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Non-periodic tilings in 2-dimensions with 4, 6, 8, 10 and 12-fold symmetries

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Abstract. The two dimensional plane can be filled with rhombuses, so as to generate non-periodic tilings with 4, 6, 8, 10 and 12-fold symmetries. Some representative tilings constructed using the rule of inflation are shown. The numerically computed diffraction patterns for the corresponding tilings are also shown to facilitate a comparison with possible X-ray or electron diffraction pictures.

Keywords. Tilings; rhombuses; symmetry; non-periodic; diffraction.

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1. Introduction

Significant progress has been made in the study of quasiperiodic structures after the advent of metallic phases showing five-fold symmetry (Shechtman *et al* 1984; Levine and Steinhardt 1984, 1986; Elser 1985; Gratias and Michel 1986). A non-periodic tiling of a plane with 5-fold symmetry was earlier envisaged by Penrose (1979) and the case was further pursued by others (Bruijn 1981; Mackay 1982). At present several methods of tiling a plane non-periodically with five-fold symmetry are available, one of them being the generalized projection method (Duneau and Katz 1985). The possibility of tiling a plane with any symmetry greater than 3 was reported by Sasisekharan (1986) in which he also showed a non-periodic tiling with 7-fold symmetry. The projection from hyperspace lattice has been a successful technique not only to generate non-periodic tilings with 5-fold symmetry but also tilings with 12-fold symmetry (Stampfli 1986). The inflation rule method originally initiated by Penrose in the 5-fold tiling of a plane, has been successfully used for the 8-fold tiling (Watanabe 1986). In this article we show in a simple manner how to tile a plane non-periodically using inflation rules with 4, 6, 8, 10, 12-fold symmetries.

It is well known that in order to generate a one-dimensional quasilattice we have to use at least two length-scales (A, B) and the sequence of arrangement of these two length scales is determined by a substitution rule. To generate a one-dimensional

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Fibonacci quasi-lattice the following substitution rule is applied;

B becomes A; A becomes AB.

The successive generation of the one-dimensional Fibonacci quasilattice can be shown schematically as B; A; AB; ABA; $ABAAB$; $ABAABABA$; $ABAABABAABAAB$... This substitution rule can be represented mathematically as

$$X_1 = TX_0, \tag{1}$$

where X_0 represents a column vector $\begin{pmatrix} B \\ A \end{pmatrix}$, and T the matrix representing the recursion rule is given by

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In general this idea can be extended to more than two length scales, i.e.

$$X_0 = \begin{pmatrix} P \\ Q \\ R \\ \vdots \\ S \end{pmatrix};$$

thereby the matrix T is also extended,

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & \cdots & t_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ t_{m1} & \cdots & \cdots & t_{mm} \end{pmatrix}.$$

Thus we have infinite one-dimensional quasilattices based on the choices of X_0 and T. The property of the matrix T has been worked out by Lu *et al* (1986) so as to generate either a quasilattice or a lattice.

2. Inflation of 2-dimensional objects

This understanding of one-dimensional quasilattices can be generalized to N-dimensions. For example the elements of X_0 will be area-scales in two dimensions and volume-scales in three dimensions. Our interest at present will be on two dimensional tilings. Hence the elements of X_0 will be area-scales i.e. $P, Q, R, \ldots, S$ will be areas of planar figures. Let us say there are m-elements in X_0. Accordingly the matrix T operating on X_0 will be $m \times m$.

Let us call this transfer matrix T as the inflation operator since it inflates areas to generate a 2-dimensional tiling. Now the following aspects have to be understood:

(i) The rotational symmetry of a tiling generated by T by operating on X_0.

(ii) The elements of T to generate a particular tiling.

(iii) The values of $P, Q, R, \ldots, S$ and the number of plane figures required for a particular tiling.

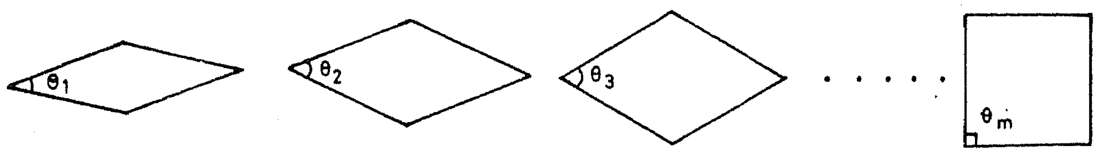


Figure 1. Rhombuses as building blocks.

For the sake of simplicity let us take the areas P, Q, $R, \ldots S$ to be the areas of rhombuses whose included angles are θ_1, θ_2, $\theta_3, \ldots, \theta_m$ and whose side-lengths are unity (figure 1). Again let us take $\theta_1 < \theta_2 < \theta_3 \cdots < \theta_m$ so that $P < Q < R \cdots < S$. From figure 1 we get

$$P = \sin\theta_1; \quad Q = \sin\theta_2; \quad R = \sin\theta_3 \cdots S = \sin\theta_m.$$

Now

$$X_0 = \begin{pmatrix} P \\ Q \\ R \\ \vdots \\ S \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \\ \cdots \\ \sin\theta_m \end{pmatrix}.$$

As regards to the inflation operator T, we shall denote its eigen-value by λ. (The number of values λ takes is the order of T).

Now the first stage in the application of the recursion rule can be written as

$$X_1 = TX_0.$$

$$X_1 = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & \cdots & t_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ t_{m1} & \cdots & \cdots & t_{mm} \end{pmatrix} \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \cdots \\ \cdots \\ \sin\theta_m \end{pmatrix} = \lambda \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \cdots \\ \cdots \\ \sin\theta_m \end{pmatrix}. \tag{2}$$

The above equation shows that each area P, Q, $R, \ldots, S$ has been inflated by a factor λ giving rise to similar rhombuses respectively. In order to find the relations between the various elements of T, a general derivation can be obtained along the following lines: From (2) for a tiling with 3 rhombuses we get,

$$\begin{aligned} t_{11}\sin\theta_1 + t_{12}\sin\theta_2 + t_{13}\sin\theta_3 &= \lambda\sin\theta_1. \\ t_{21}\sin\theta_1 + t_{22}\sin\theta_2 + t_{23}\sin\theta_3 &= \lambda\sin\theta_2. \\ t_{31}\sin\theta_1 + t_{32}\sin\theta_2 + t_{33}\sin\theta_3 &= \lambda\sin\theta_3. \end{aligned} \tag{2a}$$

