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Study of self-similar and steady flows near singularities

II. A case of multiple characteristic velocity†

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We consider here a system of first-order quasilinear partial differential equations in two independent variables: t, time and x, spatial coordinate. In many physically realistic problems in fluid mechanics, a singularity of the system of ordinary differential equations representing the steady solutions represents a critical state where one of the characteristic velocities vanishes (e.g. sonic point in fluid mechanics). Kulikovskii & Slobodkina (1967) have shown that the stability of all the steady solutions near a singularity can be studied with the help of a simple first-order quasi-linear partial differential equation. The simplicity of their method lies in the fact that all the results can be deduced from the phase-plane of the steady equations. The analysis of Kulikovskii & Slobodkina is valid for any system of equations, totally hyperbolic or mixed type with the only assumption that the characteristic velocity under consideration is real and not multiple. We have earlier (1970, to be referred to as part I) extended their treatment to self-similar flows. In this paper we have shown that in the case of a characteristic velocity of multiplicity s ($s > 1$), it is still possible to approximate the system provided there exists exactly s linearly independent eigenvectors corresponding to this characteristic velocity. The approximate system consists of s quasi-linear equations and we have to consider the $s+1$ dimensional phase-space of the steady equations. In the end we have also discussed two illustrative examples.

1. INTRODUCTION

In this paper we consider a system of first-order quasi-linear partial differential equations in two independent variables. In order to identify our results with waves propagating in one dimension, we shall denote one independent variable by x and the other by t. However, the results can be applied to a two-dimensional steady phenomenon or anywhere else with proper interpretation of the terms 'steady solutions' or 'non-steady solutions', etc., used in this paper, provided there are only two independent variables. Thus we consider the system

$$A'_{ij}(u_k, x)\frac{\partial u_j}{\partial t}+B'_{ij}(u_k, x)\frac{\partial u_j}{\partial x}+C'_i(u_k, x) = 0 \quad (i, j = 1, 2, 3, \ldots, n), \tag{1.1}$$

where the elements of the matrices $A' \equiv [A'_{ij}]$, $B' \equiv [B'_{ij}]$ and the column vector $C' \equiv [C'_i]$ do not depend on time t explicitly. The set of equations (1.1) admits two

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distinct types of solutions: (i) steady or similarity solutions where in a steady solution the dependent variables are functions of x only and in a similarity solution they depend on t and a properly chosen similarity variable $\xi = \xi(x, t)$, and (ii) general non-steady solutions. In the case of a steady or similarity solution, the system (1.1) reduces to a system of ordinary differential equations. In the case of a self-similar solution a singular point of these ordinary differential equations corresponds to states on one of the characteristics of the system (1.1) (Zeldovich & Raizer 1967) and in the case of a steady solution it corresponds to the point where a characteristic velocity vanishes (Kulikovskii & Slobodkina 1967). These authors have shown that the propagation of small perturbations of steady solutions in the neighbourhood of a singular point is governed by a first-order quasilinear partial differential equation

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = \alpha c + \beta x, \tag{1.2}$$

where c is the characteristic velocity under consideration and α and β are two constants independent of the steady solution. Bhatnagar & Prasad (1970) have shown that Kulikovskii & Slobodkina's method can be immediately generalized to study the stability of self-similar flows in the neighbourhood of a singular point. Following this method we can show (Tagare & Prasad 1970) that the similarity solutions by Hunter (1963) for the flow into a cavity with variable speed of the cavity boundary are unstable for radially symmetric disturbances. However, when a steady or self-similar flow is stable in the neighbourhood of a singular point, we cannot conclude definitely that the flow is stable since the source of instability may be in a region away from the singular point. We come across this situation in many important problems (Bhatnagar & Prasad 1970; Tagare 1970).

The simplicity and hence the importance of Kulikovskii & Slobodkina's method lies in the fact that the propagation, the growth and the decay of perturbations can be studied from the phase-plane of the steady-state equations. Further, their approximate equation (1.2) conforms to the general feeling that it is possible to approximate the system (1.1) in the neighbourhood of certain curves in (x, t)-plane (which may be characteristic curves corresponding to lower order terms) by an equation of the type

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = \Gamma\left(x, \phi, \frac{\partial \phi}{\partial x}, \ldots, \frac{\partial^p \phi}{\partial x^p}\right), \tag{1.3}$$

where

$$c = c(x, \phi) \tag{1.4}$$

is the characteristic velocity and p is a positive integer. The investigations of Whitham (1959), Taniuty & Wei (1968) and Asano & Taniuty (1969) are aimed to achieve such an approximation. The analysis of Kulikovskii & Slobodkina is valid for any system of the type (1.1), totally hyperbolic or mixed type, with the only assumption that the characteristic velocity under consideration is real and not multiple.

In this paper we have shown that in the case of a characteristic velocity of multiplicity s (> 1), it is still possible to approximate the system (1.1) in the neighbourhood of the singular point of the steady equations provided there exist exactly s generalized eigenvectors corresponding to the characteristic velocity c, i.e. the rank of the matrix $B'-cA'$ is exactly $n-s$. In this case, the system (1.1) reduces to a simple system of s quasi-linear equations. Throughout the paper, we follow the convention that a repeated suffix in any term will represent a summation over the range $1, 2, \ldots, n$. Whenever we come across a summation over a different range (e.g. $1, 2, \ldots, p$) we indicate it clearly by the symbol $\sum\limits_{i=1}^{p}$.

2. Approximate equations

We consider a system of n first-order partial differential equations (1.1). The characteristic velocities are roots of the polynomial equation

$$|B'_{ij}(u_k, x) - \lambda A'_{ij}(u_k, x)| = 0 \tag{2.1}$$

in λ. We assume that $\lambda = c(u_k, x)$ is a real root of this equation of multiplicity s over some domain D of variables x and u_k. We do not make any assumption about the other roots except that they are distinct from $c(u_k, x)$ so that they may be real or complex. We further assume that there is a continuous steady solution, given by $u_k = u_{k0}(x)$, of the system (1.1) such that the points $(x, u_{k0}(x))$ lie in the interior of the domain D and $c(u_{k0}(x), x)$ vanishes at some point in D which we choose to be $(0, u_{k0}(0))$ without any loss of generality. The $u_{k0}(x)$ satisfy

$$B'_{ij}(u_{k0}, x)\frac{\mathrm{d}u_{j0}}{\mathrm{d}x} + C'_i(u_{k0}, x) = 0. \tag{2.2}$$

The matrix $[B'_{ij} - cA'_{ij}]$ is singular and *we assume that its rank in the domain D is $n-s$*. Then there exist s generalized left eigenvectors

$$\beta^{(k)} \equiv [\beta^{(k)}_1, \beta^{(k)}_2, \ldots, \beta^{(k)}_n],$$

satisfying
$$\beta^{(k)}_i B'_{ij} = c\beta^{(k)}_i A'_{ij} \quad (k = 1, 2, \ldots, s; j = 1, 2, \ldots, n). \tag{2.3}$$

