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Study of self-similar and steady flows near singularities

BY P. L. BHATNAGAR† AND PHOOLAN PRASAD
Department of Applied Mathematics, Indian Institute of Science, Bangalore-12, India

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One-dimensional steady state flow or a self-similar flow is represented by an integral curve of the system of ordinary differential equations and, in many important cases, the integral curve passes through a singular point. Kulikovskii & Slobodkina (1967) have shown that the stability of a steady flow near the singularity can be studied with the help of a simple first-order partial differential equation. In §2 of this paper we have used their method to study steady transonic flows in radiation-gas-dynamics in the neighbourhood of the sonic point. We find that all possible one-dimensional steady flows in radiation-gas-dynamics are locally stable in the neighbourhood of the sonic point. A continuous disturbance on a steady flow, while decaying and propagating, may develop a surface of discontinuity within it. We have determined the conditions for the appearance of such a discontinuity and also the exact position where it appears. In §3 we have shown that their method can be easily generalized to study the stability of self-similar flows. As an example we have considered the stability of the self-similar flow behind a strong imploding shock. In this case we find that the flow is stable with respect to radially symmetric disturbances.

1. INTRODUCTION

Self-similar flows and one-dimensional steady-state flows in fluid dynamics share a common interesting feature that, some times, a singular point or a singular surface appears in the phase space of the flow variables. The one-dimensional steady-state flows, where the flow variables depend on just one spatial coordinate, form a particular case of self-similar flows. The singularities of the system of ordinary differential equations describing these flows can be classified in two groups. The singularities of the first group represent equilibrium states or 'uniform flows' and they can form the starting points or end points of the steady flows at infinite distance. The examples of such singularities are the Rankine–Hugoniot points in the steady-state flows (Ludford 1951; Prasad 1969; von Mises 1950) describing shock structure and we can call them 'natural singularities' since their appearance is independent of the dissipative terms, such as viscosity or heat conduction terms, included in the equations of motion. The appearance of the other group of singularities depends on the dissipative terms included in the equations of motion and hence we call them 'pseudo-singularities'. A singularity of the second group in the case of a self-similar flow corresponds to states on one of the characteristics of the flow and in the case of a steady-state flow it corresponds to those points where a characteristic velocity of original equations of motion becomes zero, i.e. where the particle velocity equals a sound velocity. For example, in the steady flow through a Laval nozzle, a singularity

† University of Rajasthan; on leave from Indian Institute of Science, Bangalore (India).

appears at the critical point where the particle velocity equals isentropic sound velocity. Finally, if we consider any steady flow with viscosity as the only dissipative mechanism, no singularity of the second group will appear, because the equations of motion with the viscous terms included are parabolic in nature with infinite speed of propagation for disturbances. Thus, the singularities in the second group depend on our physical assumptions, and they will automatically appear in similarity solutions. However, they play a very important role in determining the existence and uniqueness of a possible self-similar motion. Zel'dovich & Raizer (1967) have given an illuminating discussion of the self-similar motion of the second kind and also of the role played by pseudo-singularities in determining the unknown but unique exponent δ of the similarity variable $\xi = r/t^{\delta}$. In these self-similar motions of second kind, the equations of motion and the initial and boundary conditions contain only one independent dimensional constant so that they are not sufficient to determine the exponent, but it can be uniquely determined from a necessary condition that the integral curve must pass through the singular point.

Recently, Slobodkina (1966) has made a qualitative study of steady m.h.d. flows in channels and shown that the equations exhibit all types of singularities at the sonic point. Kulikovskii & Slobodkina (1967) have developed a general method for discussing stability of equilibrium of arbitrary steady flows in the transonic region from the nature of the singular point in the steady flow. Their analysis is valid for any system of equations of motion, hyperbolic or mixed type, with the only assumption that the characteristic speed under consideration is real and not multiple. Their basic undisturbed flow is a steady flow with a sonic transition through a singular point at $x = 0$, where x is the spatial coordinate. In the steady flow, one of the characteristic velocities, say C, of the original system of equations vanishes at $x = 0$. They perturb the original system of equations describing unsteady motion about the steady flow and retain the most dominant terms keeping in view that they wish to study only those waves which remain in the neighbourhood of the sonic point for a time interval of the order of unity, i.e. those waves whose velocity of propagation is of the order of the magnitude of C. Therefore, they approximate the perturbed equations over a length scale of the order of the magnitude of C in the neighbourhood of $x = 0$ and over a time scale of the order of unity. Finally, they get a simple first-order partial differential equation

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = \alpha C + \beta x, \tag{1.1}$$

where t is the time and α and β are constants. It is remarkable that their approximate equation (1.1) governs not only the propagation of a perturbation of the steady flow but also the steady flow in the neighbourhood of the sonic point. Therefore, to obtain the constants α and β, we do not have to work out the complicated algebra of the general theory. They are easily determined from an approximate form of the ordinary differential equations describing the steady flow. As in the case of the original system of equations, the approximate equation (1.1) is quasi-linear and hence the nonlinear effects producing a modification of the waveform are fully taken into account by it. In fact weak shocks do appear within initially continuous disturbances.

In §2 of this paper we have studied steady transonic flows in radiation-gas-dynamics and then we have used Kulikovskii & Slobodkina's method for studying non-stationary propagation of transonic waves. In §3 of this paper we have shown that Kulikovskii & Slobodkina's method can be easily generalized to study stability of self-similar flows in the neighbourhood of a singular point. We have used it to study the stability of the self-similar flow (studied by Guderley (1942) and also by Landau & Stanyukovich (see Stanyukovich 1960)) behind a strong imploding shock.