Dividing the first two equations by the third and substituting $\alpha = \sin\theta_1/\sin\theta_2$ and $\beta = \sin\theta_2/\sin\theta_3$ (2a) is simplified to

$$\begin{aligned} t_{11}\alpha + t_{12}\beta + t_{13} &= \alpha(t_{31}\alpha + t_{32}\beta + t_{33}). \\ t_{21}\alpha + t_{22}\beta + t_{23} &= \beta(t_{31}\alpha + t_{32}\beta + t_{33}). \end{aligned} \tag{2b}$$

When the actual values of α and β are available (2b) can be rearranged so that the coefficients of all elements on one side are rational while those on the other side are irrational. Thus if the elements of T are to be integers, both sides of (2b) after rearrangement must vanish identically. Subsequently it can be shown that only two elements of T are independent for a tiling with two rhombuses and five elements are independent for a tiling with three rhombuses.

3. 4-fold symmetry

Let us take the simplest of the cases where T is a 2×2 matrix.

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.$$

From (2) we get

$$\begin{aligned} t_{11} \sin \theta_1 + t_{12} \sin \theta_2 &= \lambda \sin \theta_1 \\ t_{21} \sin \theta_1 + t_{22} \sin \theta_2 &= \lambda \sin \theta_2 \end{aligned}.$$

Solving these equations,

$$\sin \theta_1 / \sin \theta_2 = t_{12}/(\lambda - t_{11}) = (\lambda - t_{22})/t_{21}. \tag{3}$$

We refer to one of our earlier results (Sasisekharan 1986) for the list containing the required set of rhombuses for a particular rotational symmetry and how to obtain the rhombuses from a self-similarity principle. We reproduce the list here for completeness (table 1). From table 1 we see that two rhombuses are required for a 4-fold non-periodic tiling. These two rhombuses are $P = \sin 45°$, $Q = \sin 90°$.

It is generally agreed that there is one and only one global n-fold origin for a given non-periodic tiling with n-fold symmetry. In this article the inflation rules are applied at the global origin which is shown in the center of the tiling. From (3) we get,

$$\sin 45°/\sin 90° = 1/\sqrt{2} = t_{12}/(\lambda - t_{11}) = (\lambda - t_{22})/t_{21}. \tag{4}$$

There are infinitely many sets of values for $(t_{11}, t_{12}, t_{21}, t_{22})$ which satisfy (4). But we saw in (2b) that the elements of T are not totally independent given the areas of

Table 1. Minimum set of rhombuses required to fill 2-dimensional space for a few non-crystallographic axes of symmetry. As the polygon is taken to have $2n$ edges, there exists in each polygon a $2n$-fold axes of symmetry. Note that for both n and $2n$ values, therefore, the same set of rhombuses can be used for generating non-periodic lattices.

n	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	θ
4	$(0, 3\theta)$	$(2\theta, 2\theta)$	—	—	—	—	45°
5	$(0, 4\theta)$	$(2\theta, 3\theta)$	—	—	—	—	36°
6	$(0, 5\theta)$	$(2\theta, 4\theta)$	$(3\theta, 3\theta)$	—	—	—	30°
7	$(0, 6\theta)$	$(2\theta, 5\theta)$	$(3\theta, 4\theta)$	—	—	—	25·7°
8	$(0, 7\theta)$	$(2\theta, 6\theta)$	$(3\theta, 5\theta)$	$(4\theta, 4\theta)$	—	—	22·5°
9	$(0, 8\theta)$	$(2\theta, 7\theta)$	$(3\theta, 6\theta)$	$(4\theta, 5\theta)$	—	—	20°
10	$(0, 9\theta)$	$(2\theta, 8\theta)$	$(3\theta, 7\theta)$	$(4\theta, 6\theta)$	$(5\theta, 5\theta)$	—	18°
12	$(0, 11\theta)$	$(2\theta, 10\theta)$	$(3\theta, 9\theta)$	$(4\theta, 8\theta)$	$(5\theta, 7\theta)$	$(6\theta, 6\theta)$	15°

rhombuses. For a tiling with 4-fold symmetry (2b) reduces to

$$t_{21} - 2t_{12} = \sqrt{2}(t_{11} - t_{22})$$

and therefore

$$t_{11} = t_{22} \text{ and } t_{21} = 2t_{12}.$$

The above result enables one to find solutions to (4) and the smallest integral solution with non-vanishing elements to (4) is

$$T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

It is worth noting that the elements of T need not be integers.

The inflation factor $\lambda = 1 + \sqrt{2}$ for the above T.

The inflation operator T operates recursively on X_0 to generate an infinite tiling.
The first stage of inflation can be shown as

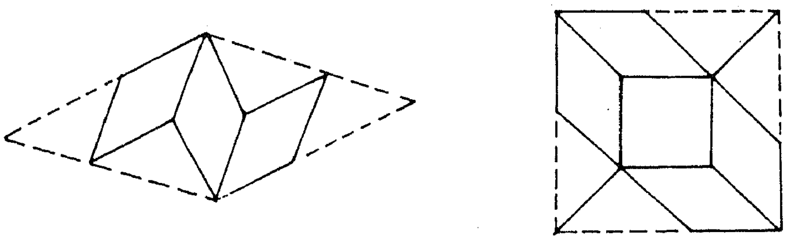
$$X_1 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \sin 45^\circ \\ \sin 90^\circ \end{pmatrix}. \tag{5}$$

The non-periodic tiling generated by such repeated operations is shown in figure 2. The successive operation of T on X_0 is given by (1). It is also of interest to note that the possibility of a non-periodic tiling with 4-fold symmetry has not been reported so far. The tiling shown in figure 2 contains more than 1000 vertices. An important question that arises here is the many possible decorations inside the inflated pattern. In fact, the larger the first inflated pattern, the more the number of possible decorations. When there are many possible decorations each will give rise to a different final tiling but with the same inflation rule. The non-periodic tiling shown in figure 2 is one such possible tiling with 4-fold symmetry and definitely not a unique tiling for the symmetry. At first, the tiling shown in figure 2 would be regarded as glassy, since more than one possible decoration inside the inflated pattern has been used to construct the tiling. The inflation rule spells out only the number of rhombuses of each type required for tiling non-periodically, but it does not indicate any unique way of arranging these rhombuses inside the inflated pattern. Various arrangements (decorations) of the rhombuses are sometimes found to be necessary to preserve the required symmetry, as seen in figure 2. Further in order to check the non-periodic nature of inflation we can compute T^n. After diagonalizing the matrix T, it is easy to arrive at T^n.

