In order to prove condition (1) in the definition of PES-morphism, let $e \in E_1$ be such that $f(e)$ is defined and suppose $e' \in \downarrow f(e)$. It must be proved that $e' \in f(\downarrow e)$. If $e' = f(e)$ then we are done, so assume that $e' \neq f(e)$. Let $\rho \in SFS_{ES_1}$ be such that $\rho e \in PI_{ES_1}$. By condition (U1) in the definition of the unique occurrence property such ρ exists. Then $alph(\rho e) \in C_1$ and hence $f(alph(\rho e)) \in C_2$ because f is an LES-morphism. Since $f(e) \in f(alph(\rho e))$ this implies that $e' \in f(alph(\rho))$ because $e' \leq_2 f(e)$ and $e' \neq f(e)$. Let $e'' \in alph(\rho)$ be such that $f(e'') = e'$. If $e'' \leq_1 e$, then $e' = f(e'') \in f(\downarrow e)$.

In order to prove that $e'' \leq_1 e$, define $R \subseteq PI_{ES_1} \times PI_{ES_1}$ by: $\rho_1 e_1 \, R \, \rho_2 e_2$ iff ($e_1 = e_2 \neq e$ or ($e_1 = e_2 = e$ and ($e'' \in alph(\rho_1) \Leftrightarrow e'' \in alph(\rho_2)$))). Assume that R is an equivalence relation which is SFS_{ES_1}-consistent. Then $\sim_{ES_1} \subseteq R$ because $\sim_{ES_1}$ is the least equivalence relation which is SFS_{ES_1}-consistent. Since $\rho e \in PI_{ES_1}$, $e'' \in alph(\rho)$, and ES_1 has the unique occurrence property it then follows that $e'' \in alph(\rho_1)$ for all $\rho_1 e \in PI_{ES_1}$. Hence $e'' \in c$ for all $c \in C_1$ such that $e \in c$ and thus $e'' \leq_1 e$.

Consequently, what remains to be proved is that R is an equivalence relation which satisfies (C1) and (C2).

Clearly, R is an equivalence relation. In order to prove that R satisfies (C1), suppose $\rho_1 u \in SFS_{ES_1}$ and $e_1 \in u$. If $e_1 \neq e$ then it is clear that $\rho_1 e_1 \, R \, \rho_1(u - e_1)e_1$, so assume that $e_1 = e$. If $e'' \notin u$ then it is clear that $\rho_1 e_1 \, R \, \rho_1(u - e_1)e_1$. We now show that $e'' \in u$ leads to a contradiction. To see this, suppose that $e'' \in u$. Since $alph(\rho_1 e_1) \in C_1$ and f is an LES-morphism, we must have that $f(alph(\rho_1 e_1)) = alph(f(\rho_1)) \cup f(e) \in C_2$. Combining this with $e' \leq_2 f(e)$ and $e' \neq f(e)$ yields that $e' \in alph(f(\rho_1))$. On the other hand, we also have that $alph(\rho_1) \vdash_1 e''$ and hence by the definition of LES-morphism $f(alph(\rho_1)) \vdash_2 f(e'')$. This leads to a contradition, because $f(e'') = e' \in alph(f(\rho_1)) = f(alph(\rho_1))$. We can now conclude that $e'' \in u$ is not possible. This proves that R satisfies (C1).

Now in order to prove that R satisfies (C2), let $\rho_1 e_1, \rho_2 e_1 \in PI_{ES_1}$ be such that $past_R(\rho_1) = past_R(\rho_2)$. If $e_1 \neq e$ then we immediately have that $\rho_1 e_1 \, R \, \rho_2 e_1$. If $e_1 = e$, then $\rho_1 e_1 \, R \, \rho_2 e_1$ because $past_R(\rho_1) = past_R(\rho_2)$ implies that also $alph(\rho_1) = alph(\rho_2)$. This proves that R satisfies (C2).