Multiplying (1.1) by $\beta^{(k)}_i$ and contracting with respect to the suffix i, we obtain the following s characteristic combinations in the form

$$A_{ij}\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\}u_j + C_i = 0 \quad (i = 1, 2, \ldots, s), \tag{2.4}$$

where
$$A_{ij} = \beta^{(i)}_k A'_{kj} \quad \text{and} \quad C_i = \beta^{(i)}_k C'_k. \tag{2.5}$$

Since the rank of the matrix, whose element in the ith row and kth column is $\beta^{(i)}_k$, is s we can always solve

$$\beta^{(i)}_k A'_{kj} = A_{ij} \quad \text{and} \quad \beta^{(i)}_k C'_k = C_i \quad (i = 1, 2, \ldots, s; j = 1, 2, \ldots, n)$$

for A'_{kj} and C'_k $(k = 1, 2, \ldots, s; j = 1, 2, \ldots, n)$. Using these values of A'_{kj} we can solve the system (2.3) for B'_{ij} $(i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, n)$. Thus the system (1.1) is equivalent to another system of equations in which the first s equations can be replaced by equations (2.4). Therefore, in the remainder of the paper, we shall consider the following system of n equations

$$A_{ij}\frac{\partial u_j}{\partial t}+B_{ij}\frac{\partial u_j}{\partial x}+C_i=0 \quad (i,j=1,2,\ldots,n) \tag{2.6a}$$

with
$$B_{ij}=cA_{ij} \quad (i=1,2,\ldots,s;\, j=1,2,\ldots,n). \tag{2.6b}$$

We arrange the variables $u_1, u_2, \ldots, u_n$ in such a manner that the $(n-s)$th order matrix

$$M=\begin{bmatrix} B^*_{s+1,\,s+1} & B^*_{s+1,\,s+2} & \cdots & B^*_{s+1,\,n} \\ B^*_{s+2,\,s+1} & B^*_{s+2,\,s+2} & \cdots & B^*_{s+2,\,n} \\ \cdots & \cdots & \cdots & \cdots \\ B^*_{n,\,s+1} & B^*_{n,\,s+2} & \cdots & B^*_{n,\,n} \end{bmatrix} \tag{2.7a}$$

is of rank $n-s$. Here
$$B^*_{ij}=B_{ij}(u_{k0}(0),\,0). \tag{2.7b}$$

This is possible since by substituting $x=0$ in the matrix $[B'_{ij}-cA'_{ij}]$ we get the nth order matrix $[B'^*_{ij}]$ with rank $n-s$.

The functions $C_i(u_{k0}(x),x)$ (for $i=1,2,\ldots,s$) must vanish at $x=0$ in order that the dependent variables $u_{k0}(x)$ in the steady solution are continuous functions at $x=0$. Behaviour of perturbations

$$v_k(x,t)=u_k(x,t)-u_{k0}(x) \tag{2.8}$$

of the steady solution is described by the system of equations

$$A_{ij}(u_k,x)\frac{\partial v_j}{\partial t}+B_{ij}(u_k,x)\frac{\partial v_j}{\partial x}+F_i=0 \quad (i=1,2,\ldots,n), \tag{2.9}$$

where
$$F_i(v_k,x)=C_i(u_{k0}+v_k,x)+c(u_{k0}+v_k,x)\,A_{ij}(u_{k0}+v_k,x)\,\mathrm{d}u_{j0}/\mathrm{d}x \quad (i=1,2,\ldots,s). \tag{2.10a}$$

and
$$F_i(v_k,x)=C_i(u_{k0}+v_k,x)+B_{ij}(u_{k0}+v_k,x)\,\mathrm{d}u_{j0}/\mathrm{d}x \quad (i=s+1,s+2,\ldots,n). \tag{2.10b}$$

We wish to retain the most dominant terms in (2.9) keeping in view that we wish to study only those waves which remain in the neighbourhood of the singular point for a time interval of the order of unity, i.e. those waves whose velocity of propagation is of the order of the magnitude of c which vanishes in the steady solution at $x=0$. Therefore, if δ is a small quantity of first order, the quantities

$$v'_i=\frac{v_i}{\delta},\quad x'=\frac{x}{\delta},\quad t'=t \tag{2.10}$$

are all of the order of unity. The various terms of equations (2.9) have the following expansions

$$A_{ij}(u_k,x)=A^*_{ij}+O(\delta), \tag{2.11}$$

$$\begin{aligned} c(u_k,x)&=mx+m_i(u_i-u^*_i)+O(\delta^2)\\ &=mx+m_i\left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}x}\right)^* x+m_iv_i+O(\delta^2), \end{aligned} \tag{2.12}$$

$$C_i(u_k, x) = \alpha_i x + \alpha_{ik}(u_k - u_k^*) + O(\delta^2) \quad (i = 1, 2, \ldots, s), \tag{2.13}$$

$$F_i(v_k, x) = \alpha_i x + \alpha_{ik}(u_k - u_k^*) + \{mx + m_k(u_k - u_k^*)\} A_{ij}^* \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* + O(\delta^2)$$
$$= \left\{\alpha_{ik} + m_k A_{ij}^* \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^*\right\} v_k + O(\delta^2) \quad (i = 1, 2, \ldots, s) \tag{2.14a}$$

and
$$F_i(v_k, x) = C_i^* + \alpha_i x + \alpha_{ik}(u_k - u_k^*) + [B_{ij}^* + \gamma_{ij} x$$
$$+ \gamma_{ijk}(u_k - u_k^*)] \left[\left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* + \left(\frac{\mathrm{d}^2 u_{j0}}{\mathrm{d}x^2}\right)^* x\right] + O(\delta^2)$$
$$= \left[\alpha_{ik} + \gamma_{ijk} \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^*\right] v_k + O(\delta^2) \quad (i = s+1, s+2, \ldots, n), \tag{2.14b}$$

where m, m_i, α_i, α_{ij}, γ_{ij} and γ_{ijk} are constants and the star on any quantity denotes its value in the steady solution at $x = 0$. In deducing (2.14) we have made use of the steady equations in the form

$$\{mx + m_k(u_{k0} - u_k^*)\} A_{ij}^* \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* + \alpha_i x + \alpha_{ik}(u_{k0} - u_k^*) = 0 \quad (i = 1, 2, \ldots, s) \tag{2.15a}$$

and

$$C_i^* + \alpha_i x + \alpha_{ik}(u_{k0} - u_k^*) + B_{ij}^* \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* + \gamma_{ij} \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* x$$
$$+ B_{ij}^* \left(\frac{\mathrm{d}^2 u_{j0}}{\mathrm{d}x^2}\right)^* x + \gamma_{ijk}(u_{k0} - u_k^*) \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^*$$
$$= 0 \quad (i = s+1, s+2, \ldots, n). \tag{2.15b}$$

Substituting (2.10) in (2.9) we get

$$A_{ij} \frac{\partial v_j'}{\partial t'} + \frac{1}{\delta} B_{ij} \frac{\partial v_j'}{\partial x'} + F_i' = 0, \tag{2.16}$$

where $F_i' = (1/\delta) F_i$ is of the order of unity.