For the given matrix $T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$,

$$T^n = \begin{pmatrix} \frac{1}{2}(\lambda_1^n + \lambda_2^n) & \frac{1}{2\sqrt{2}}(\lambda_1^n - \lambda_2^n) \\ \frac{1}{\sqrt{2}}(\lambda_1^n - \lambda_2^n) & \frac{1}{2}(\lambda_1^n + \lambda_2^n) \end{pmatrix}. \tag{6}$$

It is easy to calculate, similar to the case of one-dimensional Fibonacci quasilattice,



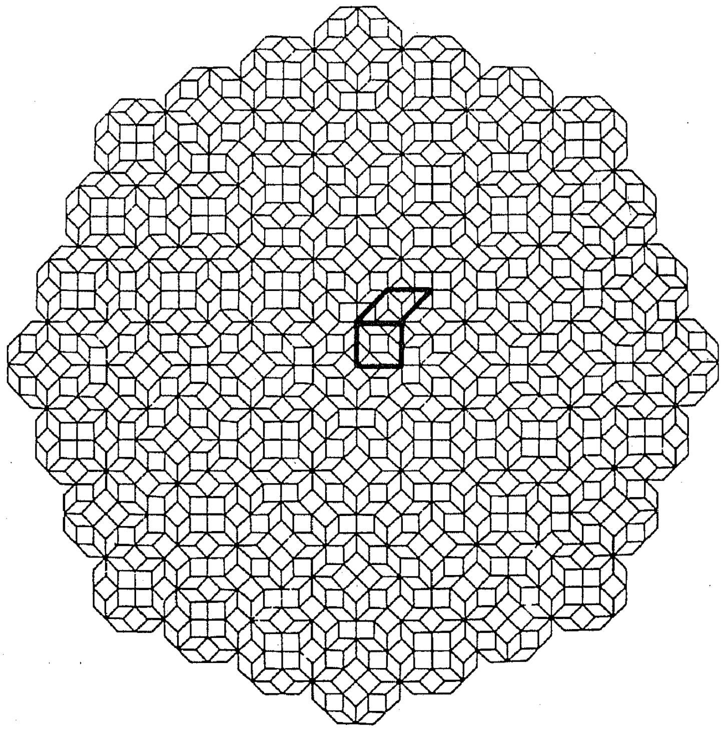


Figure 2. Non-periodic tiling with 4-fold rotational symmetry; Note that only the second stage of inflation is shown in the figure. The first stage of inflation will have only some of the vertices of the tiling. This second stage of inflation is given by,

$$T = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$

the ratio of the number of P rhombuses to the number of Q rhombuses, from T^n for a large tiling with 4-fold symmetry.

$$\lim_{n \to \infty} \frac{\text{number of } P\text{'s}}{\text{number of } Q\text{'s}} = \sqrt{2},$$

such that $T^n \neq kT$ where k is any integer. Thus the inflation is non-periodic and we have a non-periodic tiling with 4-fold rotational symmetry.

The Fourier transform of such a tiling gives the pattern obtained in the diffraction of X-rays or electrons from such a structure. The latter are commonly used in the experimental studies. Therefore the diffraction pattern for the non-periodic tiling with 4-fold symmetry at the origin has been calculated by placing unit scatterers at the vertices of the tiling shown in figure 2. The calculation has been numerically

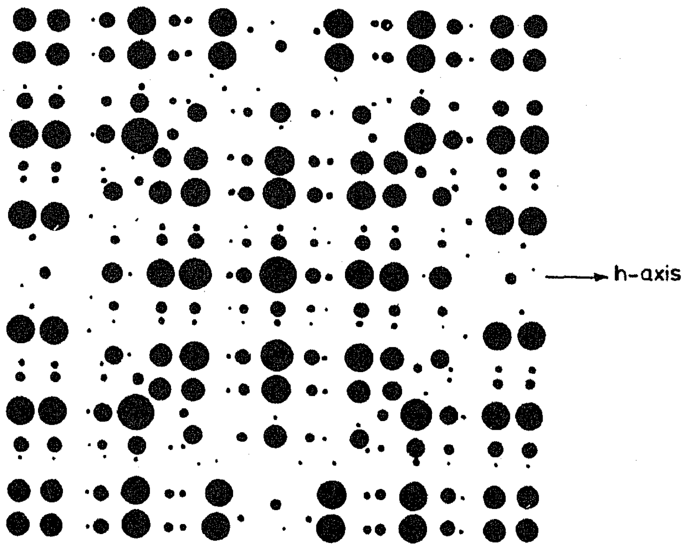


Figure 3. Computed diffraction pattern of the 4-fold non-periodic tiling.

performed by

$$F(h,k)=\sum_j f_j \exp(2\pi i(hx_j+ky_j)),$$

where x_j, y_j are coordinates of the unit scatterers with f_j set equal to unity. The numerical calculations have been carried out in DEC 1090 and VAX 11/730 systems. The computed diffraction pattern for figure 2 is shown in figure 3.

One can see that although the tiling could be regarded as glassy, there are long range correlations which lead to a distinct diffraction pattern. In fact, different tilings with 4-fold symmetry generated using different inflation rules and decorations, are found to coherently diffract although each diffraction pattern is subtly different from one another.

4. 5-fold symmetry

An analysis similar to that of 4-fold symmetry but with rhombuses P (36°) and Q (72°) gives the inflation rule,

$$T=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}. \tag{7}$$

This is the Penrose inflation rule. The tiling with 5-fold symmetry using this inflation rule has been studied by many authors (Bruijn 1981; Mackay 1982). Some aspects of the Fourier transform of finite size tilings that are perfect and imperfect, have been studied by us earlier (Baranidharan *et al* 1986) and will be communicated separately.