Thus R is an equivalence relation satisfying (C1) and (C2) which completes the proof of condition (1) in the definition of PES-morphism.

In order to prove condition (2), let $e_1, e_2 \in E_1$ be such that $f(e_1)$ and $f(e_2)$ are defined and $\neg(e_1 \#_1 e_2)$. Then by Lemma 6.17 there exists $c \in C_1$ such that $e_1, e_2 \in c$. Since f is an LES-morphism $f(c) \in C_2$ and hence $\neg(f(e_1) \#_2 f(e_2))$ by the definition of $\#_2$.

Finally, condition (3) in the definition of PES-morphism follows easily from Lemma 1.6 and Lemma 6.17. $\square$

The following result now follows immediately from Lemma 6.22 and Lemma 6.24.

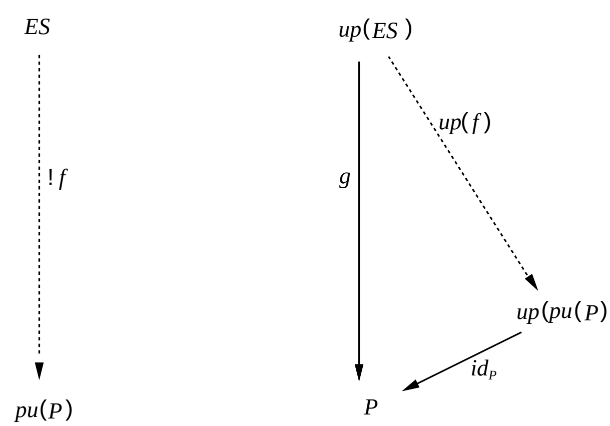
Theorem 6.25
up is a functor from $\mathcal{ULES}$ to $\mathcal{PES}$. $\square$

Now we prove that up and pu form an adjunction. The co-unit of this adjunction is given by the identity arrows id_P for each prime event structure P. Note that the co-unit is a PES-isomorphism because $P = up(pu(P))$ for each prime event structure P. Hence the adjunction is a reflection.

Theorem 6.26
$up : \mathcal{ULES} \to \mathcal{PES}$ and $pu : \mathcal{PES} \to \mathcal{ULES}$ form a reflection with up the left adjoint and the identity arrows id_P as co-unit.

Proof.
Let $ES = (E, C, \vdash)$ be an UL-event structure, let $P = (E', \leq', \#')$ be a prime event structure, and let g be a PES-morphism from $up(ES) = (E, \leq, \#)$ to P.

We must prove that there exists a unique LES-morphism f from ES to $pu(P) = (E', C', \vdash')$ such that the following diagram commutes.



Since up is the identity on arrows, it is sufficient to prove that g is an LES-morphism from ES to $pu(P)$. Suppose $c \vdash u$. Then $c \cap u = \emptyset$ and $c \cup v \in C \subseteq C_{up(ES)}$, for all $v \subseteq u$ by (A2). Since g is a PES-morphism from $up(ES)$ to P we now have by Lemma 6.17 that $g(c) \cup g(v) \in C_P$ for all $v \subseteq u$ and $g(c) \cap g(u) = \emptyset$. Hence $g(c) \vdash' g(u)$. $\square$

Discussion

In this paper we have proposed an event structure semantics for the general class of Petri nets. We have achieved this by identifying a new class of event structures called UL-event structures which turn out to be a proper and very generous generalization of the well-known prime event structures. Our event structure semantics is also a strictly conservative extension of the classic prime event structure semantics for 1-safe Petri nets constructed in [NPW]. Our results are restricted in that we use set-based event structures and only step firing sequences of Petri nets, thus effectively "filtering" out auto-concurrency. It should be noted however that even without auto-concurrency, due to a multiplicity of tokens, intuition concerning basic notions such as causality, concurrency and conflict break down for Petri nets. Hence working out a satisfactory event structure semantics even in this restricted setting turns out to be a non-trivial task.