The terms of the order of $1/\delta$ give

$$B_{ij}^* \frac{\partial v_j}{\partial x} = 0 \quad (i = 1, 2, \ldots, n), \tag{2.17}$$

whose general solution is

$$v_j = \sum_{i=1}^{s} w_i(x, t) b_j^{(i)*} + g_j(t), \tag{2.18}$$

where $b^{(i)}$ $(i = 1, 2, \ldots, s)$ are s generalized right eigenvectors corresponding to the generalized eigenvalue $\lambda = c$ of multiplicity s and $w_i(x, t)$ and $g_j(t)$ are arbitrary functions of their arguments. At the point $x = 0$ the right eigenvectors $b^{(k)}$ $(k = 1, 2, \ldots, s)$ satisfy

$$B_{ij}^* \, b_j^{(k)*} = 0 \quad (i = 1, 2, \ldots, n). \tag{2.19}$$

The equation (2.18) shows that in the neighbourhood of the singular point, the significant variation in $v_1, v_2, \ldots, v_n$ with x is only due to the variation of s new dependent variables $w_1, w_2, \ldots, w_s$. Since the first s equations of (2.19) are satisfied for arbitrary values of $b_j^{(k)*}$ and the matrix M is non-singular, we can choose the eigenvector $b^{(k)*}$ in the form

$$b^{(k)*} = [\delta_{1k}, \delta_{2k}, \ldots, \delta_{sk}, b_{s+1}^{(k)*}, \ldots, b_n^{(k)*}] \tag{2.20a}$$

where δ_{ik} are Kronecker deltas. The last $n-s$ components of $b^{(k)*}$ are determined from

$$\sum_{j=s+1}^{n} B_{ij}^* b_j^{(k)*} = -B_{ik}^* \quad (i = s+1, s+2, \ldots, n; k = 1, 2, \ldots, s). \tag{2.20b}$$

The terms of order of unity in the first s equations of the system (2.16) give us

$$A_{ij}^* \frac{\partial v_j}{\partial t} + \Delta B_{ij} \frac{\partial v_j}{\partial x} + F_{iv_k}^* v_k = 0, \tag{2.21}$$

where
$$\Delta B_{ij} = B_{ij}(u_k, x) - B_{ij}^*. \tag{2.22}$$

Using (2.6b) we can write (2.21) in the form

$$A_{ij}^* \left[\frac{\partial v_j}{\partial t} + c\frac{\partial v_j}{\partial x}\right] + F_{iv_k}^* v_k = 0 \quad (i = 1, 2, \ldots, s). \tag{2.23}$$

Substituting (2.18) in (2.23) we get

$$\sum_{k=1}^{s} K_{ik} \left[\frac{\partial w_k}{\partial t} + \left\{c_x x + \sum_{j=1}^{s} c_{w_j} w_j + \phi(t)\right\} \frac{\partial w_k}{\partial x}\right] = \sum_{k=1}^{s} L_{ik} w_k + f_i(t) \quad (i = 1, 2, \ldots, s) \tag{2.24a}$$

$$\left.\begin{aligned} \text{where}\quad K_{ik} &= A_{ij}^* b_j^{(k)*}, \quad L_{ik} = -F_{iv_j}^* b_j^{(k)*}, \quad c_x = m + m_i \left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}x}\right)^*, \\ c_{w_i} &= m_j b_j^{(i)*}, \quad \phi(t) = m_i g_i(t), \quad f_i(t) = -\left\{A_{ij}^* \frac{\mathrm{d}g_j(t)}{\mathrm{d}t} + F_{iv_k}^* g_k(t)\right\}. \end{aligned}\right\} \tag{2.24b}$$

Thus the s equations (2.24) govern the behaviour of perturbations of the steady solution in the neighbourhood of the singular point. In this region we can replace the n dependent variables v_k by a new system of n variables $w_1, w_2, \ldots, w_s, \ldots, w_n$ which are linear combinations of v_k and in the second set only $w_1, w_2, \ldots, w_s$ vary significantly with x. The functions $w_1, w_2, \ldots, w_s$ are the s Riemann invariants corresponding to the characteristic velocity c. The other variables $w_{s+1}, w_{s+2}, \ldots, w_n$ are functions of time only and they can be determined from the solution of the problem falling outside the small neighbourhood of the singular point. The functions $f_i(t)$ and $\phi(t)$ in (2.24) can be expressed in terms of $w_{s+1}, w_{s+2}, \ldots, w_n$. If we wish to get n equations governing the variation of all the n variables w_i $(i = 1, 2, \ldots, n)$ we have to make use of the last $(n-s)$ equations in (2.9). However, if we are interested in disturbances bounded in space and in the neighbourhood of the singular point, the functions $f_i(t)$ and $\phi(t)$, being zero outside the small neighbourhood of $x = 0$, can be neglected from the equations (2.24) and the history of all such disturbances can be obtained from the equations

$$\sum_{r=1}^{s} K_{ir} \left[\frac{\partial w_r}{\partial t} + \left(c_x x + \sum_{j=1}^{s} c_{w_j} w_j\right) \frac{\partial w_r}{\partial x}\right] = \sum_{r=1}^{s} L_{ir} w_r \quad (i = 1, 2, \ldots, s). \tag{2.25}$$

It is possible to get rid of the functions $f_i(t)$ and $\phi(t)$ in (2.24) by the following transformation:

$$w_k = w_k' + w_k^*(t), \quad x = \xi + x^*(t), \tag{2.26}$$

where $w_k^*(t)$ and $x^*(t)$ satisfy the ordinary differential equations

$$\sum_{r=1}^{s} K_{ir} \frac{\mathrm{d}w_r^*}{\mathrm{d}t} = \sum_{r=1}^{s} L_{ir} w_r^* + f_i(t) \tag{2.27}$$

and

$$\frac{\mathrm{d}x^*(t)}{\mathrm{d}t} = c_x x^* + \sum_{k=1}^{s} c_{w_k} w_k^* + \phi(t). \tag{2.28}$$

The constants c_x and L_{ir} in (2.25) contain the derivatives $(\mathrm{d}u_{i0}/\mathrm{d}x)^*$ and hence the equations (2.25) for the perturbations of a steady solution depend on the particular steady solution. In the next section we shall show that it is possible to remove this dependence by a suitable transformation.