5. 6-fold symmetry

As before we shall borrow from table 1 the rhombuses required to generate a non-periodic tiling with 6-fold rotational symmetry. Here we find that we need three

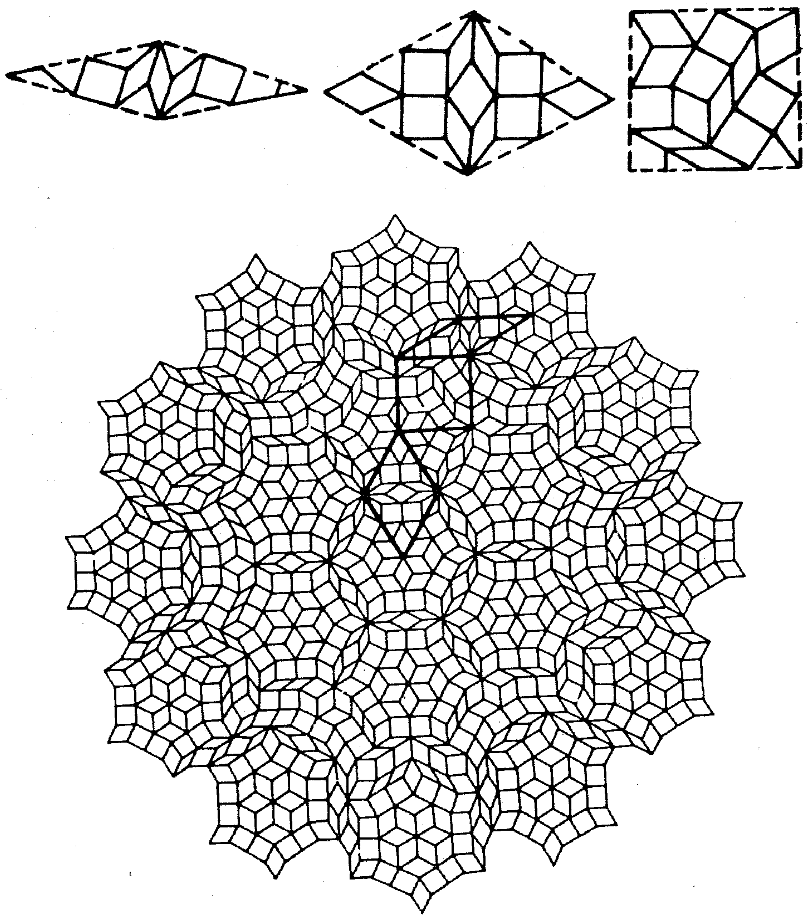


Figure 4. Non-periodic tiling with 6-fold rotational symmetry.

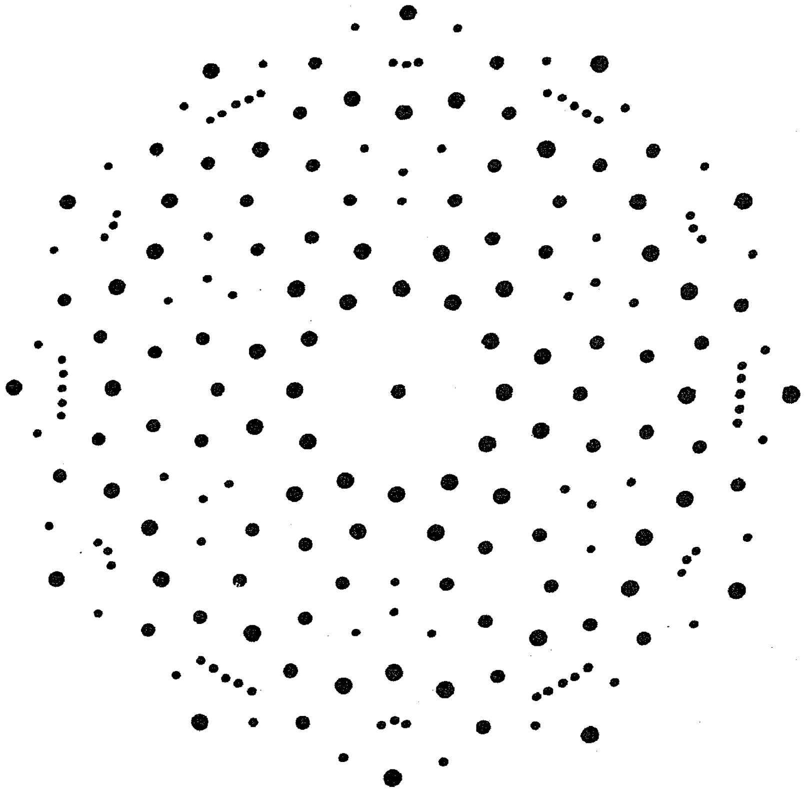


Figure 5. Computed diffraction pattern of the 6-fold non-periodic tiling.

rhombuses. These three rhombuses are

$$P = \sin 30° = 1/2; \quad Q = \sin 60° = \sqrt{3}/2; \quad R = \sin 90° = 1.$$

Substituting the values for α and β in (2b) and looking for a set of integral, non-vanishing elements of T, we get

$$T = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 7 & 4 \\ 4 & 8 & 5 \end{pmatrix}$$

with $\lambda = (2 + \sqrt{3})^2$. The non-periodic tiling with 6-fold symmetry is shown in figure 4 and its diffraction pattern is shown in figure 5. It is important to note that the above solution is one of the many possible ones. As before one can calculate T^n and find that the tiling is non-periodic.

6. 8-fold symmetry

Although table 1 tells us that we require four rhombuses, one finds that it is sufficient to use two rhombuses that were used for the 4-fold symmetry. This is because the vertex angle of the triangle used to derive the rhombuses is $180°/n$ for a n-fold axis while the n-fold axis itself is preserved about 360°. An extension of this idea leads to the fact that for a tiling with 16-fold symmetry it is enough to use the four rhombuses derived for the 8-fold case and so on. The inflation rule that was used to generate the tiling with 4-fold symmetry can be used here also but with the appropriate choice of the origin. Nevertheless for pedagogical reasons we choose the second set of small integers as a solution to T in (3) and we get,

$$T = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \quad \lambda = 2 + \sqrt{2}.$$

The inflation rule is superimposed on the 8-fold tiling shown in figure 6. The T^n has the same form as in the 4-fold case with the corresponding λ values. The diffraction pattern for this 8-fold tiling is given in figure 7.