We have also shown that the behaviour of Petri nets, when auto-concurrency is filtered out, is strongly related to the larger class of L-event structures. In particular, the map en associates a Petri net $en(ES) = N$ with each L-event structure ES so that $SFS_{ES} = MFS_N (= SFS_N)$. Thus the behaviour of N will be as rich as that of ES. Since L-event structures are not required to satisfy any global properties, this result suggests that the behaviour of Petri nets is also equally unstructured in a global sense.

The key technical idea introduced in this paper is condition (C2) used for identifying prime intervals. Once this idea is available, the means for going back and forth between L-event structures and Petri nets is established. More importantly, it leads to an, in our opinion, intuitively satisfactory event structure semantics for a variety of "problematic" examples. In case of 1-safe Petri nets it is sufficient to demand (C1) and a simplified version of (C2), see, e.g., [NPW, WN].

Turning now to the "universality" of our constructions, it turns out that we can not mimic the pleasant co-reflection between prime event structures and 1-safe Petri nets in this setting. The problem is that due to auto-concurrency, $\mathcal{PN}$ is too rich in terms of objects *and* arrows. We have shown that by cutting down on the objects, i.e. considering co-safe Petri nets, we can obtain a co-reflection between $\mathcal{ULES}$ and $\mathcal{PNS}$. One pleasant consequence of this result is that we have a complete event structure semantics for the class of co-safe Petri nets.

One can easily lift the notion of L-event structures to handle (finite) multisets by allowing multisets of events as configurations and by allowing multisets of events to become enabled at a configuration. In this way an adjunction can be obtained between the resulting category of event structures and the category of *all* Petri nets. The details can be found in [H]. The trouble with this more general approach is that this adjunction is not a co-reflection. To solve this problem it seems that we must somehow find a way of distinguishing between multiple occurrences of the same transition due to auto-concurrency on the one hand and due to causality on the other hand. It is not at all obvious at present how this can be achieved.

Also [MMS] proposes an extension of Winskel's results to general Petri nets. To this end unfoldings of Petri nets are defined and by an adjunction related to occurrence nets, and therefore to prime event structures. This adjunction is an extension of the corresponding co-reflection of Winskel. A central feature of [MMS] is that tokens are treated as coloured entities. As a result, one is forced to record *which* tokens were used in the occurrence of a transition, and thus a great deal of conflict is injected into the semantics. This is even the case for Petri nets which do not have any shared places, where conflicts may be introduced between different occurrences of the same transition. Such a colouring of tokens is often undesirable, see, e.g., [BD]. An approach similar to [MMS] is followed in [E] where also occurrence nets are used to describe the behaviour of Petri nets. Hence in both approaches 1-safe Petri nets and general Petri nets have the same expressive power in terms of event structures, whereas our semantics is a strictly conservative extension of the event structure semantics of 1-safe Petri nets.

The classes of L-event structures and UL-event structures introduced in this paper seem to be of independent interest. In particular, we have shown that prime event structures may be viewed as UL-event structures and Winskel's general event structures and their stable subclass may be viewed as L-event structures, but not as UL-event structures. The relationship between prime event structures and UL-event structures, and the relationship between L-event structures and Winskel's general and stable event structures are stated in terms of reflections in a categorical framework. Note that by composing the functors between $\mathcal{PNS}$ and $\mathcal{ULES}$ and the functors between $\mathcal{ULES}$ and $\mathcal{PES}$, we also have functors between $\mathcal{PNS}$ and $\mathcal{PES}$. Since both the functor from $\mathcal{ULES}$ to $\mathcal{PNS}$ and the functor from $\mathcal{ULES}$ to $\mathcal{PES}$ are the left adjoint of the corresponding adjunctions, this does however not yield an adjunction between $\mathcal{PNS}$ and $\mathcal{PES}$.

Another important class of event structures is formed by the flow event structures [BC]. In [B] it has been shown that the class of flow event structures is included in the class of stable event structures. Hence our results also show how to view each flow event structure as an L-event structure (which is not necessarily an UL-event structure).