3. Transformation to a system independent of the particular steady solution

Let the inverse of the matrix M, defined by $(2.7a)$, be

$$P = \begin{bmatrix} p_{s+1,\,s+1} & p_{s+1,\,s+2} & \cdots & p_{s+1,\,n} \\ p_{s+2,\,s+1} & p_{s+2,\,s+2} & \cdots & p_{s+2,\,n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n,\,s+1} & p_{n,\,s+2} & \cdots & p_{n,\,n} \end{bmatrix}. \tag{3.1}$$

Solving (2.20) we get

$$b_i^{(k)*} = -\sum_{j=s+1}^{n} p_{ij} B_{jk}^* \quad (k = 1, 2, \ldots, s;\ i = s+1, s+2, \ldots, n). \tag{3.2}$$

From the last $n-s$ equations of the system $(2.6a)$ we get

$$\sum_{j=s+1}^{n} B_{ij}^* \left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* = -C_i^* - \sum_{k=1}^{s} B_{ik}^* \left(\frac{\mathrm{d}u_{k0}}{\mathrm{d}x}\right)^* \quad (i = s+1, s+2, \ldots, n) \tag{3.3}$$

so that

$$\left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}x}\right)^* = -\sum_{j=s+1}^{n} p_{ij} \left\{C_j^* + \sum_{k=1}^{s} B_{jk}^* \left(\frac{\mathrm{d}u_{k0}}{\mathrm{d}x}\right)^*\right\} \quad (i = s+1, s+2, \ldots, n). \tag{3.4}$$

From the expression for c_{w_i} in $(2.24b)$ we get

$$c_{w_i} = m_i - \sum_{j,\,k=s+1}^{n} m_k p_{kj} B_{ji}^*. \tag{3.5}$$

The expression in $(2.24b)$ for c_x becomes

$$c_x = R + \sum_{i=1}^{s} c_{w_i} \left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}x}\right)^*, \tag{3.6}$$

where

$$R = m - \sum_{i,\,j=s+1}^{n} m_i C_j^* p_{ij}. \tag{3.7}$$

We define a new set of dependent variables $\pi_1, \pi_2, \ldots, \pi_s$ in terms of $w_1, w_2, \ldots, w_s$ by

$$\pi_i = \left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}x}\right)^* x + w_i \quad (i = 1, 2, \ldots, s). \tag{3.8}$$

Substituting (3.8) in (2.25) we get

$$\sum_{r=1}^{s} K_{ir}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\pi_r = R\left[\sum_{r=1}^{s} K_{ir}\left(\frac{\mathrm{d}u_{r0}}{\partial x}\right)^*\right]x$$
$$+ \sum_{j=1}^{s}\left[L_{ij} + c_{w_j}\sum_{r=1}^{s} K_{ir}\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right]\pi_j - \sum_{k=1}^{s} L_{ik}\left(\frac{\mathrm{d}u_{k0}}{\mathrm{d}x}\right)^* x. \tag{3.9}$$

Using the expressions for L_{ij} and K_{ij} we can write (3.9) in the form

$$\sum_{r=1}^{s} K_{ir}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\pi_r + \sum_{r=1}^{s}(\alpha_{ij} b_j^{(r)*})\,\pi_r = R\left[\sum_{r=1}^{s} K_{ir}\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right]x$$
$$+ \sum_{j=1}^{s} c_{w_j}\left\{\sum_{r=1}^{s}\left(\sum_{k=s+1}^{n} A_{ik}^* b_k^{(r)*}\right)\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^* - \sum_{r=s+1}^{n} A_{ir}^*\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right\}\pi_j$$
$$+ \left\{\sum_{r=1}^{s}(\alpha_{ij} b_j^{(k)*})\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right\}x + \left\{\sum_{r=1}^{s} c_{w_r}\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right\}A_{ij}^*\left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* x. \tag{3.10}$$

In the neighbourhood of $x = 0$, the steady equations corresponding to the first s equations of (2.6a) give

$$\sum_{r=1}^{s}\{\alpha_{ij} b_j^{(r)*}\}\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^* + \left\{\sum_{r=1}^{s} c_{w_r}\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^*\right\}A_{ij}^*\left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* = -RA_{ij}^*\left(\frac{\mathrm{d}u_{j0}}{\mathrm{d}x}\right)^* + R_i, \tag{3.11}$$

where
$$R_i = -\alpha_i + \sum_{j,k=s+1}^{n} \alpha_{ik} p_{kj} C_j^*. \tag{3.12}$$

Substituting (3.11) in (3.10) and rearranging some terms in the result we find that the right-hand side of (3.10) can be written as

$$\left\{\sum_{r=1}^{s}\left(\sum_{k=s+1}^{n} A_{ik}^* b_k^{(r)*}\right)\left(\frac{\mathrm{d}u_{r0}}{\mathrm{d}x}\right)^* - \sum_{k=s+1}^{n} A_{ik}^*\left(\frac{\mathrm{d}u_{k0}}{\mathrm{d}x}\right)^*\right\}\left\{\sum_{j=1}^{s} c_{w_j}\pi_j + Rx\right\} + R_i x. \tag{3.13}$$

If we use the expressions (3.2) of $b_i^{(k)*}$ and (3.4) for $(\mathrm{d}u_{i0}/\mathrm{d}x)^*$, the coefficient of $\sum_{j=1}^{s} c_{w_j}\pi_j + Rx$ in (3.13) becomes

$$\sum_{j,k=s+1}^{n} A_{ik}^* p_{kj} C_j^*.$$

Therefore, equation (3.10) finally reduces to

$$\sum_{r=1}^{s} K_{ir}\left(\frac{\partial \pi_r}{\partial t} + c\frac{\partial \pi_r}{\partial x}\right) = \sum_{r=1}^{s} g'_{ir}\pi_r + h'_i x \quad (i = 1, 2, \ldots, s) \tag{3.14}$$

where
$$c = Rx + \sum_{j=1}^{s} c_{w_j}\pi_j, \tag{3.15}$$

$$\left.\begin{aligned} g'_{ir} &= -\left[(\alpha_{ij} b_j^{(r)*}) + \left(\sum_{j,\,k=s+1}^{n} A^*_{ik} p_{kj} C^*_j\right) c_{w_r}\right] \\ \text{and} \qquad h'_i &= R_i + R \sum_{j,\,k=s+1}^{n} A^*_{ik} p_{kj} C^*_j. \end{aligned}\right\} \tag{3.16}$$

In the system (3.14) of s equations, the coefficients are independent of the particular steady solution passing through the singular point. Therefore, this system governs the propagation of perturbations of any steady solution through the singular point. Up to the first-order terms

$$u_i - u_i^* \approx (\mathrm{d}u_{i0}/\mathrm{d}x)^* x + v_i. \tag{3.17}$$

Substituting the expression (2.18) with $f_i \equiv 0$ we get for $i \leqslant s$

$$u_i - u_i^* \approx (\mathrm{d}u_{i0}/\mathrm{d}x)^* x + w_i = \pi_i. \tag{3.18}$$

Thus, up to the first-order terms the variables $\pi_1, \pi_2, \ldots, \pi_s$ are equal to the deviations of the first s dependent variables from their values at the singular point. When $w_i = 0\ (i = 1, 2, \ldots, s)$, π_i becomes equal to $u_{i0} - u_i^*$ up to the first order. We have been able to identify π_i with the first-order term in $u_i - u_i^*$ $(i = 1, 2, \ldots, s)$ due to a proper ordering of the variables in such a way that the matrix M in (2.7a) is non-singular. The system (3.14) governs not only the propagation of the perturbations of a steady solution but also the steady solution in the neighbourhood of the singular point. In fact, for the perturbations bounded in space and in a small neighbourhood of the singular point, the system (3.14) is equivalent to the original system (2.6a). The system (2.6a) need not be totally hyperbolic but the system (3.14) is totally hyperbolic.