2. Transonic flows in radiation-gas-dynamics

If we neglect radiation pressure, viscosity and heat conduction, the one-dimensional equations of mass, momentum and energy are:

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2.1}$$

$$\rho(u_t + uu_x) + p_x = 0, \tag{2.2}$$

and

$$(p_t + up_x) - (\gamma p/\rho)(\rho_t + u\rho_x) + (\gamma - 1) F_x = 0, \tag{2.3}$$

where u is particle velocity, ρ mass density, p the gas pressure, γ is the ratio of specific heats and F radiation flux in positive x direction. Under the Milne–Eddington approximation and the assumption of the local thermodynamic equilibrium, the radiative transfer equation (for a grey gas) becomes

$$F_x = \kappa\rho(4\sigma T^4 - cU) \tag{2.4}$$

and

$$cU_x + 3\kappa\rho F = 0, \tag{2.5}$$

where c is the speed of light in vacuum, U radiation energy density (with equilibrium value $(4\sigma/c)\, T^4$, where σ is Stefan's constant), T temperature and κ the mass absorption coefficient. In the frame of reference in which the motion is steady equations (2.1) to (2.5) give us

$$\rho_0 u_0 = m, \tag{2.6}$$

$$mu_0 + p_0 = mc_1, \tag{2.7}$$

$$\frac{\gamma p_0}{(\gamma-1)\rho_0} + \tfrac{1}{2}u_0^2 + \frac{F_0}{m} = mc_2, \tag{2.8}$$

$$\frac{du_0}{dx} = (\gamma - 1)\,\kappa_0 \frac{(4\sigma/R_\mu^4)\{u_0(c_1 - u_0)\}^4 - cU_0}{u_0^2 - a_{s0}^2} \tag{2.9}$$

and

$$c\,dU_0/dx = -3\kappa_0\rho_0 F_0, \tag{2.10}$$

where we have used a suffix 0 to denote the value of the variables in the steady flow, m, c_1, and c_2 are constants and R_μ is the gas constant appearing in the equation of state $p = R_\mu \rho T$. We also assume that x is measured from the singular point where

$$u_0 = a_{s0} = \gamma c_1/(\gamma+1) \equiv a_{s0}^* \quad \text{(say)} \tag{2.11}$$

and

$$U_0 = (4\sigma/cR_\mu^4)\{(c_1 - a_{s0}^*)\, a_{s0}^*\}^4 \equiv U_0^* \text{ (say)}. \tag{2.12}$$

Therefore, at $x = 0$ the numerator and denominator of the right-hand side of (2.9) vanish and the value of the flux at that point is

$$F_0^* \equiv \frac{\gamma^2 m^2 c_1^2}{2(\gamma^2-1)} \left\{ \frac{2(\gamma^2-1)\, mc_2}{\gamma^2 c_1^2} - 1 \right\}. \tag{2.13}$$

If M represents the mach number, defined by

$$M = u/a_s \tag{2.14}$$

we can easily find the value of $M_0' \equiv \mathrm{d}M_0/\mathrm{d}x$ from (2.6) to (2.9) in terms of the values of the flow variables and we can show that its value at the sonic point $x = 0$ is given by the two roots of

$$(M_0')^2 - \alpha_1(M_0') - \beta_1 = 0, \tag{2.15}$$

where
$$\alpha_1 = -\frac{8\sigma\kappa_0^* \rho_0^* (\gamma-1)^2 \gamma^2 c_1^6}{(\gamma+1)^6 m R_\mu^4} \tag{2.16}$$

and
$$\beta_1 = \tfrac{3}{8}(\gamma+1)^2 (\kappa_0^* \rho_0^*)^2 \left\{ \frac{2m(\gamma^2-1)\, c_2}{\gamma^2 c_1^2} - 1 \right\}. \tag{2.17}$$

The behaviour of perturbations of the above steady-state solution in the neighbourhood of the sonic point can be studied (see Kulikovskii & Slobodkina 1967) from the Lagrange's equation

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = \alpha C + \beta x \tag{2.18}$$

for the characteristic velocity C defined by

$$C = u - a_s, \tag{2.19}$$

which vanishes at the sonic point in the steady flow. Here α and β are constants and depend only on m, c_1 and c_2. Solution of the equation (2.18) can be obtained by integrating the characteristic equations

$$\left.\begin{aligned} \mathrm{d}C/\mathrm{d}t &= \alpha C + \beta x, \\ \text{and} \qquad \mathrm{d}x/\mathrm{d}t &= C. \end{aligned}\right\} \tag{2.20}$$

In the neighbourhood of the sonic point we also have $C = u - a_s \approx a_{s0}^* (M-1)$. The equation (2.18) describes both steady and non-steady solutions near the critical point. In the steady case, system (2.20) gives a solution $C_0(x)$ of equation (2.18) in the form $C_0 = C_0(t)$, $x_0 = x_0(t)$. The general solution of equation (2.20) is

$$C = A\lambda_1 \mathrm{e}^{\lambda_1 t} + B\lambda_2 \mathrm{e}^{\lambda_2 t}, \quad x = A\,\mathrm{e}^{\lambda_1 t} + B\,\mathrm{e}^{\lambda_2 t}, \tag{2.21}$$

where λ_1 and λ_2 are the two roots of the equation

$$\lambda^2 - \alpha\lambda - \beta = 0. \tag{2.22}$$

In the case of steady flows through the sonic point we have, in the neighbourhood of (0, 0), $C_0 = \lambda_1 x$ or $C_0 = \lambda_2 x$ and since

$$C_0(x) = u_0 - a_{s0} \approx a^*_{s0}(M_0 - 1) \approx a^*_{s0} M'_0 x,$$

we obtain by substituting $\lambda = a^*_{s0} M'_0$ in (2.22)

$$a^{*2}_{s0}(M'_0)^2 - \alpha a^*_{s0} M'_0 - \beta = 0, \tag{2.23}$$

which must be the same as the equation (2.15). This gives us the values of α and β in the form

$$\alpha = a^*_{s0}\alpha_1 \tag{2.24}$$

and

$$\beta = a^{*2}_{s0}\beta_1. \tag{2.25}$$

The steady flows are described by the equations (2.20) with t as an auxiliary variable. As discussed in our previous papers (Prasad 1969; Prasad & Pandey 1970), we can classify steady flows into two classes according as $\{2mc_2(\gamma^2-1)/(\gamma^2 c_1^2)\}$ is less than or greater than one.

When $\{2mc_2(\gamma^2-1)/\gamma^2 c_1^2\} < 1$, we find that the elimination of p_0 and ρ_0 between (2.6), (2.7) and (2.8) and substitution of $F_0 \equiv 0$ in the resulting equation leads to a quadratic equation in u_0 which has two real roots. Thus, in one of the two classes of steady flows, it is possible to have two real uniform states ($p, \rho, u \equiv$ constants and $F = \mathrm{d}F/\mathrm{d}x \equiv 0$) satisfying the Rankine–Hugoniot conditions. In this class we find that

$$\frac{4\beta}{\alpha^2} \equiv -\frac{3(\gamma+1)^{14} m^2 R^8_\mu}{128\gamma^4(\gamma-1)^4\sigma^2 c_1^{12}}\left\{1-\frac{2mr(\gamma^2-1)\,c_2}{\gamma^2 c_1^2}\right\}^2 \tag{2.26}$$

is always negative and we can have two subclasses:

(i) When $4\beta/\alpha^2 < -1$, the singular point $x = 0$, $C_0 = 0$ of the steady-state equations (2.20) is a focus and as we have discussed earlier (Prasad 1969), there does not exist any steady flow with continuous sonic transition. A supersonic state can be joined to a subsonic state only through an embedded shock. This case corresponds to a physical situation where the temperature of the medium is low so that the effect of radiation flux on the flow is small (Prasad 1969).