Prime event structures with binary conflicts as we have used here correspond to the behaviour of 1-safe Petri nets. Their domain theoretic characterization has been given in [NPW]. Flow event structures yield the same class of domains [B]. Winskel has shown [W3] that stable event structures yield the same class of domains as prime event structures with arbitrary conflicts. The domains corresponding to W-event structures have been characterized in [Dr], see also [W3]. For L-event structures and UL-event structures however, it is not yet clear how one should go about obtaining a domain theoretic characterization.

Acknowledgments

The authors would like to thank Mogens Nielsen for many helpful suggestions.

The authors acknowledge support from the ESPRIT BRA No. 6067 CALIBAN, and the Dutch National Concurrency Project REX sponsored by NFI.

References

[BD] Best, E., and Devillers, R., (1987), Sequential and concurrent behaviour in Petri net theory, Theoretical Computer Science 55, 87-136.

[BDH] Best, E., Devillers, R., and Hall, J., (1992), The Box Calculus: a new causal algebra with multi-label communication, Lecture Notes in Computer Science 609, 21-69.

[B] Boudol, G., (1990), Flow event structures and flow nets, Lecture Notes in Computer Science 469, 62-95.

[BC] Boudol, G., and Castellani, I., (1988), Permutation of transitions: an event structure semantics for CCS and SCCS, Lecture Notes in Computer Science 354, 411-427.

[De] Devillers, R., (1993), Construction of S-invariants and S-components for refined Petri Boxes, Lecture Notes in Computer Science 691, 242-261.

[Dr] Droste, M., (1989), Event structures and domains, Theoretical Computer Science 68, 37-47.

[ER] Ehrenfeucht, A., and Rozenberg, G., (1990), Partial (Set) 2-structures; Part II: State spaces of concurrent systems, Acta Informatica, v. 27, 343-368.

[E] Engelfriet, J., (1991), Branching processes of Petri nets, Acta Informatica, v. 28, 575-591.

[H] Hoogers, P.W., Forthcoming PhD thesis.

[HKT1] Hoogers, P.W., Kleijn, H.C.M., and Thiagarajan, P.S., (1992), A trace semantics for Petri nets, Lecture Notes in Computer Science 623, 595-604.

[HKT2] Hoogers, P.W., Kleijn, H.C.M., and Thiagarajan, P.S., (1993), Local event structures and Petri nets, Lecture Notes in Computer Science 715, 462-476.

[MM] Meseguer, J., and Montanari, U., (1990), Petri nets are monoids, Information and Computation 88, 105-155.

[MMS] Meseguer, J., Montanari, U., and Sassone, V., (1992), On the semantics of Petri nets, Lecture Notes in Computer Science 630, 286-301.

[M] Mukund, M., (1992), Petri nets and step transition systems, International Journal of Foundations of Computer Science Vol. 3 No. 4, 443-478.

[NPW] Nielsen, M., Plotkin, G., and Winskel, G., (1981), Petri nets, event structures and domains, Part I, Theoretical Computer Science 13, 85-108.

[NRT] Nielsen, M., Rozenberg, G., and Thiagarajan, P.S., (1992), Elementary transition systems, Theoretical Computer Science 96, 3-33.

[W1] Winskel, G., (1984), Categories of models for concurrency, Lecture Notes in Computer Science 197, 246-267.

[W2] Winskel, G., (1987), Petri nets, algebras, morphisms, and compositionality, Information and Computation 72, 197-238.

[W3] Winskel, G., (1987), Event structures, Lecture Notes in Computer Science 255, 325-392.

[W4] Winskel, G., (1988), An introduction to event structures, Lecture Notes in Computer Science 354, 364-397.

[WN] Winskel, G., and Nielsen, M., (1992), Models for concurrency, to appear in S. Abramsky, D.M. Gabbay, T.S.E. Maibaum eds., Handbook of Logic in Computer Science.