The non-periodic tiling (figure 6) is not the same tiling given by Watanabe *et al* (1986). They generated a different tiling with

$$T = \begin{pmatrix} 6 & 4 \\ 8 & 6 \end{pmatrix}, \quad \lambda = 6 + 4\sqrt{2}.$$

That tiling can also be generated by

$$T = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \quad \lambda = 3 + 2\sqrt{2}.$$

The tiling generated by the above T is given in figure 8. The diffraction pattern of figure 8 is shown in figure 9. While the two tilings in figure 6 and figure 8 are very different, their diffraction patterns are somewhat similar. The tiling shown in figure 8 contains regions of local 8-fold symmetry and so does its transform. The

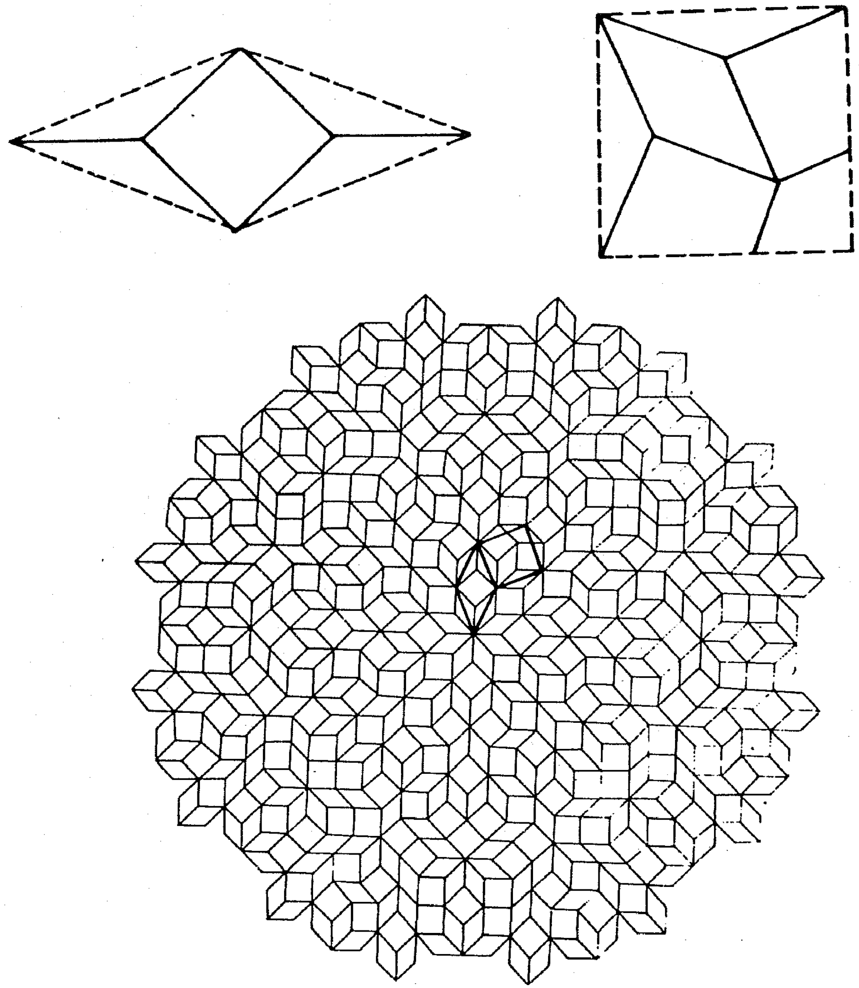


Figure 6. 8-fold non-periodic tiling.

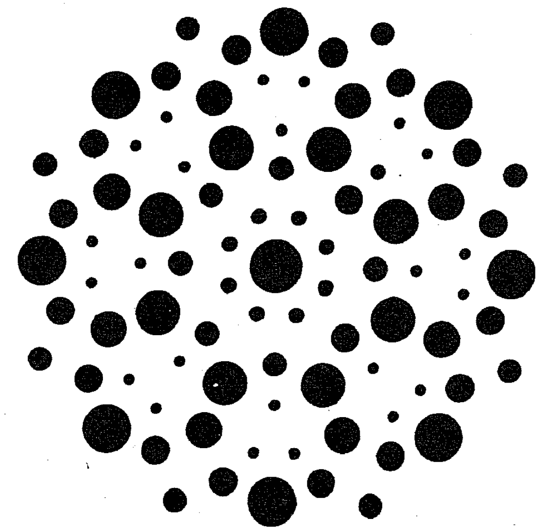


Figure 7. Computed diffraction pattern of the 8-fold non-periodic tiling shown in figure 6.