(ii) When $-1 < 4\beta/\alpha^2 < 0$ (which implies $\{2m(\gamma^2-1)c_2/\gamma^2c_1^2\} < 1$), the singular point is a node ($\lambda_2 < \lambda_1 < 0$) and the nature of the integral curves in (x, t) plane for $\alpha = -1, \beta = -0.1875$ are shown in figure 1. This case corresponds to a medium at high temperatures and the effect of radiation flux is very significant on the flow. We remark here that, for any value of α and β, the phase plane of the system (2.20) remains unchanged under an affine transformation $\bar{C} = kC$, $\bar{x} = kx$. Therefore, figure 1 (or 2) will remain unchanged if the scales of measurements of C and x are reduced or magnified by the same factor. The arrows on the integral curves represent the direction in which a disturbance propagates. A portion of an integral curve in which there are two values of C for a given value of x cannot represent a continuous flow. We also find that there exists a continuous flow with sonic transition which joins

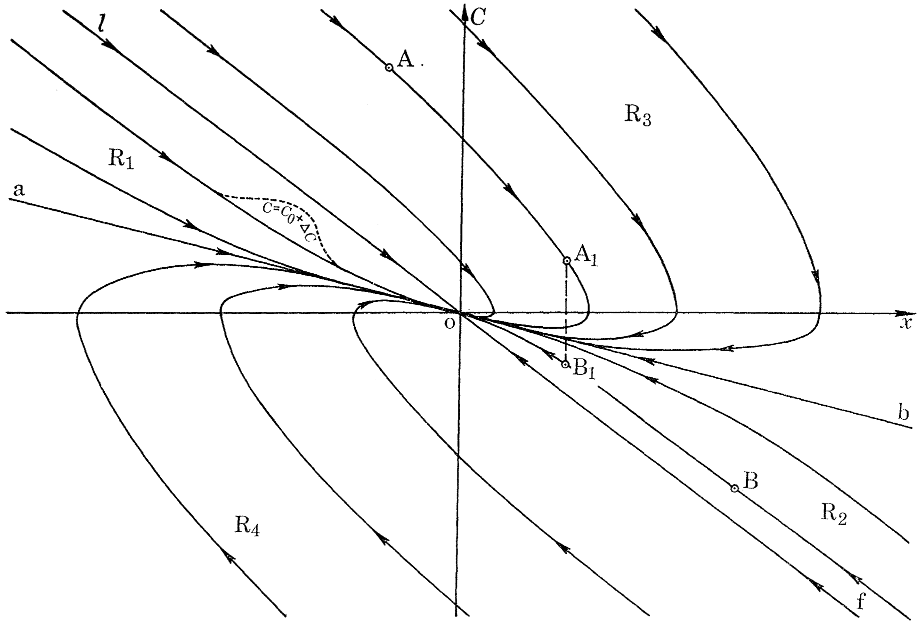

FIGURE 1. $\alpha = -1$, $\beta = -\frac{3}{16}$, $\lambda_1 = -0.25$, $\lambda_2 = -0.75$.

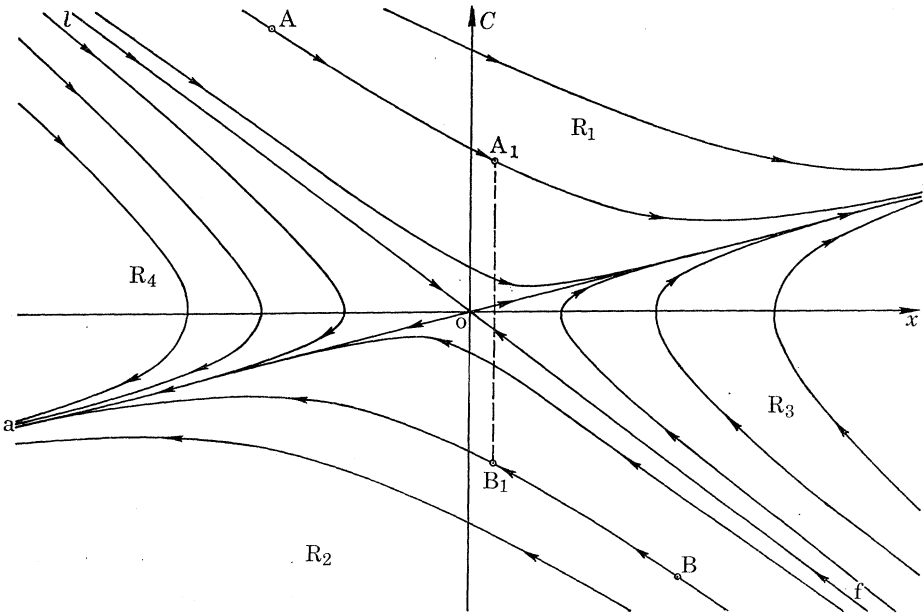

FIGURE 2. $\alpha = -\frac{1}{2}$, $\beta = \frac{3}{16}$, $\lambda_1 = +0.25$, $\lambda_2 = -0.75$.

any point in the region R_1 to any other point in the region R_2 where we have divided (x, C) plane in four regions R_1, R_2, R_3, R_4 by straight lines $C = \lambda_1 x$ and $C = \lambda_2 x$ as shown in figures 1 and 2. There is a continuous sonic transition (Prasad 1969) for shock-wave structure in this case and the wave propagation velocity changes sign at $x = 0$.

The other class of steady flows is the one for which $\{2mc_2(\gamma^2-1)/\gamma^2c_1^2\} > 1$. In this case the Rankine–Hugoniot points are complex conjugates and there does not exist any steady flow (with constants m, c_1, c_2 satisfying the above inequality) which can either terminate in or originate from a uniform state. The singular point $(0, 0)$ is a saddle point $(\lambda_2 < 0 < \lambda_1)$ as shown in figure 2 and it is possible to have just four steady flows with continuous sonic transition. These flows are aob, $l$of, aof, bo$l$. In the first two steady flows C changes sign at $x = 0$, in $l$ob it attains a minimum value at $x = 0$ and in aof a maximum at $x = 0$.