If the matrix $[K_{ir}]$ is non-singular, we can write equations (3.14) in the form

$$\frac{\partial \pi_i}{\partial t} + c\frac{\partial \pi_i}{\partial x} = \sum_{r=1}^{s} g_{ir}\pi_r + h_i x \quad (i = 1, 2, \ldots, s) \tag{3.19}$$

where the constants g_{ir} and h_i can be expressed in terms of K_{ir}, g'_{ir} and h'_i. To obtain the coefficients g_{ir}, h_i it is not necessary to work out the complicated algebra of the general theory given here. It is sufficient to eliminate the derivatives $(\mathrm{d}u_{i0}/\mathrm{d}x)$ $(i = s+1, s+2, \ldots, n)$ from the steady equations and solve for

$$\mathrm{d}u_{i0}/\mathrm{d}x \quad (i = 1, 2, \ldots, s).$$

Approximating the equations thus obtained in the neighbourhood of the singular point and comparing with

$$c_0 \frac{\mathrm{d}u_{i0}}{\mathrm{d}x} = \sum_{r=1}^{s} g_{ir}(u_{r0} - u_r^*) + h_i x,$$

we can easily determine the constants. The system of characteristic equations for (3.19) are

$$\frac{\mathrm{d}\pi_i}{\mathrm{d}t} = \sum_{r=1}^{s} g_{ir}\pi_r + h_i x \quad (i = 1, 2, \ldots, s) \tag{3.20a}$$

and
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{r=1}^{s} c_{w_r}\pi_r + Rx. \tag{3.20b}$$

The equations (3.19) and (3.20) remain unchanged under the transformation

$$\pi_i' = \pi_i/\delta \quad \text{and} \quad x' = x/\delta$$

and, therefore, the variables π_i $(i = 1, 2, \ldots, s)$ and x can be taken to be of the order of unity.

It is well known that any linear approximation in the neighbourhood of the sonic point (a singular point of the steady equations) breaks down. But the system (3.19) is quasi-linear and the nonlinear effects of the system (1.1) will be fully taken into account by this approximate system.

4. General discussion about the stability of steady solutions

A solution of the characteristic equations (3.20) in the form $\pi_i = \pi_i(t)$, $x = x(t)$ gives a steady solution of the equation (3.19) in the parametric form and represents a curve in $(s+1)$-dimensional space $(\pi_1, \pi_2, \ldots, \pi_s, x)$. (A detailed discussion of the results in this paragraph can be found in part I and in the paper of Kulikovskii & S'lobodkina for $s = 1$.) Any integral curve of (3.20) which is single-valued in x (i.e. for which no two values of t give the same value for $x(t)$) can be taken to be a physically realistic steady solution. Any integral curve which crosses the hyperplane

$$c \equiv \sum_{r=1}^{s} c_{w_r}\pi_r + Rx = 0$$

will cease to be single valued in x except when it crosses the plane at the singular point which is the origin here (figure 1). A perturbation of a steady solution

$$\pi_i = \pi_{i0}(x)$$

at any fixed time will be represented by a curve

$$\pi_i = \pi_{i0}(x) + \Delta\pi_i(x). \tag{4.1a}$$

If the perturbation is bounded in space it can be geometrically visualized as a deformation of an integral curve of (3.20) between its intersection by two planes $x = x_1$ and $x = x_2$. For a steady solution, the space rate of change of the variable π_{i0} is given by

$$\frac{\mathrm{d}\pi_{i0}}{\mathrm{d}x} = \frac{1}{c_0}\left[\sum_{r=1}^{s} g_{ir}\pi_{r0} + h_i x\right], \tag{4.1b}$$

and the equation (3.19) tells us that in any unsteady solution the space rate of change of π_i as we move with velocity c is again the same as (4.1b). Thus, as time increases, the two end-points at $x = x_1$ and $x = x_2$ of the deformation will move along the integral curve. The various points of the deformation (4.1a) between $x_1 < x < x_2$ will move along the integral curves of (3.20). Thus the propagation of any perturbation of a steady solution can be geometrically visualized in the phase-space of the autonomous system of equations (3.20) as the movement of a curve (4.1 a) whose different points move along the integral curves. The sense of propagation of the disturbance can be shown by arrows indicating time increasing direction along the integral curves. In the domain $c > 0$ the arrows will point in the x-increasing direction and in the x-decreasing direction where $c < 0$. If an arbitrary small deformation of an integral curve tends to coincide with it or remains in an arbitrary small neighbourhood of it, as t tends to infinity, the corresponding steady solution

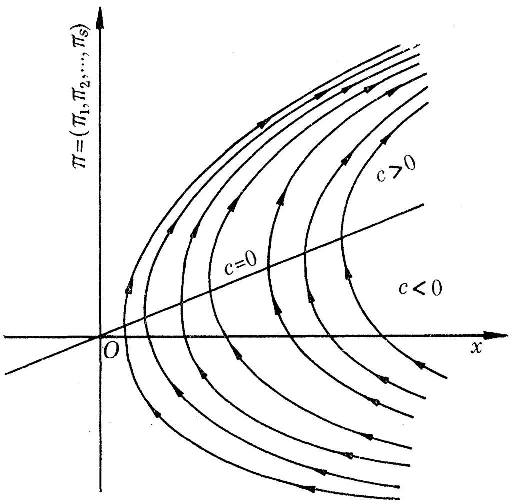

FIGURE 1. For the integral curves crossing the plane $c = 0$, π_i ceases to be a single-valued function of x.

is stable, otherwise it is unstable. In the case of $s = 1$ and $s = 2$, it is easier to visualize the propagation of perturbations, since the phase-space of the system (3.20) coincides with two- and three-dimensional spaces respectively. The case $s = 1$ has been discussed completely by Kulikovskii & Slobodkina. If we go through every minute detail, then in the case of $s = 2$, the singular point ($\pi_1 = \pi_2 = x = 0$) of the system (3.20) can be any one of the 49 types of singularities discussed by Reyn (1964). Reyn has drawn the three-dimensional phase-space for each of these 49 cases. The discussion of Kulikovskii & Slobodkina can be easily extended to each of those 49 cases where a physically realistic steady solution is possible and, therefore, we shall not go into the detailed discussion of individual particular cases.