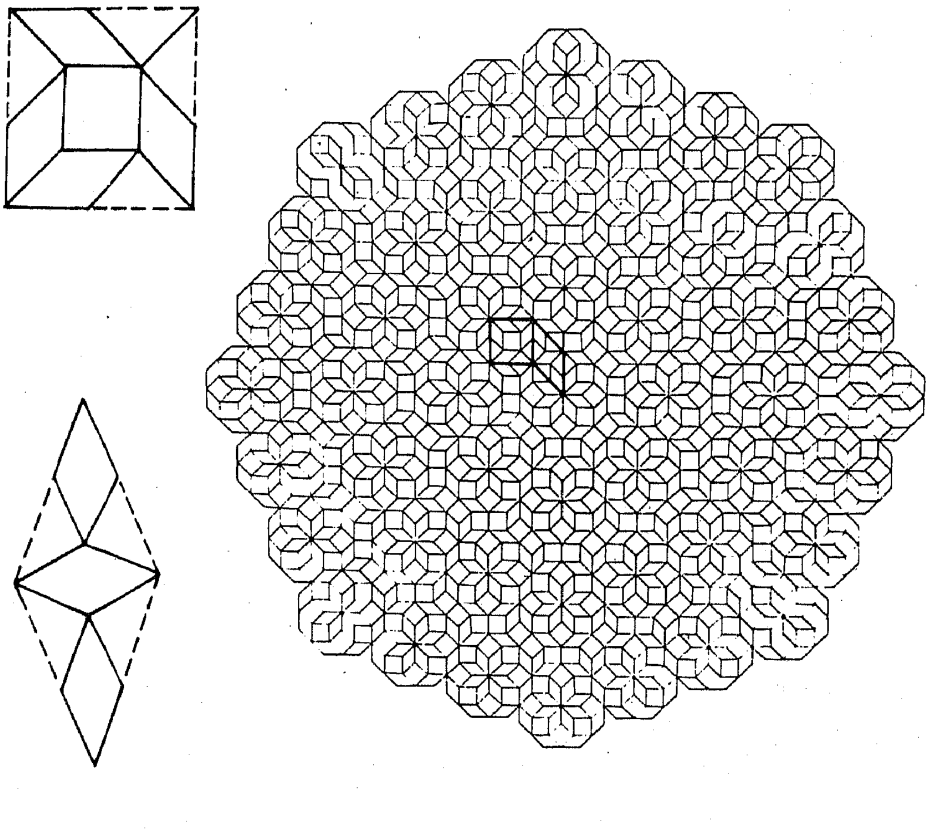


Figure 8. Non-periodic tiling with 8-fold symmetry generated by a different inflation rule.

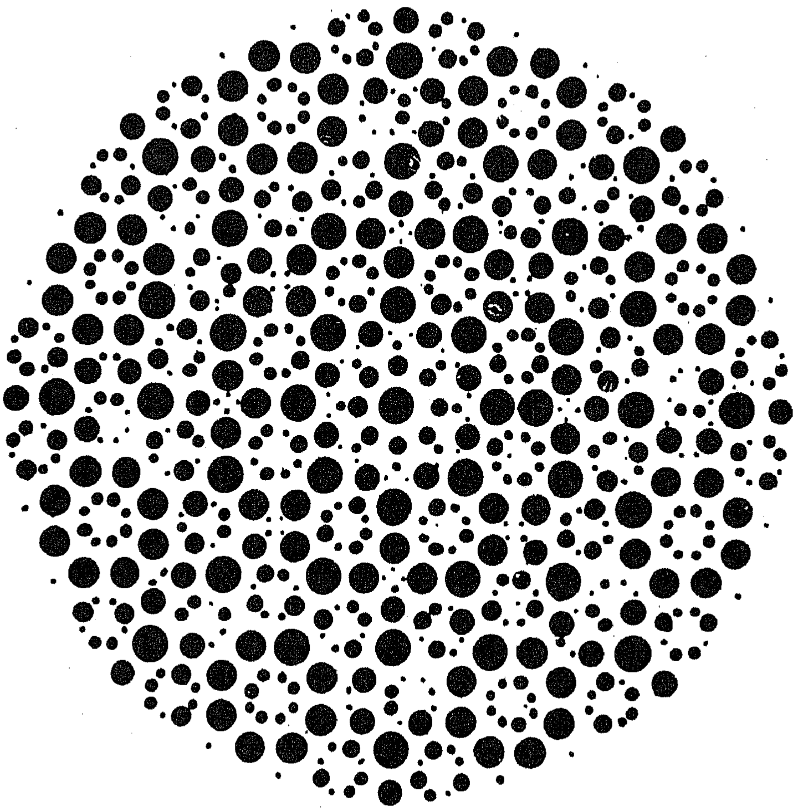


Figure 9. Diffraction pattern corresponding to the 8-fold non-periodic tiling shown in figure 8.

tiling shown in figure 6 does not have regions of local site symmetry, but such a difference is seen only in the weak peaks in its transform. Interestingly, the diffraction from the octagonal phase (Wang *et al* 1987) shows the presence of local 8-fold symmetry everywhere. We can empirically state that peaks that are not affected by the local site symmetries could be regarded as the primary peaks and the others as secondary. It must be stated that the two tilings with 8-fold symmetry are generated using different inflation rules and not using the same inflation rule with different decoration of the inflated pattern.

Now it is quite clear that by using the same basis X_0, we can generate different non-periodic tilings with the same symmetry by appropriate choice of the inflation operator T.

7. 10-fold symmetry

The same reasons with which we chose two rhombuses in the 8-fold case apply here too. We choose the two rhombuses that are required for a tiling with 5-fold symmetry although one can in principle use 5 rhombuses (table 1). The inflation operator T used here is different from the 5-fold case as shown below:

$$T=\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad \lambda=((1+\sqrt{5})/2)^4.$$

The inflation rule and the 10-fold tiling are shown in figure 10. The decomposition of rhombuses inside the inflated pattern shown is the same as that shown by Ammann (Grunbaum and Shepherd 1987). The computed diffraction pattern is given in figure 11.

Table 2. Some examples of matrices required to construct tilings with 4, 5, 6, 8, 10 and 12-fold symmetries. The number of rhombuses required for a particular tiling is given in table 1.

n	Inflation matrix (T)	Largest eigen-value (λ)
4	$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$	$1+\sqrt{2}$
5	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$((1+\sqrt{5})/2)^2$
6	$\begin{pmatrix} 3 & 4 & 2 \\ 4 & 7 & 4 \\ 4 & 8 & 5 \end{pmatrix}$	$(2+\sqrt{3})^2$
8	$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$	$2+\sqrt{2}$
10	$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$	$((1+\sqrt{5})/2)^4$
12	$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}$	$2(2+\sqrt{3})$