In any case (figure 1 or 2), the sonic point o divides a steady flow into two regions in such a way that one region may be regarded as independent of the other in a restricted sense, since the wave propagation velocity is zero at the sonic point. This means that we can change suitably the boundary conditions in one of the regions (or give small disturbance in one of the regions), without affecting the flow in the other region. If we consider the flow through a shock structure from a uniform supersonic state at $x = -\infty$ to a uniform subsonic state at $x = +\infty$, then

$$C_0 = u_0 - a_{s0} > 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad C_0 < 0 \quad \text{for} \quad x > 0.$$

Any small amplitude disturbance of the steady flow at any point will move towards the sonic point at $x = 0$ (or an embedded shock near $x = 0$) and if it is not amplified during its propagation (which we shall show to be true in the transonic region of our problem) it will ultimately die out at the sonic point. Thus a shock structure is stable.

We can also have a discontinuous steady flow with an embedded shock which is uniquely and completely determined from the Rankine–Hugoniot conditions as in Prasad (1969). If $C_0^{\mathrm{B}_1}$ and $C_0^{\mathrm{A}_1}$ be the values of C_0 just ahead and just behind a weak embedded shock, we can show that $C_0^{\mathrm{A}_1}$ is approximately equal to $-C_0^{\mathrm{B}_1}$. One such flow is shown in figures 1 and 2 from A to B with a jump in the wave velocity from A_1 to B_1. In this case we find that the direction of the wave velocity in each of the regions in the front and in the back of the embedded shock is towards the embedded shock and hence the disturbances created in these regions will be ultimately fed into the embedded shock.

We can establish an important relation between Lagrange's equation (2.18) and the autonomous system of characteristic equations (2.20). Equations (2.20) give steady-state flows but the general solution of (2.18) can also be obtained from (2.20) by well-known methods. The equation (2.20) written in the form

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{\alpha C + \beta x}{C} \tag{2.27}$$

describes the space rate of change of C in a steady flow. The equation (2.18), interpreted as a directional derivative in (x, t)-plane, means that the space rate of change of C, as we move along the characteristic, is again the same quantity $(\alpha C + \beta x)/C$.

Any steady-state solution $C_0(x)$ consisting of segments of integral curves (single-valued in x) of (2.20) can be taken as an unperturbed solution. A disturbance, at any instant, of a steady flow will be represented by a curve $C = C_0(x)+\Delta C(x)$ as shown in figure 1. We shall consider only those disturbances which are bounded in space and therefore any disturbance in this case will be represented by an area bounded by a closed curve in (x, C)-plane. During the propagation different points of the boundary curve of the disturbance will move along the integral curves of (2.20) and the sense of propagation is shown by the arrows.

Consider on the (x, C)-plane, an arbitrary part of an area S bounded by a closed curve whose points move in accordance with equations (2.20). Since the vector field given by the right-hand side of (2.20) has a constant divergence

$$\frac{\partial}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}t} = \alpha = \lambda_1+\lambda_2 = \frac{1}{S}\frac{\mathrm{d}S}{\mathrm{d}t},$$

it follows that $S = S_0\,\mathrm{e}^{\alpha t}$ where S_0 is the value of S at $t = 0$. In our case α is always negative; therefore S tends to zero as t tends to $+\infty$.

To arrive at certain definite conclusions, we assume, without loss of any generality that $u_0 > 0$, i.e. the fluid particles in the steady flow move in the positive direction of x axis. $C = u-a_s = u-\surd(\gamma T)$, therefore the inequality $\Delta C > 0$ for any disturbance can be physically interpreted either as an increase in speed u when the temperature is kept the same as that in the steady flow or as a decrease in temperature when the speed is kept the same as that in the steady flow or as small changes in both u and T such that $\Delta u-\frac{1}{2}\surd(\gamma/T_0)\,\Delta T > 0$. We can interpret $\Delta C < 0$ similarly.

Since the velocity of propagation depends on the amplitude C of the disturbance $C = C_0+\Delta C$, a continuous disturbance while propagating and decaying in this manner may develop a surface of discontinuity within it. It is possible to determine the space rate of change in the magnitude $C^{(1)}(x)$ of a discontinuity in t-derivative of wave velocity C for any disturbance by following the characteristic (2.20) as in Prasad (1967) and Whitham (1959). Let us consider a continuous steady flow $C = C_0(x)$ represented by a portion of an integral curve of (2.20) and create at $t = t_0$ a small continuous disturbance bounded in space. The wave velocity C behind and near the characteristic starting from the leading front of the disturbance or ahead of and near the characteristic from the trailing front of the disturbance can be expanded in the form

$$C = C_0(x)+C^{(1)}(x)\,\tau+C^{(2)}(x)\,\tau^2+\dots, \tag{2.28}$$

where

$$\tau = (t-t_0)-\int_{x_0}^{x}\frac{\mathrm{d}x}{C_0(x)}, \tag{2.29}$$

and x_0 is the position of the front or the back of the disturbance at t_0 and τ vanishes on the characteristic. Substituting (2.28) in (2.18) and equating various powers of τ from both sides, we have

$$C_0\frac{\mathrm{d}C^{(1)}}{\mathrm{d}x}+\left(\frac{\mathrm{d}C_0}{\mathrm{d}x}-\alpha\right)C^{(1)} = \frac{\{C^{(1)}\}^2}{C_0}, \tag{2.30}$$

which gives us
$$C^{(1)}(x) = \frac{\exp\left\{\alpha\int^x \frac{\mathrm{d}x}{C_0}\right\}}{C_0\left[A-\int_{x_0}^x \mathrm{d}x\,\frac{1}{C_0^3}\exp\left\{\alpha\int^x\frac{\mathrm{d}x}{C_0}\right\}\right]}, \tag{2.31}$$

where
$$A = \frac{\exp\left\{\alpha\int^{x_0}\frac{\mathrm{d}x}{C_0}\right\}}{C^{(1)}(x_0)\,C_0(x_0)}. \tag{2.32}$$

A discontinuity appears at a point $x = X$, where $C^{(1)}(x)$ becomes infinite and hence X is given by

$$\int_{x_0}^X \frac{1}{C_0^3}\exp\left\{\alpha\int^x\frac{\mathrm{d}x}{C_0}\right\}\mathrm{d}x = \frac{\exp\left\{\alpha\int^{x_0}\frac{\mathrm{d}x}{C_0}\right\}}{C^{(1)}(x_0)\,C_0(x_0)}. \tag{2.33}$$

The position X, where a discontinuity appears in front of or behind any disturbance, depends on the particular steady flow $C_0(x)$ and the initial location of the disturbance. For the steady flows

$$C_0 = \lambda_i x \quad (i = 1, 2), \tag{2.34}$$

the position X is determined from

$$\left(\frac{X}{x_0}\right)^{\alpha/\lambda_i - 2} = 1+\left(\frac{\alpha}{\lambda_i}-2\right)\lambda_i^2\frac{x_0}{C^{(1)}(x_0)}. \tag{2.35}$$

Here $X/x_0 > 0$ since a discontinuity, if it appears in the disturbance of the steady flows (2.34), will appear before the disturbance reaches the critical point o. We discuss now the two figures 1 and 2 separately.