Let us consider an element of volume V in $(s+1)$-dimensional space $(\pi_1, \pi_2, \ldots, \pi_s, x)$ bounded by a closed simple s-dimensional hypersurface S whose points move in accordance with equations (3.20). Using Gauss-divergence theorem in $(s+1)$-dimensional space we can show that

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \operatorname{div}\left(\frac{\mathrm{d}\pi_1}{\mathrm{d}t}, \frac{\mathrm{d}\pi_2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}\pi_s}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = R + \sum_{i=1}^{s} g_{ii}, \tag{4.2}$$

which gives the rate at which elementary volumes change if the motion of the boundary points is governed by the equations (3.20). Since the system (3.20) is linear, by superposing the results for volume elements, we find that it remains valid even for finite volumes. In the case of $s = 1$, V becomes an area in a plane whose boundary may be taken to be the curve (4.1 a) and if V, given by (4.2), tends to zero as t tends to infinity, it is found that any perturbation will die out and all steady solutions are stable (Kulikovskii & S'lobodkina 1967). For $s \geqslant 2$, if V tends to zero as t tends to infinity, we cannot say definitely that the deformation ultimately coincides with the steady solution.

The theory of stability of solutions of ordinary differential equations has been studied extensively during the last few years (Hahn 1963; Cesari 1963; Lefschetz 1957; Brauer & Nohel 1969). A solution $\pi_i = \pi_i^0(t), x = x^0(t)$ of (3.20) is said to be stable in the sense of Liapunov if for every $\epsilon > 0$ and every $t_0 \geqslant 0$ there exists a $\delta > 0$ such that whenever

$$\left[\sum_{i=1}^{s} \{\pi_i^0(t_0) - \eta_i\}^2 + \{x^0(t_0) - y\}^2\right]^{\frac{1}{2}} < \delta,$$

a solution $\pi_i = \pi_i^1(t)$, $x = x^1(t)$ exists satisfying the condition $\pi_i^1(t_0) = \eta_i$, $x^1(t_0) = y$ and satisfies

$$\left[\sum_{i=1}^{s} \{\pi_i^0(t) - \pi_i^1(t)\}^2 + \{x^0(t) - x^1(t)\}^2\right]^{\frac{1}{2}} < \epsilon$$

for all $t \geqslant t_0$. Thus the stability of a solution of (3.20) in the sense of Liapunov implies the stability of the corresponding steady solution. The following example from Kulikovskii & Slobodkina shows that the converse is not necessarily true:

For $s = 1$, the equation (3.19) can be written in the form

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = \alpha c + \beta x \tag{4.3}$$

with characteristic equations

$$\mathrm{d}c/\mathrm{d}t = \alpha c + \beta x \tag{4.4a}$$

and

$$\mathrm{d}x/\mathrm{d}t = c. \tag{4.4b}$$

If $\alpha > 0$ and the two roots of the equation $\lambda^2 - \alpha\lambda - \beta = 0$ satisfy $\lambda_2 < 0 < \lambda_1$, then the steady solution $c = \lambda_1 x$ of (4.3) is stable since any perturbation of this solution ultimately coincides with it (figure 2).

The solution $c = \lambda_1 x$ of (4.3) corresponds to the solution

$$c = \lambda_1 \mathrm{e}^{\lambda_1 t}, \quad x = \mathrm{e}^{\lambda_1 t} \tag{4.5}$$

of (4.4) and all solutions of (4.4) are unstable in Liapunov sense, since one of the eigenvalues of the matrix of the coefficients on the right-hand side of (4.4) is positive.

The system (3.20) is linear, homogeneous and with constant coefficients and for such systems we have a complete theory of Liapunov stability. In fact, we have the important theorem:

If all the eigenvalues of the $(s+1)$th order matrix J of the coefficients on the right-hand side of (3.20) have negative real part, any solution of (3.20) is stable in the sense of Liapunov and, in turn, this implies the stability of all steady solutions of (3.19).

When the system (1.1) is linear, the coefficients A'_{ij}, B'_{ij}, C'_i are functions of x only. In this case

$$g_{ij} = 0, \quad c_{w_i} = 0 \quad \text{and} \quad R = m \tag{4.6}$$

and the system (3.20) reduces to

$$\frac{\mathrm{d}\pi_i}{\mathrm{d}t} = h_i x \quad (i = 1, 2, \ldots, s); \quad \frac{\mathrm{d}x}{\mathrm{d}t} = mx. \tag{4.7}$$

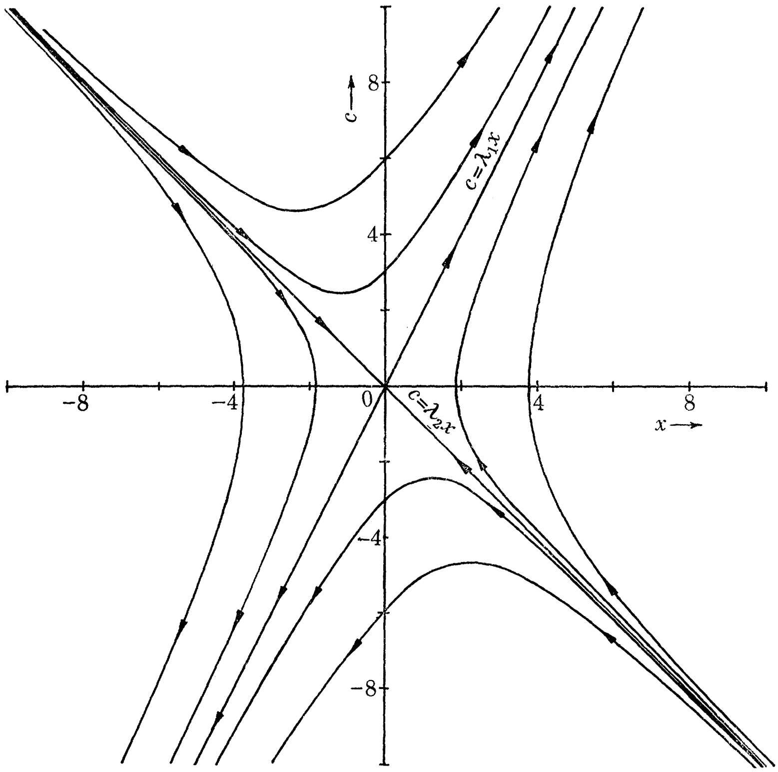

FIGURE 2. $\alpha = 1$, $\beta = 2$.