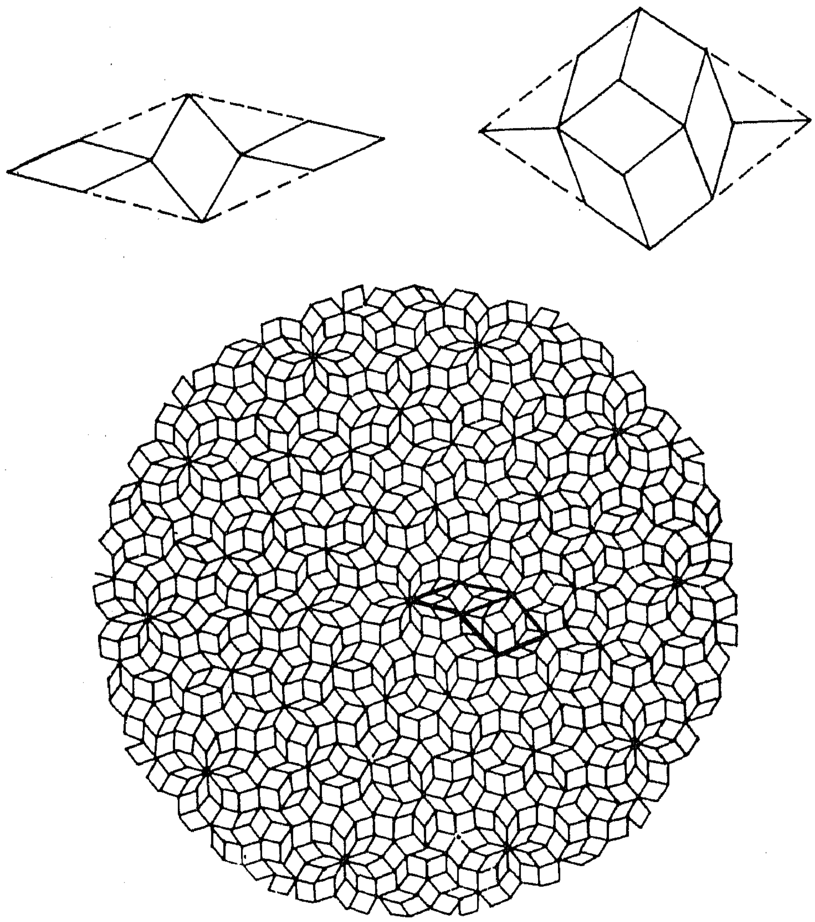


Figure 10. Non-periodic tiling with 10-fold symmetry.

A comparison with the experimentally observed diffraction from decagonal phases shown that while the positions of the peaks are correspondingly similar, the intensities of the weak peaks do not satisfactorily match. It is speculated that tilings with 10-fold symmetry constructed using different and larger inflated patterns may show some resemblance to the experimental observations (Srinivasan *et al* 1988).

8. 12-fold symmetry

The generation of dodecagonal quasilattice by the projection technique has been worked out by Stampfli (1986), who obtained triangle and square quasilattice. A dodecagonal tiling made of three rhombuses was shown by Gratias (1986) alongwith a 10-fold tiling. Here we show how to obtain a tiling with 12-fold symmetry by the inflation method. We use the same rhombuses that were used for the tiling with 6-fold symmetry. The inflation operator for the 12-fold tiling is given by,

$$T = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}, \quad \lambda = 2(2+\sqrt{3}).$$

The inflation rule and the dodecagonal tiling are given in figure 12. The diffraction

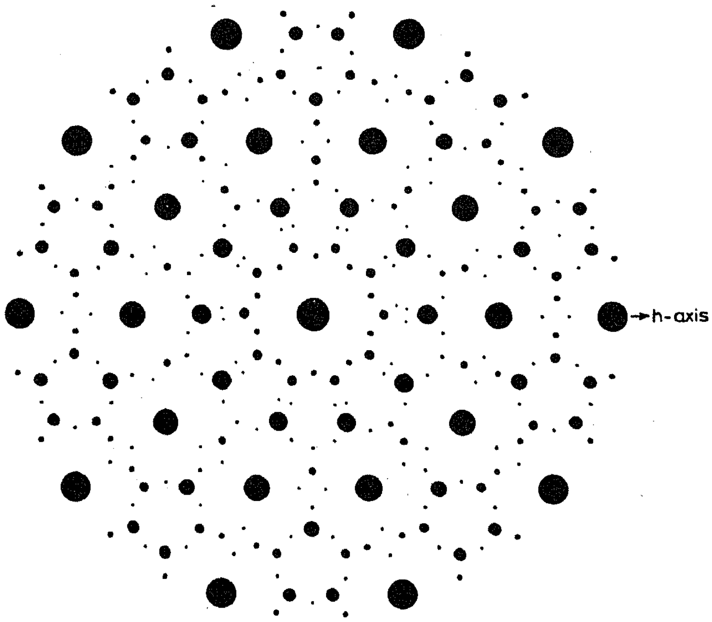


Figure 11. Computed diffraction pattern for the 10-fold non-periodic tiling given in figure 10.

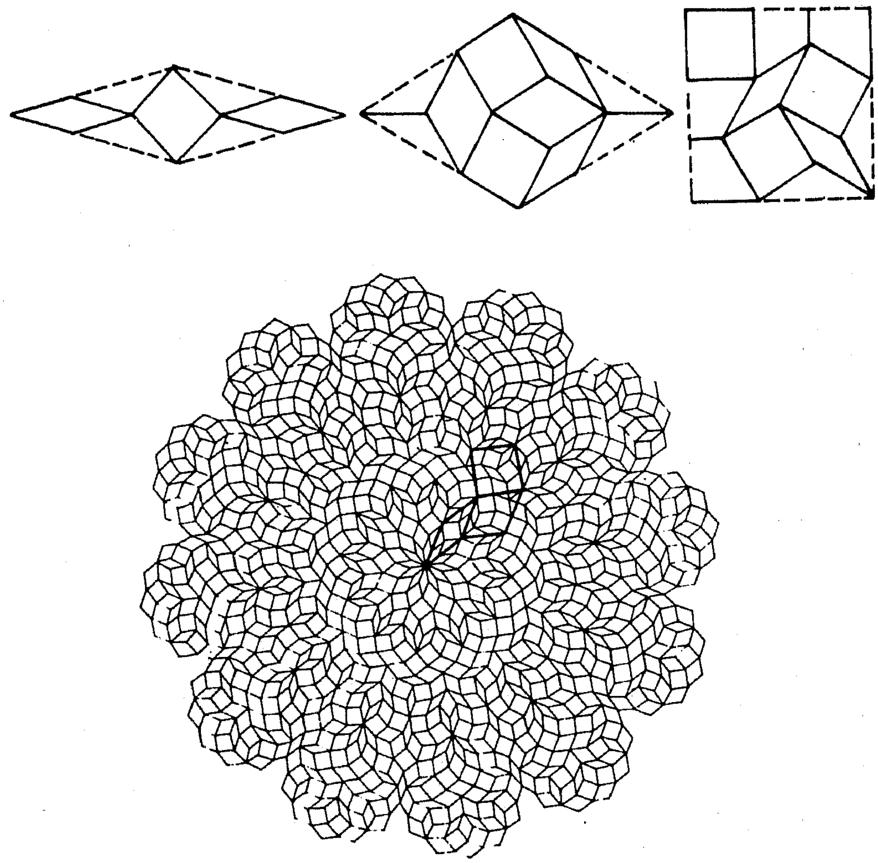


Figure 12. 12-fold non-periodic tiling.