(i) *Figure* 1. $\alpha = -1, \beta = -\frac{3}{16}, \lambda_1 = -\frac{1}{4}, \lambda_2 = -\frac{3}{4}$

We have $\lambda_2 < \lambda_1 < 0$ and the singularity is a node with negative characteristic directions. In the presence of such a singularity, the area and amplitude of any perturbation bounded in space tends to zero, while the leading and trailing fronts move towards the singular point. Thus all possible steady flows are stable.

For the steady flow aob, $\lambda_i = \lambda_1 = -\frac{1}{4}$ and equation (2.35) reduces to

$$\left(\frac{X}{x_0}\right)^2 = 1+\frac{x_0}{8C^{(1)}(x_0)}. \tag{2.36}$$

Since the wave propagates towards the point o, we have $O < X/x_0 < 1$ and, therefore, a surface discontinuity appears only if

$$-1 < \frac{x_0}{8C^{(1)}(x_0)} < 0. \tag{2.37}$$

As $C^{(1)}(x_0)$ tends to $-\frac{1}{8}x_0$, $X \to 0$ and as $C^{(1)}(x_0) \to \infty$, $X \to x_0$. The equation (2.37) shows that x_0 and $C^{(1)}(x_0)$ are of opposite sign if the discontinuity appears. $C^{(1)}(x_0)$

represents the discontinuity in the time derivative $\partial C/\partial t$ and therefore we have the following results:

(*a*) When $x_0 > 0$ a discontinuity in C appears only if the initial discontinuity in $\partial C/\partial t$ satisfies $C^{(1)}(x_0) < -\frac{1}{8}x_0$ and it appears at the leading front of the wave when $\Delta C(x) < 0$ and at the trailing front when $\Delta C(x) > 0$. Therefore, for the waves which are approaching the singular point from right, a discontinuity appears only if the disturbance is so strong that $|C^{(1)}(x_0)| > \frac{1}{8}x_0$ is initially satisfied.

(*b*) When $x < 0$, a discontinuity appears only if $C^{(1)}(x_0) > -\frac{1}{8}x_0$ and it appears at the leading front of the wave when $\Delta C(x) > 0$ and at the trailing front of the wave when $\Delta C(x) < 0$.

In the case $|C^{(1)}(x_0)| < \frac{1}{8}|x_0|$ i.e. if the initial disturbance is sufficiently weak, a continuous disturbance remains continuous and finally dies out at the critical point o.

In figure 3, the dotted curves show the type of the disturbances for which the discontinuity appears at the leading front and the curves with dashes and dots represent those disturbances for which the discontinuity appears at the trailing front.

For the steady flow *l*of, $\lambda_i = \lambda_2 = -\frac{3}{4}$ and the equation (2.35) becomes

$$\left(\frac{X}{x_0}\right)^{-\frac{2}{3}} = \left\{1 - \frac{3}{8}\frac{x_0}{C^{(1)}(x_0)}\right\}. \tag{2.38}$$

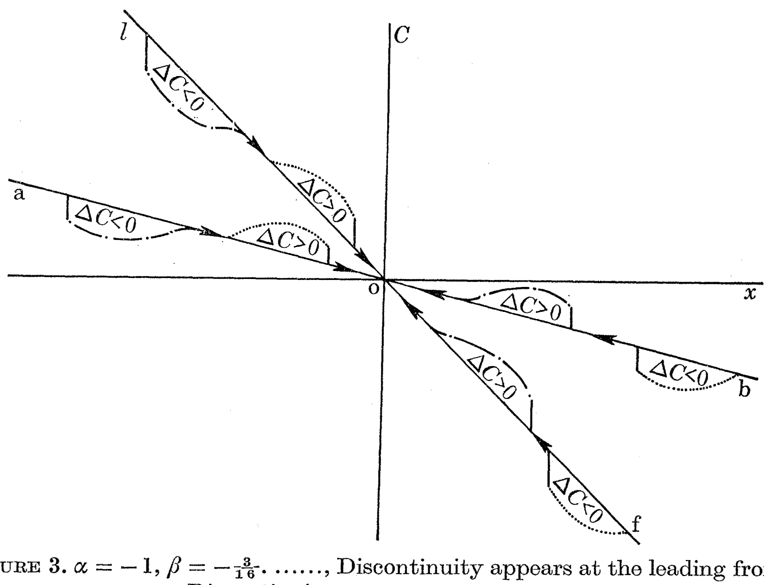

FIGURE 3. $\alpha = -1$, $\beta = -\frac{3}{16}$. ......, Discontinuity appears at the leading front; —·—, Discontinuity appears at the trailing front.

In this case also $0 < X/x_0 < 1$ and, therefore, a surface discontinuity appears if

$$\frac{x_0}{C^{(1)}(x_0)} < 0. \tag{2.39}$$

Here X tends to x_0 or zero according as $C^{(1)}(x_0)$ tends to ∞ or 0. x_0 and $C^{(1)}(x_0)$ are of opposite sign so that a discontinuity appears at the leading front of the disturbance when $x_0 > 0$ and $\Delta C < 0$ or $x_0 < 0$ and $\Delta C > 0$ are satisfied and it appears at the trailing front of the disturbance when $x_0 > 0$ and $\Delta C > 0$ or $x_0 < 0$ and $\Delta C < 0$. In this case there is no non-zero lower limit to $|C^{(1)}(x_0)|$ for the appearance of the discontinuity and a continuous disturbance, however weak, always ends into a discontinuous one before reaching the point o. These results are shown in figure 3.