The general integral of (4.7) is

$$\frac{\pi_1 - a_1}{h_1} = \frac{\pi_2 - a_2}{h_2} = \ldots = \frac{\pi_s - a_s}{h_s} = \frac{x}{m} = a_0 \,\mathrm{e}^{mt} \tag{4.8}$$

where $a_1, a_2, \ldots, a_s$, a_0 are arbitrary constants. The integral curves (4.8) form a family of parallel straight lines with direction ratios $(h_1, h_2, \ldots, h_s, m)$ in (π_i, x)-space and passing through an arbitrary point $(a_1, a_2, \ldots, a_s, 0)$ of the $x = 0$ plane. Thus the steady solutions of (3.19) are

$$\pi_{i0} = a_i + (h_i/m)\,x \quad (i = 1, 2, \ldots, s). \tag{4.9}$$

Any perturbation lying within a cylinder (a s-manifold of points) of radius ϵ with the straight line (4.9) as its axis remains within the cylinder. For $m > 0$, the perturbations move away from the singular point $x = 0$ with increasing velocity, and they spread over larger regions but the amplitude

$$\sum_{i=1}^{s} (\pi_i - \pi_{i0})^2$$

x remains constant. For $m < 0$, the perturbations tend to approach the point $x = 0$ from either side as t tends to infinity and ultimately die at $x = 0$. As the amplitude of a disturbance remains constant, all steady solutions (4.9) are stable.

5. Examples

Example 1

Let us consider the following system of three equations

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \partial u_1/\partial t \\ \partial u_2/\partial t \\ \partial u_3/\partial t \end{bmatrix} + \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ cA_{21} & cA_{22} & cA_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \\ \partial u_3/\partial x \end{bmatrix}$$
$$+ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0, \tag{5.1}$$

where
$$c = m_1 u_1 + m_2 u_2 + m_3 u_3 \tag{5.2}$$

and A_{ij}, B_{ij} and m_i are constants. The matrix M of (2.7a) is of order one:

$$M = [B_{33}] \tag{5.3}$$

and $|M| \neq 0$ implies $B_{33} \neq 0$.

c is a double characteristic velocity so that $s = 2$.

The system (3.19) reduces to

$$\frac{\partial \pi_1}{\partial t} + c\frac{\partial \pi_1}{\partial x} = g_{11}\pi_1 + g_{12}\pi_2, \tag{5.4}$$

$$\frac{\partial \pi_2}{\partial t} + c\frac{\partial \pi_2}{\partial x} = g_{21}\pi_1 + g_{22}\pi_2, \tag{5.5}$$

where

$$\left.\begin{aligned}
g_{11} &= \frac{1}{\Delta}\begin{vmatrix} A_{11} - \dfrac{B_{31}}{B_{33}}A_{13}, & \alpha_{11} - \dfrac{B_{31}}{B_{33}}\alpha_{13} \\ A_{21} - \dfrac{B_{31}}{B_{33}}A_{23}, & \alpha_{21} - \dfrac{B_{31}}{B_{33}}\alpha_{23} \end{vmatrix}, \quad
g_{12} = \frac{1}{\Delta}\begin{vmatrix} A_{11} - \dfrac{B_{31}}{B_{33}}A_{13}, & \alpha_{12} - \dfrac{B_{32}}{B_{33}}\alpha_{13} \\ A_{21} - \dfrac{B_{31}}{B_{33}}A_{23}, & \alpha_{22} - \dfrac{B_{32}}{B_{33}}\alpha_{23} \end{vmatrix} \\
g_{21} &= \frac{1}{\Delta}\begin{vmatrix} \alpha_{11} - \dfrac{B_{31}}{B_{33}}\alpha_{13}, & A_{12} - \dfrac{B_{32}}{B_{33}}A_{13} \\ \alpha_{21} - \dfrac{B_{31}}{B_{33}}\alpha_{23}, & A_{22} - \dfrac{B_{32}}{B_{33}}A_{23} \end{vmatrix}, \quad
g_{22} = \frac{1}{\Delta}\begin{vmatrix} \alpha_{12} - \dfrac{B_{32}}{B_{33}}\alpha_{13}, & A_{12} - \dfrac{B_{32}}{B_{33}}A_{13} \\ \alpha_{22} - \dfrac{B_{32}}{B_{33}}\alpha_{23}, & A_{22} - \dfrac{B_{32}}{B_{33}}A_{23} \end{vmatrix} \\
&\text{and} \\
\Delta &= \begin{vmatrix} A_{11} - \dfrac{B_{31}}{B_{33}}A_{13} & A_{12} - \dfrac{B_{32}}{B_{33}}A_{13} \\ A_{21} - \dfrac{B_{31}}{B_{33}}A_{23} & A_{22} - \dfrac{B_{32}}{B_{33}}A_{23} \end{vmatrix}.
\end{aligned}\right\} \tag{5.6}$$

The characteristic velocity c is given by

$$c = \left(m_1 - \frac{B_{31}}{B_{33}} m_3\right) \pi_1 + \left(m_2 - \frac{B_{32}}{B_{33}} m_3\right) \pi_2. \tag{5.7}$$

In order to show clearly the use of the theory presented in the previous section, we consider a very simple case. The total number of constants in (5.1) and (5.2) is 21 with only restriction that $B_{33} \neq 0$. Therefore we can choose these constants in such a manner that

$$g_{11} = 1, \quad g_{12} = 3, \quad g_{21} = 3, \quad g_{22} = 1 \tag{5.8}$$

and

$$c = \pi_1 + \pi_2. \tag{5.9}$$

The system of characteristic equations (3.20) reduces to

$$d\pi_1/\mathrm{d}t = \pi_1 + 3\pi_2, \tag{5.10}$$

$$\mathrm{d}\pi_2/\mathrm{d}t = 3\pi_1 + \pi_2 \tag{5.11}$$

and

$$\mathrm{d}x/\mathrm{d}t = \pi_1 + \pi_2, \tag{5.12}$$

with general integral

$$\chi_1 \equiv \pi_1 + \pi_2 = a_2\, \mathrm{e}^{4t}, \tag{5.13}$$

$$\chi_2 \equiv \pi_1 - \pi_2 = a_3\, \mathrm{e}^{-2t}, \tag{5.14}$$

and

$$\chi_3 \equiv \pi_1 + \pi_2 - 4x = a_1, \tag{5.15}$$

where a_1, a_2, a_3 are arbitrary constants. In this case the line $\pi_1 = 0, \pi_2 = 0$ (or $\chi_1 = 0$, $\chi_2 = 0$ in (χ_1, χ_2, χ_3)-space) is a line of saddle points (figure 3). The integral curves are plane curves and they lie in the planes parallel to the (χ_1, χ_2)-plane, i.e.

$$\pi_1 + \pi_2 - 4x \equiv 0.$$

For any integral curve which crosses the plane

$$\pi_1 + \pi_2 = 0, \tag{5.16}$$

π_1 and π_2 cease to be single-valued functions of x and hence such integral curves do not represent a physically realistic steady solution. Fortunately in this particular problem no integral curve intersects the plane (5.16) but all integral curves obtained by putting $a_2 = 0$ in (5.13) to (5.15) lie in this plane. These integral curves (for $a_2 = 0$) do not represent any steady solution as $\mathrm{d}\pi_1/\mathrm{d}x$ and $\mathrm{d}\pi_2/\mathrm{d}x$ do not exist (or are infinite) for them and they are excluded from any discussion in the following paragraphs.