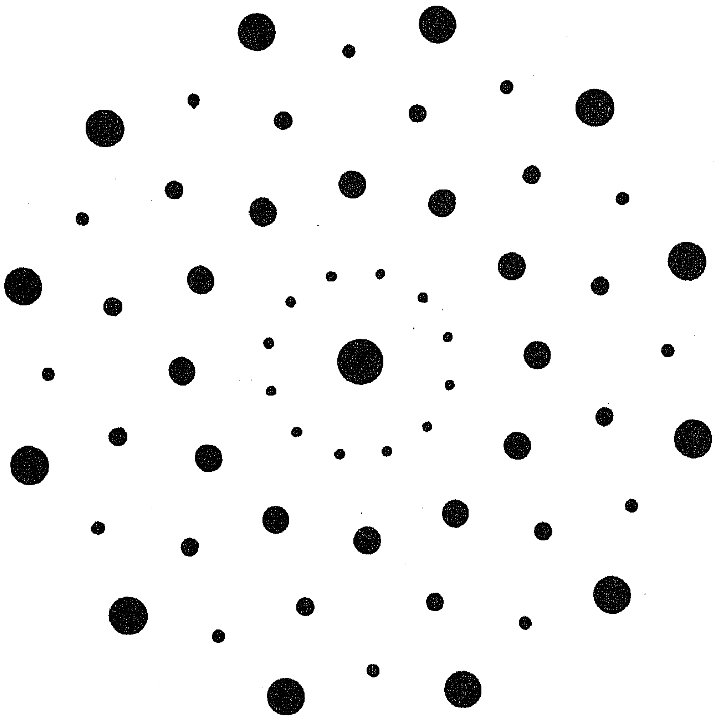


Figure 13. Diffraction pattern corresponding to the 12-fold non-periodic tiling shown in figure 12.

pattern for the 12-fold tiling is given in figure 13. Similar to the 6-fold case figure 12 is also non-periodic. One can of course use the above inflation rule to construct a tiling with 6-fold symmetry by choosing an appropriate origin which will be different from figure 4.

9. Conclusions

The discovery of the crystallographically forbidden icosahedral rotational symmetry in the electron diffraction pictures of rapidly quenched Al-Mn alloys (Shechtman *et al* 1984) has sparked off a considerable discussion on quasiperiodic and non-periodic tilings with 5-fold and various other symmetries (Steinhardt and Ostlund 1987). A non-periodic tiling with true 7-fold symmetry and its Fourier transform was shown by us earlier (Baranidharan *et al* 1988). The present analysis shows that the following observations can be made: (i) non-periodic tilings with n-fold symmetry can be generated using inflation rules; some examples of inflation rules used for constructing the various tilings shown in the article are summarised in table 2; (ii) different tilings with the same symmetry can possibly be generated by the same inflation rule but with different choice of the decoration of the inflated pattern; (iii) still different tilings with the same symmetry can be constructed by choosing different inflation rules; an example is the case of tilings with 8-fold symmetry (iv) The inflation rule for a $2n$-fold symmetry tiling can be used for a tiling with n-fold symmetry.

It is not known whether all the tilings generated using the inflation rules can be obtained by any of the projection methods. Whittaker and Whittaker (1988) has shown that only periodic tilings with 2, 3, 4 and 6-fold symmetries are obtained by the projection method. All the tilings presented here have distinct diffraction patterns shown by the numerical calculations. The tilings appear to possess long range correlations. As yet there is no mathematical proof that the tilings are quasiperiodic.

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References

Baranidharan S, Gopal E S R and Sasisekharan V 1986 *Proc. Solid State Physics symposium* (*India*) **C29** 116
Baranidharan S, Balagurusamy V S K, Srinivasan A, Gopal E S R and Sasisekharan V 1988 *Proc. Int. Conf. on modulated structures, polytypes and quasicrystals* Banaras, India (to appear in Phase Transitions)
Bruijn N G de 1981 *Proc. Konin. Ned. Acad. Weten.* **A84** 39
Duneau M and Katz A 1985 *Phys. Rev. Lett.* **54** 2688
Elser V 1985 *Phys. Rev.* **B32** 4892
Gratias D and Michel L 1986 *Proc. Int. Conf. on aperiodic crystals, J. Physique* **47** C3
Gratias D 1986 *La Recherche* **17** 788
Grunbaum B and Shepherd G C 1987 *Tilings and patterns* (New York: W H Freeman)
Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477
Levine D and Steinhardt P J 1986 *Phys. Rev.* **B34** 596
Lu J P, Odagaki T and Birman J L 1986 *Phys. Rev.* **B33** 4809
Mackay A L 1982 *Physica* **A114** 609
Penrose R 1979 *Math. Intell.* **2** 32
Sasisekharan V 1986 *Pramāṇa – J. Phys.* **26** L283
Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
Srinivasan A, Balagurusamy V S K, Baranidharan S, Sasisekharan V and Gopal E S R 1988 *Proc. Solid State Phys. Symp.* (*India*) **C31** 90
Stampfli P 1986 *Helv. Physica Acta* **59** 1260
Steinhardt P J and Ostlund S 1987 *The physics of quasicrystals* (Singapore: World Scientific)
Wang N, Chen H and Kuo K H 1987 *Phys. Rev. Lett.* **59** 1010
Watanabe Y, Ito M, Soma T and Betsumiya T 1986 *Acta Crystallogr.* **A43** 133
Whittaker E J W and Whittaker R M 1988 *Acta Crystallogr.* **A44** 105