In the above discussion we have taken some particular values for α and β but the whole discussion remains true for any values of α, β provided $-1 < 4\beta/\alpha^2 < 0$ and $\alpha < 0$ so that the two roots λ_1 and λ_2 satisfy $\lambda_2 < \lambda_1 < 0$. Under these conditions we find that

$$\frac{\alpha}{\lambda_1} - 2 = \frac{\lambda_2 - \lambda_1}{\lambda_1} > 0, \quad \frac{\alpha}{\lambda_2} - 2 = \frac{\lambda_1 - \lambda_2}{\lambda_2} < 0,$$

that condition (2.37) becomes

$$-1 < \left(\frac{\alpha}{\lambda_1} - 2\right) \lambda_1^2 \frac{x_0}{C^{(1)}(x_0)} < 0 \tag{2.40}$$

and that condition (2.30) remains unchanged.

We can also discuss the stability of discontinuous steady flows of the type AA_1B_1B with an embedded shock A_1B_1. We find that a disturbance is ultimately fed into the embedded shock where it decays.

(ii) *Figure* 2. $\alpha = -\frac{1}{2}, \beta = -\frac{3}{16}, \lambda_1 = \frac{1}{4}, \lambda_2 = -\frac{3}{4}$

In this case there are only four continuous steady flows aob, aof, *l*of and *l*ob with a sonic transition. Since $\alpha < 0$, the area of any perturbation in (x, C)-plane asymptotically tends to zero as t tends to $+\infty$. For a perturbation of any part of *l*of the leading and trailing fronts both approach the sonic point o but in the case of aob, they move away from o. Each of these four flows are stable. Any other steady flow from a supersonic state A to a subsonic state B will contain an embedded shock A_1B_1. The result in the previous paragraph about the stability of such flows again seems to be true.

For the steady flow *l*of, $\lambda_i = \lambda_2 = -\frac{3}{4}$ and equation (2.35) reduces to

$$\left(\frac{X}{x_0}\right)^{-\frac{4}{3}} = 1 - \frac{3x_0}{4C^{(1)}(x_0)} \tag{2.41}$$

and since $0 < X/x_0 < 1$ we find that a discontinuity will appear if the condition (2.39) is satisfied. A discontinuity always appears at the leading front of the disturbance when $x_0 > 0$ and $\Delta C < 0$ or $x_0 < 0$ and $\Delta C > 0$ are satisfied and it always appears at the trailing front of the disturbance when $x_0 > 0$ and $\Delta C > 0$ or $x_0 < 0$ and $\Delta C < 0$.

For the steady flow aob, $\lambda_i = \lambda_1 = \frac{1}{4}$ and equation (2.35) reduces to

$$\left(\frac{X}{x_0}\right)^{-4} = 1 - \frac{x_0}{4\,C^{(1)}(x_0)}. \tag{2.42}$$

In this case, the disturbance propagates away from the critical point o so that $X/x_0 > 1$ which leads to

$$0 < \frac{x_0}{4C^{(1)}(x_0)} < 1. \qquad (2.43)$$

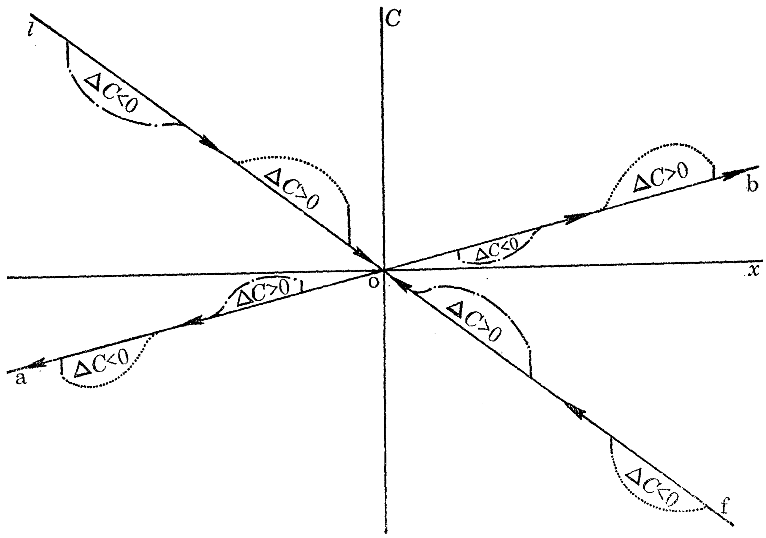

FIGURE 4. $\alpha = -\frac{1}{2}$, $\beta = -\frac{3}{16}$. ......, Discontinuity appears at the leading front; —·—, discontinuity appears at the trailing front.

As $C^{(1)}(x_0)$ tends to $\frac{1}{4}x_0$, X tends to infinity and as $C^{(1)}(x_0)$ tends to infinity, X tends to x_0. Thus a discontinuity always appears and it appears at the leading front if $\Delta C > 0$ and $x_0 > 0$ or $\Delta C < 0$ and $x_0 < 0$ and at the trailing front if $\Delta C < 0$ and $x_0 > 0$ or $\Delta C > 0$ and $x_0 < 0$.

The above results, discussed only for a particular set of values of α and β, are also true in the general case. Since $\lambda_2 < 0 < \lambda_1$, we have

$$\frac{\alpha}{\lambda_1} - 2 = \frac{\lambda_2 - \lambda_1}{\lambda_1} < 0, \quad \frac{\alpha}{\lambda_2} - 2 = -\frac{\lambda_2 - \lambda_1}{\lambda_2} < 0$$

and in the place of (2.43) we have

$$0 < -\left(\frac{\alpha}{\lambda_1} - 2\right)\lambda_1^2 \frac{x_0}{C^{(1)}(x_0)} < 1. \qquad (2.44)$$

All these results are shown in figure 4. Whether the discontinuity will appear at the leading front or at the trailing front can also be analysed intuitively by realizing that the speed of propagation is greater at a point where $|C_0 + \Delta C|$ is larger.