All other integral curves represent a possible continuous steady solution given by

$$\pi_1 = \tfrac{1}{2}[(a_1 + 4x) + a_3\{a_2/(a_1+4x)\}^{\frac{1}{2}}], \tag{5.17}$$

and

$$\pi_2 = \tfrac{1}{2}[(a_1 + 4x) - a_3\{a_2/(a_1+4x)\}^{\frac{1}{2}}]. \tag{5.18}$$

Let us consider the integral curves in a plane given by a constant value of a_1 in (4.15) and passing through the point $T(0, 0, a_1)$ in (χ_1, χ_2, χ_3)-space as shown in the figure 3. In this plane there is only one steady solution aTb with a continuous transition through the singular point T. This steady solution is given by (5.17) and

(5.18) for $a_3 = 0$. All other integral curves (except lTf) represent steady solutions for which the characteristic velocity does not change sign. If $a_2 > 0$, they extend from $x = -\frac{1}{4}a_1$ to $x = \infty$ and if $a_2 < 0$ they extend from $x = -\infty$ to $x = -\frac{1}{4}a_1$. In the parametric form (5.13) to (5.15) of the steady solution, the end $x = -\frac{1}{4}a_1$ is attained as t tends to $-\infty$. When the parameter t tends to $+\infty$, all integral curves tend to coincide with aTb:

$$\pi_1 = \pi_2 = \tfrac{1}{2}(a_1 + 4x). \tag{5.19}$$

A perturbation of a steady solution can be specified by its three independent components $\Delta\chi_1$, $\Delta\chi_2$ and $\Delta\chi_3$. $\Delta\chi_3$ being a component perpendicular to the plane

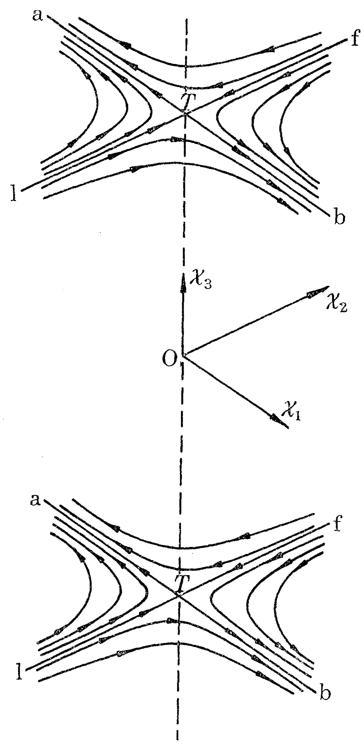

FIGURE 3. The χ_3 axis is a line of saddle points.

$\chi_3 =$ constant in (χ_1, χ_2, χ_3)-space, it remains constant in magnitude. The components $\Delta\chi_1$ and $\Delta\chi_2$ give the deformation of the steady solution in the plane $\chi_3 =$ constant. An area S bounded by a deformation in this plane changes according to the law $S = S_0 e^{2t}$, but looking at the phase-space we find that any two integral curves on the same side of lTf ultimately coincide with aTb as t tends to infinity. Therefore, if we consider a small perturbation of any steady solution such that the initial deformation due to the perturbation lies completely on the same side of the plane $\pi_1 + \pi_2 = 0$, then the deformation tends to coincide with the steady solution and therefore all steady solutions are stable.

Example 2

We consider magnetohydrodynamic flow of a perfect gas through a channel of slowly varying cross-section so that the flow can be assumed to be one dimensional. Assuming the conductivity to be infinite and that the direction of the magnetic field is perpendicular to the axis of the channel, we can write the equations of motion in the form

$$\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)-\frac{\rho}{B}\left(\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial x}\right)+\frac{\rho u}{A}\frac{\mathrm{d}A}{\mathrm{d}x}=0, \tag{5.20}$$

$$-a^2\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)+\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)=0, \tag{5.21}$$

$$\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial x}+B\frac{\partial u}{\partial x}=0 \tag{5.22}$$

and

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}+\frac{B}{\mu}\frac{\partial B}{\partial x}=0, \tag{5.23}$$

where ρ is mass density, p gas pressure, u particle velocity, B intensity of magnetic field, A cross-section of the channel, μ magnetic permeability and a is isentropic velocity of sound $=\sqrt{(rp/\rho)}$. We set

$$u_1=\rho, \quad u_2=p, \quad u_3=u, \quad u_4=B, \tag{5.24}$$

so that

$$A=\begin{bmatrix} 1 & 0 & 0 & -\rho/B \\ -a^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \end{bmatrix}, \quad B=\begin{bmatrix} u & 0 & 0 & -\rho u/B \\ -ua^2 & u & 0 & 0 \\ 0 & 0 & B & u \\ 0 & 1 & \rho u & B/\mu \end{bmatrix}. \tag{5.25}$$

The characteristic velocities given by $|B-\lambda A|=0$ are

$$u, \quad u, \quad u+\sqrt{\{a^2+(B^2/\rho\mu)\}}, \quad u-\sqrt{\{a^2+(B^2/\rho\mu)\}}.$$

Thus u is a double characteristic velocity so that $s=2$ and we take $c=u$. The vanishing of u at any point in a steady flow implies that the mass flux is zero everywhere and hence $u\equiv 0$. Thus u can be zero only in a state of equilibrium given by

$$\left.\begin{aligned} &u_0\equiv 0, \quad \rho_0=\text{arbitrary function of } x \\ \text{and} \quad &p_0+B_0^2/2\mu_0=\text{constant} \end{aligned}\right\} \tag{5.26}$$

in the channel. Therefore, we can study local stability of a gas at rest in a channel with respect to disturbances in the direction of the axis. We can choose the origin of x at any point and denote the value of the quantities at origin by a superscript '*'. Here $\alpha_{ij}=0$ for $i=1,2;\ j=1,2,3,4$ except α_{13} which is given by

$$\alpha_{13}=\frac{\rho_0^*}{A_0^*}\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^*. \tag{5.27}$$

In this case we find that

$$\left.\begin{array}{l} K_{11} = 1, \quad K_{12} = -\rho^*\mu/B^{*2}, \quad K_{21} = -a^{*2}, \quad K_{22} = 1, \\ g_{ij} = 0, \quad c_{w_i} = h_i = 0 \quad \text{for} \quad i = 1, 2; j = 1, 2, \\ \qquad\qquad\qquad\qquad R = 0, \end{array}\right\} \tag{5.28}$$

so that the equations (3.19) reduce to

$$\partial\pi_1/\partial t = 0, \quad \partial\pi_2/\partial t = 0,$$

showing that whatever disturbances are created in π_1 and π_2, they remain stationary and as time passes they do not increase or decrease in magnitude.

The above result can be explained in the following manner: π_1 and π_2 are two Riemann invariants corresponding to the characteristic velocity c. They represent deviations in density and pressure. Any perturbation in these two quantities leaves the particle velocity unchanged. Therefore, if we wish to study the propagation of disturbances only in p and ρ due to waves moving with velocity u we find that these disturbances neither propagate nor grow or decay.

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