3. Stability of self-similar motions in the neighbourhood of a critical point

The equations of motion of a polytropic gas are

$$\rho_t + u\rho_r + \rho u_r + (\nu-1)\,\rho u/r = 0, \tag{3.1}$$

$$\rho(u_t + uu_r) + p_r = 0, \tag{3.2}$$

and

$$p_t + up_r - (\gamma p/\rho)\,(\rho_t + u\rho_r) = 0, \tag{3.3}$$

where $\nu = 1, 2$ and 3 correspond to one-dimensional, axi-symmetric and spherically symmetric motions; r represents distance from a fixed plane when $\nu = 1$, from the line of symmetry when $\nu = 2$ and from the point of symmetry when $\nu = 3$ and other variables have their usual meanings. We introduce the non-dimensional variables π, g, v, ξ and τ defined by equations:

$$\left.\begin{aligned} p(r,t) = m(t)\,\dot{R}^2(t)\,\pi(\xi,\tau),\ \rho(r,t) = m(t)\,g(\xi,\tau),\\ u(r,t) = \dot{R}(t)\,v(\xi,\tau),\ \xi = r/R(t),\ \tau = \delta \ln R(t), \end{aligned}\right\} \tag{3.4}$$

where $m(t)$ and $R(t)$ are two positive functions of time with dimensions of density and length respectively and δ is a constant. The equations (3.1) to (3.3) in terms of new dependent variables π, g, v and new independent variables ξ, τ become

$$\frac{g_\tau}{\delta} + (v-\xi)\,g_\xi + gv_\xi + \frac{(\nu-1)\,vg}{\xi} + \left(\frac{\dot{m}R}{m\dot{R}}\right) g = 0, \tag{3.5}$$

$$\frac{v_\tau}{\delta} + (v-\xi)\,v_\xi + \frac{\pi_\xi}{g} + \frac{R\ddot{R}}{\dot{R}^2}\,v = 0, \tag{3.6}$$

and

$$\frac{\pi_\tau}{\delta} + (v-\xi)\,\pi_\xi - \frac{\gamma\pi}{g}\left\{\frac{g_\tau}{\delta} + (v-\xi)\,g_\xi\right\} + \left\{-(\gamma-1)\,\frac{\dot{m}R}{m\dot{R}} + 2\,\frac{R\ddot{R}}{\dot{R}^2}\right\}\pi = 0. \tag{3.7}$$

We know that similarity solutions are possible for the equations (3.1) to (3.3) and in such cases π, g, v are functions of only one similarity variable ξ so that

$$g_\tau = v_\tau = \pi_\tau \equiv 0.$$

In the case of a self-similar flow it is necessary that the two functions $R(t)$ and $m(t)$ satisfy

$$\frac{\dot{m}R}{m\dot{R}} = \text{constant} = B\,(\text{say}), \quad \frac{R\ddot{R}}{\dot{R}^2} = \text{constant} = A\ (\text{say}). \tag{3.8}$$

Therefore, if $\pi_0(\xi)$, $g_0(\xi)$, $v_0(\xi)$ represent a self-similar flow, the functions $\pi_0(\xi)$, $g_0(\xi)$ and $v_0(\xi)$ satisfy

$$\frac{v_0-\xi}{g_0}\frac{dg_0}{d\xi} + \frac{dv_0}{d\xi} + \frac{(\nu-1)\,v_0}{\xi} + B = 0, \tag{3.9}$$

$$(v_0-\xi)\frac{dv_0}{d\xi} + \frac{1}{g_0}\frac{d\pi_0}{d\xi} + Av_0 = 0 \tag{3.10}$$

and

$$(v_0-\xi)\left[\frac{d\pi_0}{d\xi} - \frac{\gamma\pi_0}{g_0}\frac{dg_0}{d\xi}\right] + [-(\gamma-1)\,B + 2A] = 0. \tag{3.11}$$

As discussed by Sedov (1959) the set of equations (3.9) to (3.11) has many singular points and there are a large number of physically realistic flows (Zel'dovich & Raizer 1967) for which the integral curves pass through a singular point, where the value ξ^* of ξ is, in general, not zero. We can immediately apply Kulikovskii & Slobodkina's method to study the stability of these self-similar flows in the neighbourhood of these singular points. Here $\xi - \xi^*$ takes the role of the spatial coordinate x and τ that of time t. Finally the self-similar flow, being independent of the new time variable τ, takes the role of the steady flow. We also find that when $m(t)$ and $R(t)$ satisfy (3.8), the coefficients in the equations (3.5) to (3.7) do not contain the new time variable explicitly.

Let us now take a simple example and apply the above method to study the stability in the neighbourhood of the critical point. We imagine a spherically symmetric flow ($\nu = 3$) in which a strong shock wave travels to the centre of the symmetry through a gas of uniform initial density ρ_0 and zero pressure. Whatever may be the origin of the wave, the above limiting motion (i.e. when the shock radius is very small) must be self-similar (Zel'dovich & Raizer 1967). This problem was first studied independently by Landau and Stanyukovich (see Stanyukovich 1960) and Guderley (1942), and has been discussed in detail as one of the self-similar motions of second kind by Zel'dovich & Raizer. The origin for time is taken to be the instant of collapse when $R(t)$, the radius of shock, is zero. Thus the time t up to the instant of collapse is negative and we can take

$$m = \text{constant} = \rho_0, \quad R(t) = A(-t)^{\delta}. \tag{3.12}$$

Instead of working with variables π, g and v we use a new system of dependent variables V, G and Z and new spatial coordinate η defined by

$$\left.\begin{aligned} \eta = \ln \xi, \quad G(\eta, \tau) = g(\xi, \tau), \\ V(\eta, \tau) = \delta \frac{v(\xi, \tau)}{\xi}, \quad Z(\eta, \tau) = r\delta^2 \frac{\pi(\xi, \tau)}{g\xi^2}, \end{aligned}\right\} \tag{3.13}$$

and the equations (3.9) to (3.11) transform to

$$\frac{\mathrm{d}V_0}{\mathrm{d}\eta} + \frac{V_0 - \delta}{G_0}\frac{\mathrm{d}G_0}{\mathrm{d}\eta} + 3V_0 = 0, \tag{3.14}$$

$$(V_0 - \delta)\frac{\mathrm{d}V_0}{\mathrm{d}\eta} + \frac{Z_0}{\gamma G_0}\frac{\mathrm{d}G_0}{\mathrm{d}\eta} + \frac{1}{\gamma}\frac{\mathrm{d}Z_0}{\mathrm{d}\eta} + \frac{2Z_0}{\gamma} + V_0(V_0 - 1) = 0 \tag{3.15}$$

and

$$\frac{(\gamma - 1)Z_0}{G_0}\frac{\mathrm{d}G_0}{\mathrm{d}\eta} - \frac{\mathrm{d}Z_0}{\mathrm{d}\eta} - 2\left[\frac{\delta - 1}{V_0 - \delta} + 1\right] Z_0 = 0. \tag{3.16}$$

We notice that when equations (3.5) to (3.7) are expressed in terms of G, V, Z, η and τ, the independent variable η also does not appear explicitly in the coefficients and hence in our results the value η^* of η at the singular point will not appear explicitly.

The characteristics of the equations (3.5) to (3.7) in (η, τ)-plane are

$$\frac{d\eta}{d\tau} = V-\delta, \quad \frac{d\eta}{d\tau} = (V-\delta) \pm \sqrt{Z}. \tag{3.17}$$

We define Mach number μ by

$$\mu = \frac{V-\delta}{\sqrt{Z}}. \tag{3.18}$$

Solving equations (3.14) to (3.16) for $dV_0/d\eta$, $1/G_0 dG_0/d\eta$ and $dZ_0/d\eta$ and using the relation (3.18) we can easily obtain

$$\frac{d\mu_0}{d\eta} = -\frac{[(\delta-1)/\gamma]+2V_0+\delta}{\sqrt{Z_0}} + \frac{(\gamma+1)(V_0-\delta)}{2Z_0^{\frac{3}{2}}}\frac{f(V_0)}{\mu_0^2-1}, \tag{3.19}$$

and

$$\frac{dV_0}{d\eta} = \frac{2\sqrt{Z_0}}{\gamma+1}\frac{d\mu_0}{d\eta} - \left\{\frac{3\gamma-1}{\gamma+1}V_0 - \frac{2}{\gamma+1}\right\}, \tag{3.20}$$

where

$$f(V_0) = \frac{2(\delta-1)}{\gamma}(V_0-\delta) + V_0(2V_0+1-3\delta). \tag{3.21}$$

In order that the solution of equations (3.14) to (3.16) satisfies correct boundary conditions at the shock and at infinity it is necessary (Zel'dovich & Raizer 1967) that the integral curve in (Z_0, V_0)-plane must pass through the singular point (Z_0^*, V_0^*) determined by the equations

$$Z_0^* = (\delta - V_0^*)^2 \quad \text{and} \quad f(V_0^*) = 0, \tag{3.22}$$

and this determines a unique value of the exponent δ. The equation $f(V_0^*) = 0$ has two roots and V_0^* is the larger of the two roots. At the critical point $\mu_0 = \mu_0^* (= -1$ in this particular problem) and $f(V_0)/(\mu_0^2-1)$ is of the form $0 \div 0$. Therefore, we differentiate the numerator and denominator of $f(V_0)/(\mu_0^2-1)$ and use the relation (3.20). This gives us the following equation for $d\mu_0/d\eta$ at the critical point

$$\left(\frac{d\mu_0}{d\eta}\right)^2 - \alpha_1\left(\frac{d\mu_0}{d\eta}\right) - \beta_1 = 0, \tag{3.23}$$

where

$$\alpha_1 = \frac{[(\delta-1)/\gamma]+2V_0^*+\delta}{Z_0^*}\left\{\frac{V_0^*-\delta}{\mu_0^*} - \sqrt{Z_0^*}\right\} + \frac{(V_0^*-\delta)(1-5\delta)}{2Z_0^*\mu_0^*} \tag{3.24}$$

and

$$\beta_1 = -\frac{(V_0^*-\delta)}{4\mu_0^* Z_0^{*\frac{3}{2}}}\left\{\frac{2(\delta-1)}{\gamma} + 4V_0^* + (1-3\delta)\right\}\{(3\gamma-1)V_0^*-2\}. \tag{3.25}$$

Therefore, the characteristic speed (in (τ, η)-plane)

$$C = V-\delta+\sqrt{Z},$$

which vanishes at the critical point in the case of the self-similar motion, satisfies the equation

$$\frac{\partial C}{\partial \tau} + C\frac{\partial C}{\partial \eta} = \alpha C + \beta(\eta-\eta^*) \tag{3.26}$$

in the neighbourhood of the critical point.

Here the constants α and β are given by

$$\alpha = \alpha_1\sqrt{Z^*} = -\tfrac{1}{2}(5\delta - 1) \tag{3.27}$$

and

$$\beta_1 = \beta_1 Z^* = -\left\{\frac{\delta-1}{\gamma} + 2V_0^* + \frac{1-3\delta}{2}\right\}\left\{\frac{3\gamma-1}{2}V_0^* - 1\right\} \tag{3.28}$$

since $\mu_0^* = -1$ and $\sqrt{Z_0^*} = \delta - V_0^*$.

In our particular problem we have $\delta = 0.638$ for $\gamma = 3$ and tends to 1 as γ tends to 1. Thus for all physically realistic values of γ we have $\alpha < 0$. The time τ decreases from ∞ to $-\infty$ continuously as t increases from $-\infty$ to 0 up to the instant of collapse.

For $\gamma = 1.4$, we have $\delta = 0.717$, $V_0^* = 0.469$, $\alpha = -1.29$ and $\beta = 0.0397$, $\lambda_1 = 0.030$, $\lambda_2 = -1.322$ and the singular point $\eta = \eta^*$, $C = 0$ of the characteristic equations $\mathrm{d}\eta/\mathrm{d}\tau = C$, $\mathrm{d}C/\mathrm{d}\tau = \alpha C + \beta(\eta - \eta^*)$ is a saddle point. The phase plane will be similar to that in the figure 2 with the only exception that the origin o of figure 2 will correspond to the point $(\eta^*, 0)$. If we integrate equations (3·14) to (3.16) numerically from the shock boundary, we can easily verify that in the neighbourhood of the critical point, the self-similar flow is represented by the line $l$of. In order to study the growth of perturbations with time we must reverse the direction of all arrows on the integral curves since, as time increases to zero, τ decreases to $-\infty$. After reversing the direction of arrows we get the direction of propagation of waves as t increases. Owing to this change in direction of arrows, our case now corresponds to that of $\alpha > 0$ of Kulikovskii & Slobodkina (1967) where only one of the four steady flows passing through the saddle point is stable. Since $\alpha < 0$, the area of a perturbation $S = S_0 e^{\alpha\tau}$ increases without limit as τ tends to $-\infty$, i.e. as t tends to zero from negative side. The leading and trailing fronts of a disturbance of $l$of moves away from the critical point and even though the area of disturbance in (η, C)-plane increases, its boundary tends to coincide with $l$of as t tends to -0 or τ tends to $-\infty$. Therefore, our self-similar flow is stable in the neighbourhood of the critical point for radially symmetric disturbances bounded in space.

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